Quantum Mechanics HT 2020: Problem Sheet 4

Transformations

4.1 *Reflection symmetry around a point* $\mathbf{x}_0$

Let $P_{\mathbf{x_0}}$ be the operator that induces reflections around a point $\mathbf{x_0}$. Argue that

$$\begin{aligned}\hat{P}_{\mathbf{x_0}}|\mathbf{x_0}+\mathbf{x}\rangle &= |\mathbf{x_0}-\mathbf{x}\rangle\ ,\\ \hat{P}_{\mathbf{x_0}}\hat{\mathbf{x}}\hat{P}_{\mathbf{x_0}} &= 2\mathbf{x_0}\mathbf{1}-\hat{\mathbf{x}}\ ,\\ \hat{P}_{\mathbf{x_0}}\hat{\mathbf{p}}\hat{P}_{\mathbf{x_0}} &= -\hat{\mathbf{p}},\end{aligned}$$

and that the transformed wave function fulfils

$$\psi'(\mathbf{x}) = \psi(2\mathbf{x_0}-\mathbf{x}).$$

4.2 For which potentials V is the Hamiltonian $H = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$ translationally invariant?

4.3 Show that the orbital angular momentum operators $\hat{L}_a$ $(a = x, y, z)$ are Hermitian.

4.4 A spinless QM system is called *rotationally invariant* if its Hamiltonian commutes with the orbital angular momentum operators $[H, \hat{L}_a] = 0$, $a = x, y, z$. Rotational invariance expresses the fact the energy measurements remain unchanged under rotations of the system. If the Hamiltonian commutes only with $\hat{L}_z$ it is called invariant under rotations around the z-axis. Consider Hamiltonians of the form

$$H = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}).$$

Show that potentials that depend only on the distance $\|\mathbf{x}\|$ lead to rotationally symmetric Hamiltonians, while potentials that depend on x and y only through the combination $x^2 + y^2$ leads to Hamiltonians that invariant under rotations around the z-axis.

4.5 Show that

$$\lim_{N\to\infty}\left[1+\frac{x}{N}\right]^N = e^x.$$

Give arguments that an analogous formula holds for operators.

4.6 *Heisenberg equations of motion for the SHO*

Derive the Heisenberg equations of motion for the creation and annihilation operators in the simple harmonic oscillator and show that their solution is

$$a(t) = a(0)e^{-i\omega t}\ , \quad a^\dagger(t) = a^\dagger(0)e^{i\omega t}\ .$$

From these, obtain equations of motion for the position and momentum operators. Comment on the relation of your results to Ehrenfest's theorem.

Orbital angular momentum

4.7 (a) Show by explicit calculation using $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ that $[\hat{L}_i, \hat{x}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{x}_k$ and $[\hat{L}_i, \hat{p}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{p}_k$.

(b) Evaluate $[\hat{L}_x, \hat{L}_y]$ by writing $\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$ and using the results from part (a) of this question.

(c) Show that in the position representation we have

$$\langle\mathbf{x}|\hat{L}_i|\psi\rangle = \hat{\mathrm{L}}_i\psi(\mathbf{x})\ ,$$

and obtain explicit expressions for the differential operators $\hat{\mathrm{L}}_i$. Show that for any differentiable function f

$$(\hat{\mathrm{L}}_x\hat{\mathrm{L}}_y - \hat{\mathrm{L}}_y\hat{\mathrm{L}}_x)f(x,y,z) = \mathrm{i}\hbar\hat{\mathrm{L}}_z f(x,y,z).$$

Since this holds for any f it can be written as an operator equation $[\hat{\mathrm{L}}_x, \hat{\mathrm{L}}_y] = \hat{\mathrm{L}}_x\hat{\mathrm{L}}_y - \hat{\mathrm{L}}_y\hat{\mathrm{L}}_x = \mathrm{i}\hbar\hat{\mathrm{L}}_z$, as you will have found in part (b). Deduce similar expressions for $[\hat{\mathrm{L}}_y, \hat{\mathrm{L}}_z]$ and $[\hat{\mathrm{L}}_z, \hat{\mathrm{L}}_x]$.

(d) Defining $\hat{\boldsymbol{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, show that (Hint: remember $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$)

$$[\hat{L}_x, \hat{\boldsymbol{L}}^2] = [\hat{L}_y, \hat{\boldsymbol{L}}^2] = [\hat{L}_z, \hat{\boldsymbol{L}}^2] = 0.$$

(e) Show that in spherical polar co-ordinates the differential operators $\hat{\mathrm{L}}_i$ take the form

$$\hat{\mathrm{L}}_x \;=\; i\sin\phi\frac{\partial}{\partial\theta} + i\cot\theta\cos\phi\frac{\partial}{\partial\phi}\ , \quad \hat{\mathrm{L}}_y = -i\cos\phi\frac{\partial}{\partial\theta} + i\cot\theta\sin\phi\frac{\partial}{\partial\phi}\ , \quad \hat{\mathrm{L}}_z = -i\frac{\partial}{\partial\phi}\ .$$

4.8 (a) Verify that the three functions $\cos\theta$, $\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}$ and $\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}$ are all eigenfunctions of $\hat{\boldsymbol{L}}^2$ and $\hat{\mathrm{L}}_z$.

(b) Find normalization constants N for each of the above functions so that

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\ \sin\theta\, N^2\, |\psi(\theta,\phi)|^2 = 1.$$

(c) Once normalized, these functions are called spherical harmonics and given the symbol $Y_\ell^m(\theta,\phi)$. Hence deduce that your results are consistent with the functions:

$$\mathrm{Y}_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta; \qquad \mathrm{Y}_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}; \qquad \mathrm{Y}_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}.$$

(Note, you can't use this method to get the signs of Y_1^0, Y_1^1 and Y_1^{-1}. The minus sign in Y_1^1 can be deduced by using a raising operator $\hat{L}_+$ on Y_1^0. This is not required.)

(d) Rewrite these functions in terms of Cartesian variables [$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$]. Sketch $|Y_1^0|^2$, $|Y_1^1|^2$ and $|Y_1^{-1}|^2$. (They are angular functions, so keep r fixed and look only at the angle dependence. A cross section in the x–z plane will do. Why?)

4.9 The angular part of a system's wavefunction is

$$\langle\theta,\phi|\psi\rangle \propto (\sqrt{2}\cos\theta + \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} - \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}).$$

What are the possible results of measurement of (a) $\hat{\boldsymbol{L}}^2$, and (b) $\hat{L}_z$, and their probabilities? What is the expectation value of $\hat{L}_z$?

4.10 A system's wavefunction is proportional to $\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}$. What are the possible results of measurements of (a) $\hat{L}_z$ and (b) $\hat{\boldsymbol{L}}^2$?

4.11 A system's wavefunction is proportional to $\sin^2\theta$. What are the possible results of measurements of (a) $\hat{L}_z$ and (b) $\hat{\boldsymbol{L}}^2$? Give the probabilities of each possible outcome.

4.12 A particle of mass m is described the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r} - e\mathcal{E}\hat{x}.$$

(a) What is the physical origin of the last term in $\hat{H}$?

(b) Which of the observables represented by the operators $\hat{\boldsymbol{L}}^2$, $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ are constants of the motion assuming (i) $\mathcal{E} = 0$; (ii) $\mathcal{E} \neq 0$ (Hint: Use the results for $[\hat{L}_i, \hat{x}]$ from [4.6].)

4.13 Show that $\hat{L}_i$ commutes with $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}}$.