Quantum Mechanics MT: Problem Sheet 1

Dirac notation

1.1 What physical phenomenon requires us to work with probability amplitudes rather than with probabilities?

1.2 Given that $|\psi\rangle = e^{i\pi/5}|a\rangle + e^{i\pi/4}|b\rangle$, express $\langle\psi|$ as a linear combination of $\langle a|$ and $\langle b|$.

1.3 An electron can be in one of two potential wells that are so close that it can 'tunnel' from one to the other. Its state vector can be written

$$|\psi\rangle = a|A\rangle + b|B\rangle, \tag{1}$$

where $|A\rangle$ is the state of being in the first well and $|B\rangle$ is the state of being in the second well and all kets are correctly normalised. What is the probability of finding the particle in the first well given that: (a) $a = i/2$; (b) $b = e^{i\pi}$; (c) $b = \frac{1}{3} + i/\sqrt{2}$?

1.4 An electron can "tunnel" between potential wells that form a linear chain, so its state vector can be written as

$$|\psi\rangle = \sum_{n=-\infty}^{\infty} a_n|n\rangle, \tag{2}$$

where $|n\rangle$ is the state of being in the n^{th} well, where n increases from left to right. Let

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{-i}{3}\right)^{\frac{|n|}{2}} e^{in\pi}. \tag{3}$$

(a) What is the probability of finding the electron in the n^{th} well?
(b) What is the probability of finding the electron in well 0 or anywhere to the right of it?

Operators

1.5 Let Q be the operator of an observable and let $|\psi\rangle$ be the state of our system.
(a) What are the physical interpretations of $\langle\psi|Q|\psi\rangle$ and $|\langle q_n|\psi\rangle|^2$, where $|q_n\rangle$ is the n^{th} eigenket of the observable Q and q_n is the corresponding eigenvalue?
(b) What is the operator $\sum_n |q_n\rangle\langle q_n|$, where the sum is over all eigenkets of Q? What is the operator $\sum_n q_n|q_n\rangle\langle q_n|$?

1.6 Which of the following operators are Hermitian, given that $\hat{A}$ and $\hat{B}$ are Hermitian:
$\hat{A}+\hat{B}$; $c\hat{A}$; $\hat{A}\hat{B}$; $\hat{A}\hat{B}+\hat{B}\hat{A}$.
Show that in one dimension, for functions which tend to zero as $|x| \to \infty$, the operator $\partial/\partial x$ is not Hermitian, but $-i\hbar\partial/\partial x$ is. Is $\partial^2/\partial x^2$ Hermitian?

1.7 Given that $\hat{A}$ and $\hat{B}$ are Hermitian operators, show that $i[\hat{A},\hat{B}]$ is a Hermitian operator.

1.8 Given that for any two operators $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$, show that

$$(\hat{A}\hat{B}\hat{C}\hat{D})^\dagger = \hat{D}^\dagger\hat{C}^\dagger\hat{B}^\dagger\hat{A}^\dagger.$$

Commutators

1.9 Show that if there is a complete set of mutual eigenkets of the Hermitian operators $\hat{A}$ and $\hat{B}$, then $[\hat{A},\hat{B}] = 0$. Explain the physical significance of this result.

1.10 Does it always follow that if a system is an eigenstate of $\hat{A}$ and $[\hat{A},\hat{B}] = 0$ then the system will be in a eigenstate of $\hat{B}$? If not, give a counterexample.

1.11 Show that
(a) $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$
(b) $[\hat{A}\hat{B}\hat{C},\hat{D}] = \hat{A}\hat{B}[\hat{C},\hat{D}] + \hat{A}[\hat{B},\hat{D}]\hat{C} + [\hat{A},\hat{D}]\hat{B}\hat{C}$. Explain the similarity with the rule for differentiating a product.
(c) $[\hat{x}^n,\hat{p}] = \mathrm{i}\hbar n\hat{x}^{n-1}$
(d) $[f(\hat{x}),\hat{p}] = \mathrm{i}\hbar\frac{\mathrm{d}f}{\mathrm{d}x}$ for any function $f(x)$.

1.12 Let A and B be two Hermitian operators. Prove that if $[A,B]=0$ there exists a complete set of simultaneous eigenstates of the operators A and B, i.e.

$$A|u_j\rangle = a_j|u_j\rangle \ , \quad B|u_j\rangle = b_j|u_j\rangle, \tag{4}$$

and $\{|u_j\rangle\}$ form a basis of the LVS on which A and B act.

1.13 Prove that two observables are compatible if and only if the commutator between the associated Hermitian operators vanishes.

1.14 What does it mean to say that two operators commute? What is the significance of two observables having mutually commuting operators?

Given that the commutator $[P,Q]\neq 0$ for some observables P and Q, does it follow that for all $|\psi\rangle \neq 0$ we have $[P,Q]|\psi\rangle \neq 0$?

1.15 Prove the following statements involving the delta-function and its derivative (and explain how these statements are to be understood):

(a)

$$\delta(cx) = \frac{1}{|c|}\delta(x) \ , \ 0 \neq c \in \mathbb{R}. \tag{5}$$

(b)

$$\delta\big(x^2-c^2\big) = \frac{1}{2|c|}\big(\delta(x-c)+\delta(x+c)\big). \tag{6}$$

(c)

$$\frac{d}{dx}\theta(x-c) = \delta(x-c) \ , \qquad \theta(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{else} \end{cases} \tag{7}$$

The function $\theta(x)$ is known as the Heaviside theta-function.

(d)

$$\delta(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\ e^{ikx}. \tag{8}$$

(e)

$$\int dx\ f(x)\delta'(x-x_0) = -f'(x_0)\ . \tag{9}$$

(f)* (starred problem for students who have already taken the complex analysis short-option)

How would you show that $\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}$?