Interacting Quantum Fields
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1 Interacting Quantum Fields

In this part of the lecture we will develop the formalism that allows one to understand the basic features of perturbatively interacting quantum field theories (QFTs). The main goal is to derive the Feynman rules for bosonic field theories and to show how to calculate matrix elements and cross sections for simple scattering processes. More formal aspects of interacting QFTs (such as *e.g.* the exponentiation of vacuum diagrams or the Lehmann-Symanzik-Zimmermann reduction formula) are only mentioned in passing.

1.1 Toy Models

In what follows we will only deal with weakly coupled QFTs, *i.e.*, theories that can be truly considered as small perturbations of the free field theory at all energies. We will look in more detail at two specific examples that serve the purpose of toy models.

The first example of a weakly coupled QFT we will study is the ϕ^4 *theory*,

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \phi)^2 - \frac{1}{2}\,m^2\phi^2 - \frac{\lambda}{4!}\,\phi^4\,, \tag{1.1}$$

where ϕ is our well-known real scalar field. For (1.1) to be weakly-coupled we have to require $\lambda \ll 1$. We can get a hint for what the effects of the additional ϕ^4 term will be. Expanding it in terms of ladder operators, we find terms like

$$a^\dagger(p)a^\dagger(p)a^\dagger(p)a^\dagger(p)\,, \qquad a^\dagger(p)a^\dagger(p)a^\dagger(p)a(p)\,, \tag{1.2}$$

etc., which create and destroy particles. This signals that the ϕ^4 Lagrangian (1.1) describes a theory in which particle number is not conserved. In fact, it is not too difficult to check that the number operator N does not commute with the Hamiltonian, *i.e.*, $[N, H] \neq 0$.

The second example we will look at is a *scalar Yukawa theory*. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) + \frac{1}{2}\,(\partial_\mu \phi)^2 - M^2 \varphi^* \varphi - \frac{1}{2}\, m^2 \phi^2 - g\, \varphi^* \varphi\, \phi\,, \tag{1.3}$$

with $g \ll M, m$. This theory couples a complex scalar φ to a real scalar ϕ. In this theory the individual particle numbers for φ and ϕ are not conserved. Yet, the Lagrangian (1.3) is invariant under global phase rotations of φ, which ensures that there will be a conserved charge Q obeying $[Q, H] = 0$. In fact, we have met this charge already in our discussion of the quantization of the complex Klein-Gordon field (see (1.76) in the script "Canonical Quantization"). In consequence, in the scalar Yukawa theory the number of φ particles minus the number of φ anti-particles is conserved. Notice also that the potential in (1.3) has a stable minimum at $\varphi = \phi = 0$, but it is unbounded from below, if $-g\phi$ becomes too large. This means that we should not mess to much with the scalar Yukawa theory.

1.2 Interaction Picture

In quantum mechanics (QM), there is a useful viewpoint called the *interaction picture*, which allows one to deal with small perturbations to a well-understood Hamiltonian. Let me briefly recall how this works. In the *Schrödinger picture*, the states evolve as $id/dt|\psi\rangle_S = H|\psi\rangle_S$, while the operators $\mathcal{O}_S$ are time independent. In contrast, in the *Heisenberg picture* the states do not evolve with time, but the operators change with time, namely one has $|\psi\rangle_H = e^{iHt}|\psi\rangle_S$ and $\mathcal{O}_H = e^{iHt}\,\mathcal{O}_S\, e^{-iHt}$. The interaction picture is a hybrid of the two. We split the Hamiltonian as follows

$$H = H_0 + H_{\rm int}\,, \tag{1.4}$$

where in the interaction picture the time dependence of operators $\mathcal{O}_I$ is governed by H_0, while the time dependence of the states $|\psi\rangle_I$ is governed by $H_{\rm int}$. While this split is arbitrary, things are easiest if one is able to solve the Hamiltonian H_0, *e.g.*, if H_0 is the Hamiltonian of a free theory. From what I have said so far, it follows that

$$|\psi\rangle_I = e^{iH_0 t}\,|\psi\rangle_S\,, \qquad \mathcal{O}_I = e^{iH_0 t}\,\mathcal{O}_S\, e^{-iH_0 t}\,. \tag{1.5}$$

