Path Integrals in Quantum Field Theory
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Uli Haisch[a]

[a]*Rudolf Peierls Centre for Theoretical Physics*
University of Oxford
OX1 3PN Oxford, United Kingdom

Please send corrections to u.haisch1@physics.ox.ac.uk.

1 Path Integrals in Quantum Field Theory

In the solid-state physics part of the lecture you have seen how to formulate quantum mechanics (QM) in terms of path integrals. This has led to an intuitive picture of the transition between classical and quantum physics. In this lecture notes I will show how to apply path integrals to the quantization of field theories. We start the discussion by recalling the most important feature of path integrals in QM.

1.1 QM Flashback

It has already been explained how QM can be formulated in terms of path integrals. One important finding was that the matrix element between two position eigenstates is given by

$$\langle x, t | x', t' \rangle = \langle x | e^{-iH(t-t')} | x' \rangle \propto \int \mathcal{D}x \, \exp\left(i \int_{t'}^{t} dt'' \mathcal{L}(x, \dot{x}) \right) . \tag{1.1}$$

Notice that here $|x, t\rangle = e^{iHt} |x\rangle_S$ and $|x', t'\rangle = e^{iHt'} |x'\rangle_S$ are Heisenberg picture states.

We also learned in the lectures on interacting quantum fields that the central objects to compute in quantum field theory (QFT) are vacuum expectation values (VEVs) of time-ordered field-operator products such as the Feynman propagator

$$D_F\,(x - y) = \langle 0 \,| T \,\phi(x) \phi(y) |\, 0 \rangle \; . \tag{1.2}$$

In QM the analog of (1.2) is simply[1]

$$\langle x_f, t_f \,| T \,\hat{x}(t_1) \hat{x}(t_2) |\, x_i, t_i \rangle \, . \tag{1.3}$$

[1]In this part of the lecture, we will use hats to distinguish operators from their classical counterparts which appear in the path integral.

Focusing on the case $t_1 > t_2$ and inserting complete sets of states we can write the latter expression as

$$\begin{aligned}\langle x_f, t_f \left| \hat{x}(t_1)\hat{x}(t_2) \right| x_i, t_i \rangle &= \langle x_f \left| e^{-iH(t_f-t_1)}\hat{x}_S e^{-iH(t_1-t_2)}\hat{x}_S e^{-iH(t_2-t_i)} \right| x_i \rangle \\ &= \int dx_1 dx_2 \langle x_f \left| e^{-iH(t_f-t_1)} \right| x_1 \rangle \langle x_1 | \hat{x}_S e^{-iH(t_1-t_2)} | x_2 \rangle \langle x_2 | \hat{x}_S e^{-iH(t_2-t_i)} \left| x_i \right\rangle . \end{aligned} \tag{1.4}$$

Using now that $\hat{x}|x\rangle = x|x\rangle$, replacing all three expectation values by (1.1) and combining the three path integrals with the integrations over x_1 and x_2 into a single path integral the result (1.4) simplifies further. One finds that

$$\langle x_f, t_f \left| T\,\hat{x}(t_1)\hat{x}(t_2) \right| x_i, t_i \rangle \propto \int \mathcal{D}x\, x(t_1)x(t_2)\, \exp\left(i \int_{t_i}^{t_f} dt \mathcal{L}(x, \dot{x}) \right) . \tag{1.5}$$

For $t_2 > t_1$ the same result holds, because time ordering is automatic in the path-integral formulation. It should also be clear that results similar to (1.5) apply for a product of an arbitrary number of operators $\hat{x}$. Furthermore, it can be shown that

$$\lim_{t_{i,f} \to \mp\infty} \langle x_f, t_f \left| T\left(\hat{x}(t_1) \ldots \hat{x}(t_n)\right) \right| x_i, t_i \rangle \propto \langle 0 \left| T\left(\hat{x}(t_1) \ldots \hat{x}(t_n)\right) \right| 0 \rangle . \tag{1.6}$$

Therefore one arrives at

$$\langle 0 \left| T\left(\hat{x}(t_1) \ldots \hat{x}(t_n)\right) \right| 0 \rangle \propto \int \mathcal{D}x\, x(t_1) \ldots x(t_n)\, e^{iS[x]} , \tag{1.7}$$

with $S[x]$ the action functional.

