Quantum Mechanics HT 2020: Problem Sheet 5

Spin and $S > 1$ Systems

5.1 Write down the expression for the commutator $[\sigma_i, \sigma_j]$ of two Pauli matrices. Show that the anticommutator of two Pauli matrices is

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}.$$

5.2 Let $\boldsymbol{n}$ be any unit vector and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the vector whose components are the Pauli matrices. Why is it physically necessary that $\boldsymbol{n} \cdot \boldsymbol{\sigma}$ satisfy $(\boldsymbol{n} \cdot \boldsymbol{\sigma})^2 = I$, where I is the 2×2 identity matrix? Let $\boldsymbol{m}$ be a unit vector such that $\boldsymbol{m} \cdot \boldsymbol{n} = 0$. Why do we require that the commutator $[\boldsymbol{m} \cdot \boldsymbol{\sigma}, \boldsymbol{n} \cdot \boldsymbol{\sigma}] = 2\mathrm{i}(\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{\sigma}$? Prove that these relations follow from the algebraic properties of the Pauli matrices. You should be able to show that $[\boldsymbol{m} \cdot \boldsymbol{\sigma}, \boldsymbol{n} \cdot \boldsymbol{\sigma}] = 2\mathrm{i}(\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{\sigma}$ for any two vectors $\boldsymbol{n}$ and $\boldsymbol{m}$.

5.3 Let $\boldsymbol{n}$ be the unit vector in the direction with polar coordinates (θ, ϕ). Write down the matrix $\boldsymbol{n} \cdot \boldsymbol{\sigma}$ and find its eigenvectors. Hence show that the state of a spin-half particle in which a measurement of the component of spin along $\boldsymbol{n}$ is certain to yield $\frac{1}{2}\hbar$ is

$$|+, \boldsymbol{n}\rangle = \sin(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi/2}|-\rangle + \cos(\theta/2)\, \mathrm{e}^{-\mathrm{i}\phi/2}|+\rangle,$$

where $|\pm\rangle$ are the states in which $\pm\frac{1}{2}$ is obtained when s_z is measured. Obtain the corresponding expression for $|-, \boldsymbol{n}\rangle$. Explain physically why the amplitudes in the previous equation have modulus $2^{-1/2}$ when $\theta = \pi/2$ and why one of the amplitudes vanishes when $\theta = \pi$.

5.4 For a spin-half particle at rest, the operator $\boldsymbol{J}$ is equal to the spin operator $\boldsymbol{S}$. Use the properties of the Pauli spin matrices to show that in this case the rotation operator $U(\boldsymbol{\alpha}) \equiv \exp(-\mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{J}/\hbar)$ is

$$U(\boldsymbol{\alpha}) = I \cos\left(\frac{\alpha}{2}\right) - \mathrm{i}\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\sigma} \sin\left(\frac{\alpha}{2}\right),$$

where $\hat{\boldsymbol{\alpha}}$ is the unit vector parallel to $\boldsymbol{\alpha}$. Comment on the value this gives for $U(\boldsymbol{\alpha})$ when $\alpha = 2\pi$.

5.5 Explain why a spin-$\frac{1}{2}$ particle in a magnetic field $\boldsymbol{B}$ has a Hamiltonian given by

$$H = -\gamma \boldsymbol{S} \cdot \boldsymbol{B},$$

where γ is the gyromagnetic ratio which you should define. In a coordinate system such that $\boldsymbol{B}$ lies along the z-axis, a proton is found to be in a eigenstate $|+, x\rangle$ of $\hat{S}_x$ at $t = 0$. Find $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ for $t > 0$.

5.6 Write down the 3×3 matrix that represents S_x for a spin-one system in the basis in which S_z is diagonal (i.e., the basis states are $|0\rangle$ and $|\pm\rangle$ with $S_z|+\rangle = |+\rangle$, etc.)