Since the Hamiltonian is itself an operator, the latter equation also applies to the interaction Hamiltonian $H_{\rm int}$. In consequence, one has

$$H_I = (H_{\rm int})_I = e^{iH_0 t}\, H_{\rm int}\, e^{-iH_0 t}\,. \tag{1.6}$$

The Schrödinger equation in the interaction picture is readily derived starting from the Schrödinger picture,

$$\begin{aligned} i\frac{d}{dt}\,|\psi\rangle_S = H|\psi\rangle_S\,, \quad &\Longrightarrow \quad i\frac{d}{dt}\left(e^{-iH_0 t}\,|\psi\rangle_I\right) = (H_0 + H_{\rm int})\, e^{-iH_0 t}\,|\psi\rangle_I\,, \\ &\Longrightarrow \quad i\frac{d}{dt}\,|\psi\rangle_I = e^{iH_0 t}\, H_{\rm int}\, e^{-iH_0 t}|\psi\rangle_I\,, \\ &\Longrightarrow \quad i\frac{d}{dt}\,|\psi\rangle_I = H_I\,|\psi\rangle_I\,. \end{aligned} \tag{1.7}$$

Dyson's Formula

In order to solve the system described by the Hamiltonian (1.4), we have to find a way of how to find a solution to the Schrödinger equation in the interaction basis (1.7). Let us write the solution as

$$|\psi(t)\rangle_I = U(t,t_0)\,|\psi(t_0)\rangle_I\,, \tag{1.8}$$

where $U(t,t_0)$ is an unitary *time-evolution operator* satisfying $U(t,t) = 1$, $U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$, and $U(t_1,t_3)\left[U(t_2,t_3)\right]^\dagger = U(t_1,t_2)$. Inserting (1.8) into the last line of (1.7), we find

$$i\frac{d}{dt}\,U(t,t_0) = H_I(t)\,U(t,t_0)\,. \tag{1.9}$$

If H_I would be a function, the solution to the differential equation (1.9) would read

$$U(t,t_0) \stackrel{?}{=} \exp\left(-i\int_{t_0}^{t} dt'\,H_I(t')\right)\,. \tag{1.10}$$

Yet, H_I is not a function but an operator and this causes ordering issues. Let's have a closer look at the exponential to understand where the trouble comes from. The exponential is defined through its power expansion,

$$\exp\left(-i\int_{t_0}^{t} dt'\,H_I(t')\right) = 1 - i\int_{t_0}^{t} dt'\,H_I(t') + \frac{(-i)^2}{2}\left(\int_{t_0}^{t} dt'\,H_I(t')\right)^2 + \dots\,. \tag{1.11}$$

When we differentiate this with respect to t, the third term on the right-hand side gives

$$-\frac{1}{2}\left(\int_{t_0}^{t} dt'\,H_I(t')\right)H_I(t) - \frac{1}{2}\,H_I(t)\left(\int_{t_0}^{t} dt'\,H_I(t')\right)\,. \tag{1.12}$$

The second term of this expression looks good since it is part of $H_I(t)U(t,t_0)$ appearing on the right-hand side of (1.9), but the first term is no good, because the $H_I(t)$ sits on the wrong side of the integral, and we cannot commute it through, given that $[H_I(t'),H_I(t)] \neq 0$ when $t \neq t'$. So what is the correct expression for $U(t,t_0)$ then?

The correct answer is provided by *Dyson's formula*,[1] which reads

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\,H_I(t')\right)\,. \tag{1.13}$$

Here T denotes time ordering as defined in (1.78) of my script "Canonical Quantization". It is easy to prove the latter statement. We start by expanding out (1.13), which leads to

$$\begin{aligned} U(t,t_0) = 1 &- i\int_{t_0}^{t} dt'\,H_I(t') \\ &+ \frac{(-i)^2}{2}\left[\int_{t_0}^{t} dt'\int_{t'}^{t} dt''\,H_I(t'')\,H_I(t') + \int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\,H_I(t')\,H_I(t'')\right] + \dots\,. \end{aligned} \tag{1.14}$$

[1] Essentially figured out by Dirac, but in its compact notation due to Dyson.