1.2 Basics of QFT Path Integrals

In order to keep the following discussion as simple as possible we will focus on the real scalar field ϕ. An extension to more complicated theories would however be straightforward. As we saw the Green's functions of the form

$$\mathcal{G}^{(n)}(x_1, \ldots, x_n) = \left\langle 0 \middle| T\left(\hat{\phi}(x_1) \ldots \hat{\phi}(x_n)\right) \middle| 0 \right\rangle , \tag{1.8}$$

play an important role in QFT. In analogy to (1.7) these objects can be written as

$$\mathcal{G}^{(n)}(x_1, \ldots, x_n) = \mathcal{N} \int \mathcal{D}\phi\, \phi(t_1) \ldots \phi(t_n)\, e^{iS[x]} , \tag{1.9}$$

where $\mathcal{N}$ is a normalization constant.

Like in QM, we introduce the *generating functional*

$$W[J] = \mathcal{N} \int \mathcal{D}\phi\, \exp\left\{ i \int d^4x \left[\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x) \right] \right\} , \tag{1.10}$$

for the Green's functions such that

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n) = \frac{i^{-n}\,\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} . \tag{1.11}$$

The value of $\mathcal{N}$ is fixed by requiring that

$$W[J]\big|_{J=0} = W[0] = \langle 0|0\rangle = 1 . \tag{1.12}$$

Recalling that in QM a second generating functional called $Z[J]$ has been introduced, we also define

$$Z[J] = -i \ln W[J] . \tag{1.13}$$

By apply n functional differentiations to $Z[J]$ we get another type of Green's functions[2]

$$G^{(n)}(x_1,\ldots,x_n) = \frac{i^{1-n}\,\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} , \tag{1.14}$$

which correspond to *connected* Feynman diagrams, so that it makes sense to call $Z[J]$ the generating functional for connected Green's functions. Notice that all information on the QFT is now encoded in the generating functionals, which are hence the primary objects to calculate. We will do this below for the simplest case of a free real scalar field.

1.3 Generating Functionals For Free Real Scalar

For the purpose of explicit calculations it turns out to be useful to introduce a *Euclidean* or *Wick-rotated* version $W_E[J]$ of the generating functional. To do this we define Euclidean 4-vectors $\bar{x} = (\bar{x}_0, \bar{\boldsymbol{x}}) = (ix_0, \boldsymbol{x})$, associated derivatives $\bar{\partial}^\mu = \partial/\partial\bar{x}_\mu$, and an Euclidean version of the Lagrangian, $\mathcal{L}_E = \mathcal{L}_E(\phi, \bar{\partial}_\mu\phi)$. To give an example,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \;\rightarrow\; \mathcal{L}_E = -\frac{1}{2}(\bar{\partial}_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 . \tag{1.15}$$

Starting from (1.10) it is easy to see that the Euclidean version of $W[J]$ is

$$W_E[J] = \mathcal{N}\int\mathcal{D}\phi\,\exp\left\{i\int d^4\bar{x}\left[\mathcal{L}_E(\phi,\bar{\partial}_\mu\phi) + J(\bar{x})\phi(\bar{x})\right]\right\} . \tag{1.16}$$

The corresponding Euclidean Green's function are then obtained by

$$\mathcal{G}_E^{(n)}(\bar{x}_1,\ldots,\bar{x}_n) = \frac{i^{-n}\,\delta^n W_E[J]}{\delta J(\bar{x}_1)\ldots\delta J(\bar{x}_n)}\bigg|_{J=0} . \tag{1.17}$$

We now want to derive $W[J]$ for the real Klein-Gordon theory. We start by writing

$$\int d^4\bar{x}\,(\bar{\partial}_\mu\phi(\bar{x}))(\bar{\partial}^\mu\phi(\bar{x})) = \int d^4\bar{x}\,d^4\bar{y}\,\phi(\bar{y})\left(\bar{\partial}^y_\mu\bar{\partial}^\mu_x\delta^{(4)}(\bar{x}-\bar{y})\phi(\bar{x})\right) . \tag{1.18}$$

[2]Realize that these Green's functions are the ones we meet already in (1.65) and (1.66) of the script "Interacting Quantum Fields".

