M.Phys Option in Theoretical Physics: C6. Revision Problem Sheet, Quantum Field Theory

1.) (Field theories with global $SO(n)$ symmetry)
a) The group $SO(n)$ consists of the real $n \times n$ matrices R satisfying $R^T R = \mathbf{1}_n$ and $\det(R) = 1$. Consider infinitesimal transformations of the form $R = \mathbf{1}_n + iT$ and determine the allowed matrices T, i.e. the Lie algebra of $SO(n)$. Show that the matrices T_{ab}, labelled by an antisymmetric pair of indices a, b and defined as $(T_{ab})_c{}^d = i(\delta_{ac}\delta_b^d - \delta_a^d\delta_{bc})$ (where $a, b, c, \ldots = 1, \ldots, n$) provide a basis for the Lie algebra. What is the dimension of this algebra?
b) A set of n scalar fields $\phi_a(x)$, where $a = 1, \ldots, n$, has a Lagrangian density $\mathcal{L} = \frac{1}{2}\sum_{a=1}^n \partial_\mu\phi_a\partial^\mu\phi_a - V$ with scalar potential $V = \frac{1}{2}m^2\sum_{a=1}^n \phi_a^2 + \frac{\lambda}{4}(\sum_{a=1}^n \phi_a^2)^2$. Show that this Lagrangian density is invariant under $SO(n)$.
c) Compute the Noether currents of this $SO(n)$ symmetry.
d) For $m^2 < 0$ and $\lambda > 0$ find the minima of the scalar potential and show that the $SO(n)$ symmetry is broken to $SO(n-1)$. What is the number of Goldstone modes?

2.) ($SU(3)$ gauge symmetry with spontaneous symmetry breaking)
The Lie group $SU(3)$ is defined as the set of complex 3×3 matrices U satisfying $U^\dagger U = \mathbf{1}_3$ and $\det(U) = 1$.
a) Show that the Lie algebra of $SU(3)$ consists of all hermitian 3×3 matrices with vanishing trace.
b) Write down a basis t^a for the Lie algebra of $SU(3)$ which contains the matrices ($i = 1, 2, 3$)

$$t^i = \begin{pmatrix} \tau^i & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tau^i \equiv \sigma_i/2$ with σ^i the usual Pauli matrices. What is the dimension of this $SU(3)$ Lie algebra?
c) Consider now an $SU(3)$ gauge theory with a scalar field ϕ in the *adjoint* representation. The covariant derivative of ϕ takes the form

$$D_\mu\phi_a = \partial_\mu\phi_a + g f_{abc} A_\mu^b \phi_c,$$

where A_μ^a denote the $SU(3)$ gauge fields and f_{abc} are the corresponding fully antisymmetric structure constants, i.e. $[t^a, t^b] = if^{abc}t^c$. Which term in the Lagrangian give rise to the masses of the gauge fields?
d) The mass term can be written more clearly by defining the quantity $\Phi \equiv \phi_a t^a$. Use this notation together with the definition of the $SU(3)$ structure constants to write the mass term as

$$\mathcal{L} \supset -g^2 \,\mathrm{tr}\left([t^a, \Phi][t^b, \Phi]\right) A_\mu^a A^{b\mu}.$$

e) Now let Φ acquire a vacuum expectation value (VEV). Consider the following two possibilities:

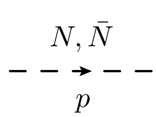
$$\langle\Phi\rangle = v\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \langle\Phi\rangle = v\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Give the symmetry-breaking pattern induced by these VEVs and calculate the mass spectrum of the gauge bosons for both cases.

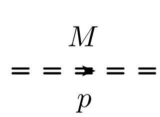
3.) (Theory with two complex scalars and local $U(1)$ symmetry)
Consider a theory with two complex scalar fields ϕ_1 and ϕ_2 with masses m_1 and m_2 . The scalar fields have charges $q_1 = 1$ and $q_2 = 2$ under a local $U(1)$ symmetry with associated vector field A_μ.
a) Write down the locally $U(1)$ invariant Lagrangian density for these fields up to order four in the fields.
b) Write down the local $U(1)$ symmetry transformations for all fields and explicitly show that the Lagrangian in a) is invariant.
c) Consider the limit where all cross-coupling terms between ϕ_1 and ϕ_2 vanish, i.e. neglect all terms in the scalar potential which involve both ϕ_1 and ϕ_2. There are four different choices for the signs of m_1^2 and m_2^2. For each choice, find the minima of the scalar potential. In which of these cases is the local $U(1)$ symmetry spontaneously broken?
d) For the broken cases, rewrite the Lagrangian in terms of a massive vector field and the remaining physical scalars. What is the mass of the vector field and the scalars in each case? Comment on your results.

4.) (Interacting scalar field theory)
Two real scalar fields ϕ_1 and ϕ_2 with masses $m_1^2 > 0$ and $m_2^2 > 0$ transform as $\phi_1 \to -\phi_1$ and $\phi_2 \to \phi_2$ under a $\mathbb{Z}_2$ symmetry.
a) Write down the most general $\mathbb{Z}_2$ invariant Lagrangian (with standard kinetic terms and up to quartic terms in the fields) for ϕ_1 and ϕ_2.
b) Derive the Feynman rules for the vertices in this theory by computing the appropriate Green's functions.
c) Based on the results in b) and assuming negligible masses m_1 and m_2, find the amplitudes and differential cross sections for $\phi_1\phi_1 \to \phi_1\phi_1$ and $\phi_1\phi_1 \to \phi_2\phi_2$ scattering.

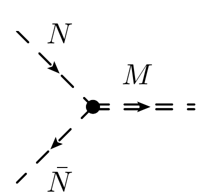
5.) (Scalar Yukawa theory)
The Lagrangian of the scalar Yukawa theory is given by $\mathcal{L} = (\partial_\mu\varphi)^\dagger(\partial^\mu\varphi) - M^2\varphi^\dagger\varphi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - g\,\varphi^\dagger\varphi\,\phi$, where φ and ϕ is a complex and a real scalar, respectively.
a) What is the dimension of the coupling constant g? When is the interaction term $-g\,\varphi^\dagger\varphi\,\phi$ a small perturbation?
b) We saw in the lecture that the quantized version of $\mathcal{L}$ describes the interactions of nucleons N and antinucleons $\bar{N}$ via the exchange of mesons M. In momentum space, the Feynman rules of the scalar Yukawa theory are



$= \tilde{D}_F(p, M)$



$= \tilde{D}_F(p, m)$



$= -ig$

with $\tilde{D}_F(p, M) = i/(p^2 - M^2 + i\epsilon)$ and $\tilde{D}_F(p, m) = i/(p^2 - m^2 + i\epsilon)$. Explain why there is no $NN \to M$ vertex?

c) Draw the leading-order Feynman graphs that describe the process $NN \to NN$ and write down the corresponding matrix element $\mathcal{M}(NN \to NN)$. Express your result in Mandelstam variables.

d) Using naive dimensional analysis estimate the size of the total cross section for elastic nucleon-nucleon scattering.