In fact, the terms in the last line are actually the same, since

$$\begin{aligned}\int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, H_I(t'')\, H_I(t') &= \int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt'\, H_I(t'')\, H_I(t') \\ &= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')\, H_I(t'')\,,\end{aligned} \tag{1.15}$$

where the range of integration in the first expression is over $t'' \geq t'$, while in the second expression one integrates over $t' \leq t''$, which is, of course, the same thing. The final expression is simply obtained by relabelling t' and t''. In fact, it is not too difficult to show that one has

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1) \ldots H_I(t_n) = \frac{1}{n!} \int_{t_0}^{t} dt_1 \ldots dt_n\, T\left(H_I(t_1) \ldots H_I(t_n)\right)\,. \tag{1.16}$$

Putting things together this means that the power expansion of (1.12) takes the form

$$U(t,t_0) = 1 - i \int_{t_0}^{t} dt'\, H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')\, H_I(t'') + \ldots\,. \tag{1.17}$$

The proof of Dyson's formula is straightforward. First, observe that under the T operation, all operators commute, since their order is already fixed by time ordering. Thus,

$$\begin{aligned} i\frac{d}{dt}\, U(t,t_0) &= i\frac{d}{dt}\left[T \exp\left(-i \int_{t_0}^{t} dt'\, H_I(t')\right)\right] = T\left[H_I(t) \exp\left(-i \int_{t_0}^{t} dt'\, H_I(t')\right)\right] \\ &= H_I(t)\, T \exp\left(-i \int_{t_0}^{t} dt'\, H_I(t')\right) = H_I(t) U(t,t_0)\,. \end{aligned} \tag{1.18}$$

Notice that since t, being the upper limit of the integral, is the latest time so that the factor $H_I(t)$ can be pulled out to the left.

Before moving on, I have to say that Dyson's formula is rather formal. In practice, it turns out to be very difficult to compute the time-ordered exponential in (1.13). The power of (1.13) comes from the expansion (1.17) which is valid when H_I is a small perturbation to H_0.

Anatomy of Scattering Processes

Let us now try to apply the interaction picture to QFT, starting with an easy example, namely the interaction Hamiltonian of the Yukawa theory,

$$H_{\text{int}} = g \int d^3x\, \varphi^* \varphi\, \phi\,. \tag{1.19}$$

Unlike the free scalar theories discussed before, this interaction does not conserve the particle number of the individual fields, allowing particles of one type to morph into others. In order to see why this is the case, we look at the evolution of the state, *i.e.*, $|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle$, in the interaction picture. If $g \ll M, m$, where M and m are the masses of

φ and ϕ, respectively, the perturbation (1.19) is small and we can approximate the full time-evolution operator $U(t, t_0)$ in (1.13) by (1.17). Notice that (1.17) is, in fact, an expansion in powers of $H_{\rm int}$. The interaction Hamiltonian $H_{\rm int}$ contains ladder operators for each type of particle. In particular, the ϕ field contains the operators $a^\dagger$ and a that create or destroy ϕ particles.[2] Let's call this particle *mesons* (M). On the other hand, from earlier discussions it follows that the field φ contains the operators $a_+^\dagger$ and a_-, which implies that it creates φ anti-particles and destroys φ particle. We will call these particles *nucleons* (N).[3] Finally, the action of $\varphi^\dagger$ is to create nucleons through $a_-^\dagger$ and to destroy anti-nucleons via a_+.

While the individual particle number is not conserved, it is important to emphasize that $Q = N_+ - N_-$ is conserved not only in the free theory, but also in the presence of $H_{\rm int}$. At first order in $H_{\rm int}$, one will have terms of the form $a_+^\dagger a_-^\dagger a$ which destroys a meson and creates a nucleon-antinucleon pair, $M \to N\bar{N}$. At second order in $H_{\rm int}$, we have more complicated processes. *E.g.*, the combination of ladder operators $(a_+^\dagger a_-^\dagger a)(a_+ a_- a^\dagger)$ gives rise to the scattering process $N\bar{N} \to M \to N\bar{N}$. The rest of this section is devoted to calculate the quantum amplitudes for such processes to occur.