It follows that

$$W_E[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ -\frac{1}{2} \int d^4\bar{x} d^4\bar{y} \, \phi(\bar{y}) A(\bar{y},\bar{x}) \phi(\bar{x}) + \int d^4\bar{x} \, J(\bar{x}) \phi(\bar{x}) \right\} , \tag{1.19}$$

with

$$A(\bar{y},\bar{x}) = \left(\bar{\partial}^y_\mu \bar{\partial}^\mu_x + m^2 \right) \delta^{(4)}(\bar{x} - \bar{y}) \, . \tag{1.20}$$

This is a *Gaussian path integral* with a source J of precisely the type you have discussed in the QM context in the solid-state part of this lecture.

Given this analogy we perform a variable transformation to find an explicit expression for (1.19). Skipping over the details of the actual calculation, one obtains

$$W_E[J] = \bar{\mathcal{N}} \exp\left\{ \frac{1}{2} \int d^4\bar{x} \, d^4\bar{y} \, J(\bar{y}) D^E_F(\bar{y} - \bar{x}) J(\bar{x}) \right\} , \tag{1.21}$$

with

$$D^E_F(\bar{y} - \bar{x}) = A^{-1}(\bar{y} - \bar{x}) \, , \tag{1.22}$$

and $\bar{\mathcal{N}}$ an appropriate normalization.

Fine, but how do we calculate the inverse of the operator A? The idea is to use Fourier transformations and then to go back to Minkowski space. We first recall that

$$\delta^{(4)}(\bar{x} - \bar{y}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \, e^{i\bar{p}(\bar{x}-\bar{y})} \, , \tag{1.23}$$

which we use to write

$$A(\bar{y},\bar{x}) = \left(\bar{\partial}^y_\mu \bar{\partial}^\mu_x + m^2 \right) \delta^{(4)}(\bar{x} - \bar{y}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \left(\bar{p}^2 + m^2 \right) e^{i\bar{p}(\bar{x}-\bar{y})} \, . \tag{1.24}$$

Now we invert A by taking the inverse inside the Fourier transformation, *i.e.*,

$$A^{-1}(\bar{y} - \bar{x}) = D^E_F(\bar{y} - \bar{x}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{1}{\bar{p}^2 + m^2} \, e^{i\bar{p}(\bar{x}-\bar{y})} \, . \tag{1.25}$$

To go to Minkowski space we introduce $p = (p_0, \boldsymbol{p}) = (i\bar{p}_0, \bar{\boldsymbol{p}})$. Putting things together one finds for the generating functional in Minkowski space

$$W[J] = \exp\left\{ -\frac{1}{2} \int d^4x \, d^4y \, J(y) D_F(y - x) J(x) \right\} , \tag{1.26}$$

where

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \, e^{-ip(x-y)} \, , \tag{1.27}$$

is our *Feynman propagator*. Notice that we have chosen $\bar{\mathcal{N}} = 1$ so that (1.12) holds.

From (1.26) we can now derive Green's functions effortless. *E.g.*, for the 2-point function we get from (1.11)

$$\mathcal{G}^{(2)}(x,y) = - \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \bigg|_{J=0} = D_F(x - y) \, , \tag{1.28}$$

which agrees with the result that we got using *canonical quantization*.

From (1.13) and (1.26) we also find

$$Z[J] = \frac{i}{2} \int d^4x\, d^4y\, J(y) D_F(y-x) J(x)\,, \tag{1.29}$$

for the generating functional $Z[J]$ of connected Green's functions. It is important to bear in mind that the results (1.26) and (1.29) hold for the free field theory only.