A beam of spin-one particles emerges from an oven and enters a Stern–Gerlach filter that passes only particles with $J_z = \hbar$. On exiting this filter, the beam enters a second filter that passes only particles with $J_x = \hbar$, and then finally it encounters a filter that passes only particles with $J_z = -\hbar$. What fraction of the particles stagger right through?

5.7 A system that has spin angular momentum $\sqrt{6}\hbar$ is rotated through an angle ϕ around the z-axis. Write down the 5×5 matrix that describes a rotation by an angle ϕ around the z-axis.

5.8 * *Vector operators* (Optional, off-syllabus problem)

(a) Show that expectation values of the position operators in a state $|\psi\rangle$ transform like a classical vector under a rotation around an axis $\mathbf{n}$ by an angle α.
(b) Show that the commutation relations $[\hat{V}_j, \hat{J}_k] = i\hbar\epsilon_{jkl}\hat{V}_l$ are equivalent to

$$\hat{V}_j + \frac{d\alpha}{i\hbar}[\hat{V}_j, \mathbf{n} \cdot \hat{\mathbf{J}}] = \sum_k \left(R(\mathbf{n}d\alpha)\right)_{jk} \hat{V}_k\ ,$$

where $\mathbf{n}$ is a unit vector and $R(\mathbf{n}d\alpha)$ is the rotation matrix around the axis $\mathbf{n}$ by an angle $d\alpha$.

Composite systems

5.10 A system AB consists of two non-interacting parts A and B. The dynamical state of A is described by $|a\rangle$, and that of B by $|b\rangle$, so $|a\rangle$ satisfies the TDSE for A and similarly for $|b\rangle$. What is the ket describing the dynamical state of AB? In terms of the Hamiltonians $H_{\rm A}$ and $H_{\rm B}$ of the subsystems, write down the TDSE for the evolution of this ket and show that it is automatically satisfied. Do $H_{\rm A}$ and $H_{\rm B}$ commute? How is the TDSE changed when the subsystems are coupled by a small dynamical interaction $H_{\rm int}$? If A and B are harmonic oscillators, write down $H_{\rm A}$, $H_{\rm B}$. The oscillating particles are connected by a weak spring. Write down the appropriate form of the interaction Hamiltonian $H_{\rm int}$. Does $H_{\rm A}$ commute with $H_{\rm int}$? Explain the physical significance of your answer.

5.11 Explain what is implied by the statement that "the physical state of system A is correlated with the state of system B." Illustrate your answer by considering the momenta of cars on (i) London's circular motorway (the M25) at rush-hour, and (ii) the road over the Nullarbor Plain in southern Australia in the dead of night.

5.12 Consider a system of two particles of mass m that each move in one dimension along a given rod. Let $|1;x\rangle$ be the state of the first particle when it's at x and $|2;y\rangle$ be the state of the second particle when it's at y. A complete set of states of the pair of particles is $\{|xy\rangle\} = \{|1;x\rangle|2;y\rangle\}$. Write down the Hamiltonian of this system given that the particles attract one another with a force that's equal to C times their separation.

Suppose that the particles experience an additional potential $V(x,y) = \frac{1}{2}C(x+y)^2$. Show that the dynamics of the two particles is now identical with that of a single particle that moves in two dimensions in a particular potential $\Phi(x,y)$, and give the form of Φ.

5.13 In the lectures we considered measurements by Alice and Bob on an entangled pair of spins prepared in a singlet state. Bob measures the component of spin along an axis that is inclined by angle θ to that used by Alice. Given the expression

$$|-,\boldsymbol{b}\rangle = \cos(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2}|-\rangle - \sin(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2}|+\rangle,$$

for the state of a spin-half particle in which it has spin $-\frac{1}{2}$ along the direction $\boldsymbol{b}$ with polar angles (θ,ϕ), with $|\pm\rangle$ the states in which there is spin $\pm\frac{1}{2}$ along the z-axis, calculate the amplitude that Bob finds the positron's spin to be $-\frac{1}{2}$ given that Alice has found $+\frac{1}{2}$ for the electron's spin. Hence show that the corresponding probability is $\cos^2(\theta/2)$.