In order to calculate the amplitude, we have to make an important, but slightly dodgy, assumption. We require that the initial state $|i\rangle$ at $t \to -\infty$ (final state $|f\rangle$ at $t \to \infty$) is an eigenstate of the free theory described by the Hamiltonian H_0. At some level, this sounds like a reasonable approximation. If at $t \to \mp\infty$ the particles are well separated they do not feel the effects of each other. Moreover, we intuitively expect that the states $|i\rangle$ and $|f\rangle$ are eigenstates of the individual number operators N and $N_\pm$. These operators commute with H_0, but not with $H_{\rm int}$. As the particles approach each other, they interact briefly, before departing again, each going on its own way. The amplitude to go from $|i\rangle$ to $|f\rangle$ is given by

$$\lim_{t_\mp \to \mp\infty} \langle f|U(t_+, t_-)|i\rangle = \langle f|S|i\rangle \,, \tag{1.20}$$

where the unitary operator S is known as the *scattering* or simply *S-matrix*.

There are a number of reason why the assumption of non-interacting initial and final states $|i\rangle$ and $|f\rangle$ is shaky. First, one cannot describe bound states. *E.g.*, naively this formalism cannot deal with the scattering of an electron (e^-) and proton (p) which collide, bind, and leave as a hydrogen atom. Second, and more importantly, a single particle, a long way from its neighbors, is never alone in field theory. This is true even in classical electrodynamics, where the electron sources the electromagnetic field from which it can never escape. In quantum electrodynamics (QED), a related fact is that there is a cloud of virtual photons surrounding the electron. This line of thought gets us into the issues of *renormalization*. Although both problems can be circumvent, let me, for the time being, simply use the assumption of non-interacting *asymptotic states*. After developing the basics of scattering theory, we will briefly revisit the latter problem.

[2]The argument p of the ladder operators $a^\dagger$ and a *etc.* is dropped hereafter in the text.

[3]Of course, in reality nucleons are spin-1/2 particles, and do not arise from the quantization of a scalar field. Our scalar Yukawa theory is therefore only a toy model for nucleons interacting with mesons.

1.3 Wick's Theorem

Using Dyson's formulas (1.13) and (1.17), we can now in principle[4] calculate all matrix elements of the form

$$\langle f|T\left(H_I(x_1)\dots H_I(x_n)\right)|i\rangle\,, \tag{1.21}$$

where $|i\rangle$ and $|f\rangle$ are assumed to be asymptotically free states. The ordering of the operators H_I is fixed by time ordering. However, since the interaction Hamiltonian contains certain creation and annihilation operators, it would be convenient if we could start to move all annihilation operators to the right, where they can start eliminating particles in $|i\rangle$ (recall that this is the definition of normal ordering). *Wick's theorem* tells us how to go from time-ordered to normal-ordered products. Before stating Wick's theorem in its full generality, let's keep it simple and try to rederive something that we have seen already before. This is always a very good idea.

Case of Two Fields

The most simple matrix element of the form (1.21) is

$$\langle 0|T(\phi_I(x)\phi_I(y))|0\rangle\ . \tag{1.22}$$

We already calculated this object in Chapter 1.4 of the script "Canonical Qunatization" and gave it the name Feynman propagator. What we want to do now is to rewrite it in such a way that it is easy to evaluate and to generalize the obtained result to the case with more than two fields. We start by decomposing the real scalar field in the interaction picture as

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)\,, \tag{1.23}$$

with[5]

$$\phi_I^+(x) = \int d^3\tilde{p}\ a(p)\,e^{-ipx}\,, \qquad \phi_I^-(x) = \int d^3\tilde{p}\ a^\dagger(p)\,e^{ipx}\,. \tag{1.24}$$

This decomposition can be done for any free field. It is useful since

$$\phi_I^+(x)|0\rangle = 0\,, \qquad \langle 0|\phi_I^-(x) = 0\,. \tag{1.25}$$

Now we consider the case $x^0 > y^0$ and compute the time-order product of the two scalar fields,

$$\begin{aligned} T(\phi_I(x)\phi_I(y)) &= \phi_I(x)\phi_I(y) = \big(\phi_I^+(x) + \phi_I^-(x)\big)\big(\phi_I^+(y) + \phi_I^-(y)\big) \\ &= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^-(y) \\ &\quad + \big[\phi_I^+(x), \phi_I^-(y)\big] \\ &= :\phi_I(x)\phi_I(y): + \big[\phi_I^+(x), \phi_I^-(y)\big]\ , \end{aligned} \tag{1.26}$$

[4]The actual calculations turn out to be rather cumbersome. An simple example of the direct application of Dyson's formula can be found in Chapter 4.3 of my script *Quantum Field Theory I* available at http://wwwthep.physik.uni-mainz.de/∼uhaisch/QFTI10/QFTI.pdf.