1.4 Effective Action

Path integrals also provide an intuitive picture for the transition between classical and quantum physics. In order to illustrate this property we define the *classical field* ϕ_c by

$$\phi_c(x) = \frac{\delta Z[J]}{\delta J(x)}\,. \tag{1.30}$$

We have

$$\phi_c(x) = \frac{\delta}{\delta J(x)} \left(-i \ln W[J]\right) = -\frac{i}{W[J]} \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\hat{\phi}(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \tag{1.31}$$

Here we have defined *VEVs in the presence of* J as follows

$$\langle 0|0\rangle_J = W[J]\,, \qquad \langle 0|\hat{\phi}(x)|0\rangle_J = -i\,\frac{\delta W[J]}{\delta J(x)}\,. \tag{1.32}$$

The final result for ϕ_c shows that the classical field is the suitably normalized VEV of the field operator $\hat{\phi}$, which from a physical standpoint sounds quite reasonable. Recall also that $W[J] = \exp\left(iZ[J]\right)$ which suggest that the generating functional of connected Green's functions is something like the action in our path integrals $\int \mathcal{D}\phi\, \ldots\, e^{iS[\phi]}$. This suggest that $Z[J]$ is some sort of effective action.

To remove the effect of the source term that is present in $Z[J]$ we use a Legendre transform. We define the *effective action* as

$$\Gamma[\phi_c] = Z[J] - \int d^4x\, J(x) \phi_c(x)\,. \tag{1.33}$$

In fact, with this definition one has

$$\frac{\delta \Gamma[\phi_c]}{\delta J(y)} = \frac{\delta Z[J]}{\delta J(y)} - \int d^4x\, \frac{\delta J(x)}{\delta J(y)}\, \phi_c(x) = \phi_c(y) - \int d^4x\, \delta^{(4)}(x-y)\, \phi_c(x) = 0\,, \tag{1.34}$$

so $\Gamma[\phi_c]$ is independent of the source J.

To further see that the definition (1.33) is meaningful, let us discuss the free field case. We begin by deriving an explicit expression for the classic field:

$$\begin{aligned}\phi_c(x) &= \frac{\delta}{\delta J(x)}\,\frac{i}{2}\int d^4y d^4z\, J(y)D_F(y-z)J(z) \\ &= \frac{i}{2}\left\{\int d^4y d^4z\left[\delta^{(4)}(x-y)\,D_F(y-z)J(z)+\delta^{(4)}(x-z)\,D_F(y-z)J(y)\right]\right\} \\ &= i\int d^4y\, D_F(x-y)J(y)\,.\end{aligned} \tag{1.35}$$

Since

$$\left(\Box_x+m^2\right)D_F(x-y) = -i\delta^{(4)}(x-y)\,, \tag{1.36}$$

we arrive at

$$\left(\Box_x+m^2\right)\phi_c(x) = J(x)\,, \tag{1.37}$$

which means that $\phi_c(x)$ is a solution to the Klein-Gordon equation with source $J(x)$. This is exactly what one would expect for a classical field coupled to J. Furthermore, inserting (1.29) and (1.35) into the effective action (1.33) it follows that

$$\begin{aligned}\Gamma[\phi_c] &= \frac{i}{2}\int d^4x d^4y\, J(x)D_F(x-y)J(y) - \int d^4x\, J(x)\phi_c(x) \\ &= -\frac{1}{2}\int d^4x\, \phi_c(x)J(x) = -\frac{1}{2}\int d^4x\, \phi_c(x)\left(\Box_x+m^2\right)\phi_c(x) \\ &= \int d^4x\left[\frac{1}{2}\left(\partial_\mu\phi_c(x)\right)^2-\frac{1}{2}m^2\phi_c^2(x)\right]\,,\end{aligned} \tag{1.38}$$

where in the last step we have used integration by parts. In the free field case the effective action hence coincides with the classic action of the real scalar field.

For interacting theories, the generating functional can typically not be calculated exactly. Yet, one can evaluate the path integral (1.10) in the *saddle point approximation.* The solution ϕ_0 to the classical equations of motions (EOMs) is determined from

$$\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} = \frac{\delta S}{\delta\phi(x)}[\phi_0] = -J(x)\,. \tag{1.39}$$

Then to leading order in the saddle point approximation one has

$$\begin{aligned}W[J] &= \mathcal{N}\exp\left[iS[\phi_0]+i\int d^4x\, J(x)\phi_0(x)\right]\,, \\ Z[J] &= S[\phi_0]+\int d^4x\, J(x)\phi_0(x)\,.\end{aligned} \tag{1.40}$$

Comparing these results to (1.30) and (1.33) it is readily seen that

$$\phi_c(x)=\phi_0(x)\,,\qquad \Gamma[\phi_c]=S[\phi_0]\,. \tag{1.41}$$

Hence to lowest order the effective action $\Gamma[\phi_c]$ is simply the classic action $S[\phi_0]$. Beyond the leading order the effective action will however receive quantum corrections and as a result one has generically $\Gamma[\phi_c] \neq S[\phi_0]$.

Let me add that the above formalism allows one to shed some light on a point that we glossed over in our discussion of spontaneous symmetry breaking in Section 1.4 of the script "Classical Field Theory". In this discussion we only talked about classic theories. However, one should ask whether the same or similar results would be obtained in the corresponding quantum theories. In fact, as it turns out spontaneous symmetry breaking should be analyzed with the effective action $\Gamma[\phi_c]$ (or more precisely the *effective potential*) rather than the classic action $S[\phi_0]$. If this is done, one can convince oneself that our discussion based on the classic theory makes sense even in the quantum theory. Yet, the classical analysis has to be viewed as a leading-order approximation. Since in weakly-coupled theory quantum corrections are always suppressed, the classic analysis of spontaneous symmetry breaking is therefore typically a good approximation to the full story.

1.5 Feynman Integrals from Path Integrals

In order to develop perturbation theory the path integral formalism, we split the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \,, \tag{1.42}$$

where $\mathcal{L}_0$ is the Lagrangian of the free theory, while $\mathcal{L}_{\text{int}}$ contains all interactions. *E.g.*, in the case of ϕ^4 theory one has $\mathcal{L}_{\text{int}} = -\lambda/4!\,\phi^4$. The generating functional associated to $\mathcal{L}_0$ is called $W_0[J]$, while we will denote the full generating functional by $W[J]$. Explicitly, one has

$$\begin{aligned} W_0[J] &= \mathcal{N}_0 \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left(\mathcal{L}_0 + J(x)\phi(x) \right) \right] \,, \\ W[J] &= \mathcal{N} \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_{\text{int}} + J(x)\phi(x) \right) \right] \,. \end{aligned} \tag{1.43}$$

It follows that we can write $W[J]$ as

$$\begin{aligned} W[J] &= \mathcal{N} \exp\left[i \int d^4x \, \mathcal{L}_{\text{int}} \left(-i \, \frac{\delta}{\delta J(x)} \right) \right] W_0[J] \\ &= \mathcal{N} \left[1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, \mathcal{L}_{\text{int}} \left(-i \, \frac{\delta}{\delta J(x_1)} \right) \dots \mathcal{L}_{\text{int}} \left(-i \, \frac{\delta}{\delta J(x_n)} \right) \right] W_0[J] \,. \end{aligned} \tag{1.44}$$

Here

$$\mathcal{N}^{-1} = \exp\left[i \int d^4x \, \mathcal{L}_{\text{int}} \left(-i \, \frac{\delta}{\delta J(x)} \right) \right] W_0[J] \bigg|_{J=0} \,, \tag{1.45}$$

to ensure that (1.12) is satisfied. The result (1.44) shows that $W[J]$ is a perturbative series in terms of $W_0[J]$. But from (1.26) we know that

$$W_0[J] = \exp\left\{ -\frac{1}{2} \int d^4x \, d^4y \, J(y) D_F(y-x) J(x) \right\} \,, \tag{1.46}$$

so in fact all functional derivatives can be carried out explicitly and lead to Feynman diagrams.