[5]The superscripts "±" do not make much sense, but I just follow Pauli and Heisenberg here. If you have to, complain with them.

where we have normal ordered the last line, *i.e.*, brought all ϕ_I^+'s to the right. To get rid of the $\phi_I^+(x)\phi_I^-(y)$ term, we have added the commutator $\left[\phi_I^+(x), \phi_I^-(y)\right]$ in the second line. In the case $y^0 > x^0$, we find, repeating the above exercise,

$$T(\phi_I(x)\phi_I(y)) = :\phi_I(x)\phi_I(y): + \left[\phi_I^+(y), \phi_I^-(x)\right] . \tag{1.27}$$

In order to combine the results (1.26) and (1.27) into one equation, we define the *contraction* of two fields,

$$\overbrace{\phi_I(x)\phi_I(y)} = \begin{cases} \left[\phi_I^+(x), \phi_I^-(y)\right] , & x^0 > y^0 , \\ \left[\phi_I^+(y), \phi_I^-(x)\right] , & y^0 > x^0 . \end{cases} \tag{1.28}$$

This definition implies that the contraction of two ϕ_I fields is nothing but the Feynman propagator:

$$\overbrace{\phi_I(x)\phi_I(y)} = D_F(x-y) . \tag{1.29}$$

For a string of field operators ϕ_I, the contraction of a pair of fields means replacing the contracted operators with the Feynman propagator, leaving all other operators untouched. Equipped with the definition (1.28), the relation between time-ordered and normal-ordered products of two fields can now be simply written as

$$T(\phi_I(x)\phi_I(y)) = :\phi_I(x)\phi_I(y): + \overbrace{\phi_I(x)\phi_I(y)} . \tag{1.30}$$

Let me emphasize that while both $T(\phi_I(x)\phi_I(y))$ and $:\phi_I(x)\phi_I(y):$ are operators, their difference is a complex function, namely the Feynman propagator or the contraction of two ϕ_I fields.

The formalism of contractions is also straightforwardly extended to our complex scalar field φ_I. One has

$$T(\varphi_I(x)\varphi_I^\dagger(y)) = :\varphi_I(x)\varphi_I^\dagger(y): + \overbrace{\varphi_I(x)\varphi_I^\dagger}(y) , \tag{1.31}$$

prompting us to define the contraction in this case as

$$\overbrace{\varphi_I(x)\varphi_I^\dagger}(y) = D_F(x-y) . \qquad \overbrace{\varphi_I(x)\varphi_I}(y) = \overbrace{\varphi_I^\dagger(x)\varphi_I^\dagger}(y) = 0 . \tag{1.32}$$

For convenience and brevity, I will from here on often drop the subscript I, whenever I calculate matrix elements of the form (1.21). There is however little room for confusion, since contractions will always involve interaction-picture fields.

Strings of Fields

With all this new notation at hand, the generalization to arbitrarily many fields is also easy to write down:

$$T\left(\phi(x_1)\dots\phi(x_n)\right) = :\left(\phi(x_1)\dots\phi(x_n) + \text{all possible contractions}\right): . \tag{1.33}$$

This identity is known as the Wick's theorem. Notice that for $n = 2$ the latter equation is equivalent to (1.31). Before proving Wick's theorem, let me tell you what the phrase "all possible contractions" means by giving an simple example.