It is straightforward to see that the Green's functions (1.11) can be written as

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n) = \mathcal{N}\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\left[1+\sum_{m=1}^{\infty}\frac{i^m}{m!}\int d^4y_1\ldots d^4y_m \right.$$
$$\left.\times\,\mathcal{L}_{\text{int}}\left(-i\,\frac{\delta}{\delta J(y_1)}\right)\ldots\mathcal{L}_{\text{int}}\left(-i\,\frac{\delta}{\delta J(y_m)}\right)\right] W_0[J]\Bigg|_{J=0}. \tag{1.47}$$

While this result looks kind of horrible it can in fact be worked out order by order in perturbation theory using *Wick's theorem*. This results in a sum over products of propagators D_F (suitably integrated) and each term can be associated to a Feynman graph. This is exactly what we have obtained before using canonical quantization, so the two approaches give at the end the same result. The path integral formalism is however more elegant.

1.6 A Simple Application

To become more familiar with the path integral formalism in QFT it seems worthwhile to consider a simple but educated example. In the following we will study a theory with a massless real scalar field described by the classic Lagrangian

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{\lambda}{3!}\,\phi^3\,. \tag{1.48}$$

Remember that the dimension of the coupling λ is $[\lambda]=1$.

In this case, $W_0[J]$ is given by (1.46) employing the massless Feynman propagator of a scalar field. The full generating functional takes the following form

$$\begin{aligned} W[J] &= \mathcal{N}\exp\left[-i\int d^4x\,\frac{\lambda}{3!}\left(-i\,\frac{\delta}{\delta J(x)}\right)^3\right] W_0[J] \\ &= \mathcal{N}\left\{\sum_{V=0}^{\infty}\frac{1}{V!}\left[-\frac{i\lambda}{3!}\int d^4x\left(-i\,\frac{\delta}{\delta J(x)}\right)^3\right]^V\right\} \\ &\quad\times\left\{\sum_{P=0}^{\infty}\frac{1}{P!}\left[-\frac{1}{2}\int d^4y d^4z\,J(y)D_F(y-z)J(z)\right]^P\right\}, \end{aligned} \tag{1.49}$$

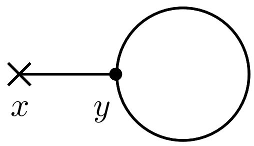
where V and P count the *number of vertices and propagators.*

In order to evaluate this expression we can use the tools of Feynman diagrams. First, we determine the number of surviving sources which is equal to the number of *external legs* L of the graph. Since each propagator connects two points (external or internal) and our scalar theory (1.48) has only 3-point vertices, the number of external legs is given in terms of V and P by the simple formula

$$L = 2P - 3V\,. \tag{1.50}$$

So instead of using V and P to order the perturbative series, we can also use V and L, and this is what we will do below.

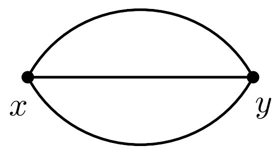
For instance, let us consider $V = L = 1$. In this case we have



$$= -\frac{\lambda}{2} \int d^4x d^4y \, J(x) D_F(x-y) D_F(y-y) \,, \tag{1.51}$$

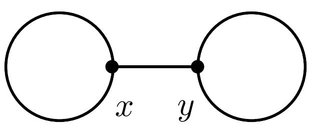
where the dot in the diagram denotes a vertex while the cross indicates a source. The factor of 2 appearing in the denominator is the symmetry factor of the diagram (the ends of the line meeting at y can be interchanged without altering the result).

In the case of $V = 2$ and $L = 0$, we get on the other hand



$$= \frac{\lambda^2}{12} \int d^4x d^4y \, D_F^3(x-y) \,, \tag{1.52}$$

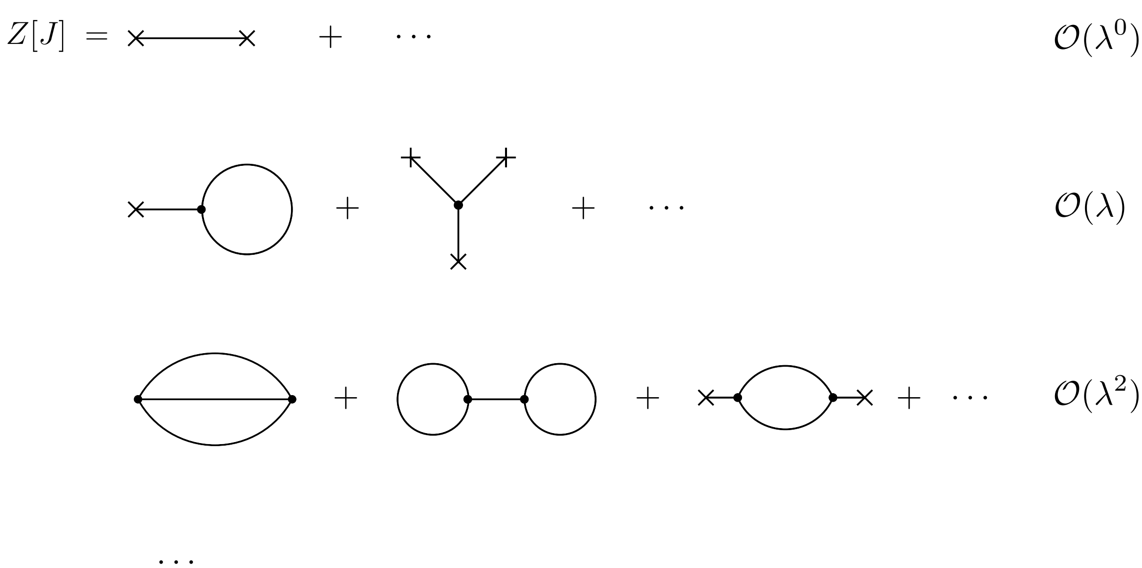
and



$$= \frac{\lambda^2}{8} \int d^4x d^4y \, D_F(x-x) D_F(x-y) D_F(y-y) \,. \tag{1.53}$$

The value of the symmetry factor is in the first case $S = 2 \cdot 3! = 12$, where the factor of 2 arises from the exchange of x with y and the factor 3! stems from the possible ways to interchange the lines joining x and y. In the second case similar arguments lead to $S = 2 \cdot 2 \cdot 2 = 8$.

Dropping the multiplicative overall factors it is also easy to give a pictorial representation of the generating functional $Z[J]$ as defined in (1.13). One has

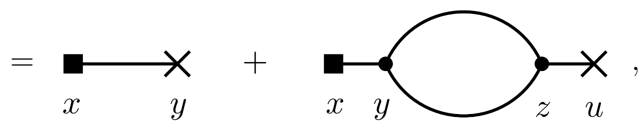


(1.54)

Notice that I have ordered the individual diagrams in the perturbative series of $Z[J]$ corresponding to their number of external legs and powers of λ (or equivalent number of vertices).

Let us now try to calculate the classic field ϕ_c as defined in (1.30). Up to and including terms of order λ^2, we get

$$\phi_c(x) = i\int d^4y\, D_F(x-y)J(y) + \frac{\lambda^2}{4}\int d^4y d^4z d^4u\, D_F(x-y)D_F^2(y-z)D_F(z-u)J(u) \tag{1.55}$$



where the black square indicates the position x of the classic field $\phi_c(x)$. The factor of 1/4 in the $\mathcal{O}(\lambda^2)$ term arises again from the symmetry factor of the associated graph.

We now compute $\Box\phi_c(x)$. Recalling that D_F fulfils the Klein-Gordon equation (1.36), we obtain

$$\Box\phi_c(x) = J(x) - \frac{i\lambda^2}{4}\int d^4y D_F^2(x-y)D_F(y-z)J(z)\,, \tag{1.56}$$

meaning that

$$J(x) = \Box\phi_c(x) + \frac{i\lambda^2}{4}\int d^4y D_F^2(x-y)D_F(y-z)J(z)\,. \tag{1.57}$$