For $n = 4$ we have, writing ϕ_i instead of $\phi(x_i)$ for brevity,

$$
\begin{aligned}
T\left(\phi_1\phi_2\phi_3\phi_4\right) = :\big(&\phi_1\phi_2\phi_3\phi_4 + \overbracket{\phi_1\phi_2}\phi_3\phi_4 + \overbracket{\phi_1\phi_2\phi_3}\phi_4 + \overbracket{\phi_1\phi_2\phi_3\phi_4} \\
&+ \phi_1\overbracket{\phi_2\phi_3}\phi_4 + \phi_1\overbracket{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overbracket{\phi_3\phi_4} \\
&+ \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4} + \overbracket{\phi_1\phi_2\phi_3\phi_4} + \overbracket{\phi_1\phi_2\phi_3\phi_4}\big): \ .
\end{aligned}
\tag{1.34}
$$

When the contracted field operator are not adjacent, we still define it to give Feynman propagator. *E.g.*,

$$
:\phi_1\overbracket{\phi_2\phi_3}\phi_4: = D_F(x_2 - x_4) \ :\phi_1\phi_3: \ . \tag{1.35}
$$

Since the vacuum expectation value (VEV) of any normal-ordered operator vanishes, *i.e.*, $\langle 0|:\mathcal{O}:|0\rangle = 0$, sandwiching any term of (1.34) in which there remain uncontracted field operators between $|0\rangle$ gives zero. This means that only the three fully contracted terms in the last line of that equation survive and they are all complex functions. We thus have

$$
\begin{aligned}
\langle 0|T\left(\phi_1\phi_2\phi_3\phi_4\right)|0\rangle = &D_F(x_1 - x_2)D_F(x_3 - x_4) \\
&+ D_F(x_1 - x_3)D_F(x_2 - x_4) \\
&+ D_F(x_1 - x_4)D_F(x_2 - x_3)\,,
\end{aligned}
\tag{1.36}
$$

which is a rather simple result and has, as we will see in the next section, a nice pictorial interpretation.

Proof of Wick's Theorem

We still like to prove Wick's theorem. Naturally this is done by induction. We have already proved the case $n = 2$. So let's assume that (1.33) is valid for $n - 1$ and try to show that the latter equation also holds for n field operators. With out loss of generality we can assume that $x_1^0 > \ldots > x_n^0$, since if this is not the case we simply relabel the points in an appropriate way. Such a relabeling leaves both sides of (1.33) unchanged. Then applying Wick's theorem to the string $\phi_2 \ldots \phi_n$, we arrive at

$$
\begin{aligned}
T\left(\phi_1 \ldots \phi_n\right) &= \phi_1 \ldots \phi_n = \\
&= \phi_1 :(\phi_2 \ldots \phi_n + \text{all contraction not involving } \phi_1): \\
&= (\phi_1^+ + \phi_1^-) :(\phi_2 \ldots \phi_n + \text{all contraction not involving } \phi_1): \ .
\end{aligned}
\tag{1.37}
$$

We now want to move the $\phi_1^\pm$'s into the $:\big(\ldots\big):$. For ϕ_1^- this is easy, since moving it in, it is already on the left-hand side and thus the resulting term is normal ordered. The term with ϕ_1^+ is more complicated because we have to bring it into normal order by commuting ϕ_1^+ to the right. *E.g.*, consider the term without contractions,

$$
\begin{aligned}
\phi_1^+ :\phi_2 \ldots \phi_n: &= :\phi_2 \ldots \phi_n: \phi_1^+ + [\phi_1^+, :\phi_2 \ldots \phi_n:] \\
&= :\phi_1^+\phi_2 \ldots \phi_n: + :\big([\phi_1^+, \phi_2^-]\phi_3 \ldots \phi_n + \phi_2[\phi_1^+, \phi_3^-]\phi_4 \ldots \phi_n + \ldots\big): \\
&= :\big(\phi_1^+\phi_2 \ldots \phi_n + \overbracket{\phi_1\phi_2}\phi_3 \ldots \phi_n + \overbracket{\phi_1\phi_2\phi_3}\phi_4 \ldots \phi_n + \ldots\big): \ .
\end{aligned}
\tag{1.38}
$$

Here we first used the fact that the commutator of a single operator and a string of operators can be written as a sum of all possible strings of operators with two adjacent operators put into a commutator. The simplest relation of this type reads $[\phi_1, \phi_2\phi_3] = [\phi_1, \phi_2]\phi_3 + \phi_2[\phi_1, \phi_3]$ and is easy to prove. In the last step we then realized that under the assumption $x_1^0 > \ldots > x_n^0$ all commutator of two operators are equivalent to a contraction of the relevant fields.

The first term in the last line of (1.38) combines with the ϕ_1^- term of (1.37) to give $:\phi_1 \ldots \phi_n:$, meaning that we have derived the first term on the right-hand side of Wick's theorem as well as all terms involving only one contraction of ϕ_1 with another field in (1.33). It is not too difficult to understand that repeating the above exercise (1.38) with all the remaining terms in (1.37) will then give all possible contractions of all the fields, including those of ϕ_1. Hence the induction step is complete and Wick's theorem is proved.

Application of Wick's Theorem

In order to see the real power of Wick's theorem let's put it to work and try to calculate $NN \to NN$ scattering in the Yukawa theory (1.19). We first write down the expressions for the initial and final states,

$$\begin{aligned} |i\rangle &= a_+^\dagger(p_1) a_+^\dagger(p_2)|0\rangle = |p_1, p_2\rangle\,, \\ |f\rangle &= a_+^\dagger(q_1) a_+^\dagger(q_2)|0\rangle = |q_1, q_2\rangle\,. \end{aligned} \tag{1.39}$$

We now look at the expansion of $\langle f|S|i\rangle$ in powers of the coupling constant g. In order to isolate the interesting part of the S-matrix, *i.e.*, the part due to interactions, we define the *T-matrix* by

$$S = 1 + iT\,, \tag{1.40}$$

where the 1 describes the situation where nothing happens. The leading contribution to iT occurs at second order in the interaction (1.19). We find

$$\frac{(-ig)^2}{2} \int d^4x\, d^4y\; T\left(\varphi^\dagger(x)\varphi(x)\phi(x)\varphi^\dagger(y)\varphi(y)\phi(y)\right). \tag{1.41}$$

Applying Wick's theorem to the time-order production entering this expression, we get (besides others) a term

$$D_F(x-y)\,:\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y):\,, \tag{1.42}$$

which features a contraction of the two ϕ fields. This term will contribute to the scattering, because the operator $:\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y):$ destroys the two nucleons in the initial state and generates those appearing in the final state. In fact, (1.42) is the only contribution to the process $NN \to NN$, since any other ordering of the field operators would lead to a vanishing matrix element. The matrix element of the normal-ordered operator in (1.42) is readily computed:

$$\begin{aligned} \langle q_1, q_2|:\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y):|p_1, p_2\rangle &= \langle q_1, q_2|\varphi^\dagger(x)\varphi^\dagger(y)|0\rangle\langle 0|\varphi(x)\varphi(y)|p_1, p_2\rangle \\ &= \left(e^{i(q_1x+q_2y)} + e^{i(q_1y+q_2x)}\right)\left(e^{-i(p_1x+p_2y)} + e^{-i(p_1y+p_2x)}\right) \\ &= e^{i[(q_1-p_1)x+(q_2-p_2)y]} + e^{i[(q_2-p_1)x+(q_1-p_2)y]} + (x \leftrightarrow y)\,, \end{aligned} \tag{1.43}$$

where, in going to the third line, we have used the fact that

$$\begin{aligned}\langle 0|\varphi(x)|p\rangle &= \langle 0|\int d^3\tilde{q}\,\left(a_+(q)e^{-iqx}+a_-^\dagger(q)e^{iqx}\right)a_+^\dagger(p)|0\rangle = \langle 0|\int d^3\tilde{q}\,[a_+(q),a_+^\dagger(p)]e^{-iqx}|0\rangle \\ &= \langle 0|\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{q}}}\,(2\pi)^3\,2\omega_{\boldsymbol{p}}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})e^{-iqx}|0\rangle = e^{-ipx}\,.\end{aligned} \tag{1.44}$$