We solve this equation recursively by making the ansatz

$$J(x) = J_0(x) + J_2(x)\lambda^2\,. \tag{1.58}$$

We get

$$J_0(x) = \Box\phi_c(x)\,, \tag{1.59}$$

and

$$\begin{aligned} J_2(x) &= \frac{i}{4}\int d^4y d^4z\, D_F^2(x-y)D_F(y-z)\Box_z\phi_c(z) \\ &= \frac{i}{4}\int d^4y d^4z\, D_F^2(x-y)\left(\Box_z D_F(y-z)\right)\phi_c(z) \\ &= \frac{1}{4}\int d^4y\, D_F^2(x-y)\phi_c(y)\,. \end{aligned} \tag{1.60}$$

To obtain the final result we have employed integration by parts twice, then used that D_F satisfies the Klein-Gordon equation with delta function source, which in turn allowed us to integrate over z. Combining (1.59) and (1.60), it follows that

$$J(x) = \Box\phi_c(x) + \frac{\lambda^2}{4}\int d^4y\, D_F^2(x-y)\phi_c(y)\,. \tag{1.61}$$

We now move our attention to the effective action (1.33). At $\mathcal{O}(\lambda^0)$ one finds in terms of $J_0(x) = \Box\phi_c(x)$ the following expression

$$\begin{aligned}\Gamma_0[\phi_c] &= \frac{i}{2}\int d^4x d^4y\, \Box\phi_c(x) D_F(x-y) \Box\phi_c(y) - \int d^4x\, (\Box\phi_c(x))\, \phi_c(x) \\ &= \int d^4x\, \frac{1}{2}\left(\partial_\mu \phi_c(x)\right)^2 .\end{aligned} \tag{1.62}$$

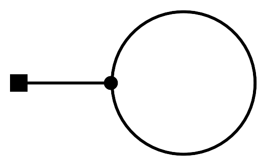
Here I have again used integration by parts and the fact that D_F is a Green's function of the Klein-Gordon equation. This result looks quite familiar. In fact, we have already derived it in (1.38). It is the kinetic term of the classic action. There is of course no mass term, because we have set $m = 0$ by hand in the classic theory (1.48).

By inserting (1.61) into the definition (1.33), one can also show that the $\mathcal{O}(\lambda)$ part of the effective action takes the form

$$\Gamma_1[\phi_c] = \int d^4x \left[-\frac{\lambda}{3!}\, \phi_c^3(x) - \frac{\lambda}{2}\, D_F(0)\, \phi_c(x) \right] . \tag{1.63}$$

The term $-\lambda/3!\, \phi_c^3$ is again part of the classic action or Lagrangian (1.48). But the term

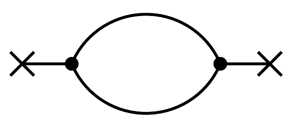
$$-\frac{\lambda}{2}\, D_F(0)\, \phi_c(x) =$$



, (1.64)

is a new contribution that arises from quantum corrections. These so-called *tadpole* contributions can be removed (by a proper renormalization) and therefore not affect physical processes. So let's forget about them and press on.

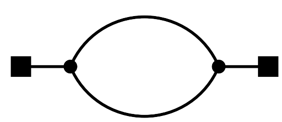
At $\mathcal{O}(\lambda^2)$ one gets a contribution to $\Gamma[\phi_c]$ from the graph



, (1.65)

i.e., the third diagram in the second line of $Z[J]$ as given in (1.54) as well as a similar contribution from $-\int d^4x\, J(x)\phi_c(x)$. Using the expansion (1.61) of the source J, one finds after some algebra

$$\Gamma_2[\phi_c] = \int d^4x \left[-\frac{\lambda^2}{4}\, \phi_c(x) \int d^4y\, D_F^2(x-y) \phi_c(y) \right] =$$



. (1.66)

This is an interesting result. Since (1.66) is bilinear in ϕ_c the $\mathcal{O}(\lambda^2)$ term of the effective action corresponds to a *loop-induced mass term* for the classic field. So our scalar field will get a mass from radiative corrections even if we start with $m = 0$.

The general lesson to learn here is that if there is *no symmetry* that forbids a specific term in the Lagrangian, one better includes it in the theory. If one does not do this, one will always get it back in the quantum theory.