Putting things together, the matrix element (1.41) takes the form

$$\frac{(-ig)^2}{2}\int\frac{d^4x\,d^4y\,d^4k}{(2\pi)^4}\left[e^{i[(q_1-p_1)x+(q_2-p_2)y]}+e^{i[(q_2-p_1)x+(q_1-p_2)y]}+(x\leftrightarrow y)\right]\frac{ie^{ik(x-y)}}{k^2-m^2+i\epsilon}\,, \tag{1.45}$$

where the term in curly brackets arises from (1.43), while the final factor stems from the expression for the Feynman propagator of the ϕ field. The $(x\leftrightarrow y)$ terms double up with the others to cancel the factor of $1/2$ in the prefactor $(-ig)^2/2$, while the x and y integrals give delta functions. One arrives at

$$\begin{aligned}(-ig)^2\int\frac{d^4k}{(2\pi)^4}\,\frac{i(2\pi)^8}{k^2-m^2+i\epsilon}\,\Big[&\delta^{(4)}(q_1-p_1+k)\delta^{(4)}(q_2-p_2-k) \\ &+\delta^{(4)}(q_2-p_1+k)\delta^{(4)}(q_1-p_2-k)\Big]\,.\end{aligned} \tag{1.46}$$

Finally, we perform the k integration using the delta functions. We obtain

$$i(-ig)^2\left[\frac{1}{(p_1-q_1)^2-m^2+i\epsilon}+\frac{1}{(p_1-q_2)^2-m^2+i\epsilon}\right](2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)\,, \tag{1.47}$$

where the delta function imposes momentum conservation.

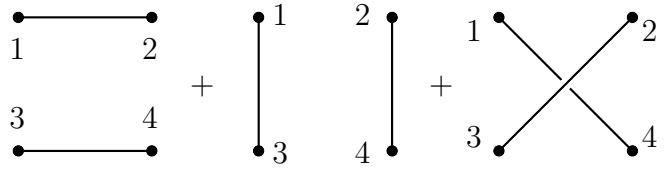
Notice that the above calculation is also relevant for the scatterings $\bar{N}\bar{N}\to\bar{N}\bar{N}$ and $N\bar{N}\to N\bar{N}$. Both reactions arise from the term (1.43) in Wick's theorem. However, we will never find a term that contributes to $NN\to\bar{N}\bar{N}$ or $\bar{N}\bar{N}\to NN$, because these transitions would violate the conservation of the charge $Q=N_+-N_-$ that is associated to the global $U(1)$ symmetry of φ.

1.4 Feynman Diagrams

As the above example demonstrates, to actually compute scattering amplitudes using Wick's theorem is (still) rather tedious. There's a much better way, which starts by drawing pretty pictures. This pictures represent the expansion of $\langle f|S|i\rangle$ and we will learn how to associate mathematical expressions with those pictures. The pictures, you probably already guessed it, are the famous *Feynman diagrams*. The Feynman-diagram approach turns out to be a powerful tool to calculate QFT amplitudes (or as Schwinger said: "Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses.").

We again start simple and consider the case of four fields, all at different space-time points, which we have already worked out in (1.36). Let us present each of the points x_1 to x_4 by a

point and the propagators $D_F(x_1 - x_2)$ *etc.* by a line joining the relevant points. Then the right-hand side of (1.36) can be represented as a sum of three Feynman diagrams,

$\langle 0|T\big(\phi_1\phi_2\phi_3\phi_4\big)|0\rangle \;=$ 

. (1.48)

While this matrix element is not a measurable quantity, the pictures suggest a physical interpretation. Two particles are generated at two points and then each propagators to one of the other points, where they are both annihilated. This can happen in three possible ways corresponding to the three shown graphs. The total amplitude for this process is the sum of the three Feynman diagrams.

Things get more interesting, if one considers expressions like (1.48) that contain field operators evaluated at the same space-time point. So let us have a look at the expansion of the propagator (1.22) of the real scalar field,

$$\langle 0|T\left\{\phi(x)\phi(y) + \phi(x)\phi(y)\left[-i\int dt\, H_I(t)\right] + \ldots\right\}|0\rangle\,, \tag{1.49}$$

in the presence of the interaction term $H_I = -\lambda/(4!)\,\phi^4$ of the ϕ^4 theory (1.1). The first term gives the free-field result, $\langle 0|T(\phi(x)\phi(y))|0\rangle = D_F(x-y)$, while the second term takes the form

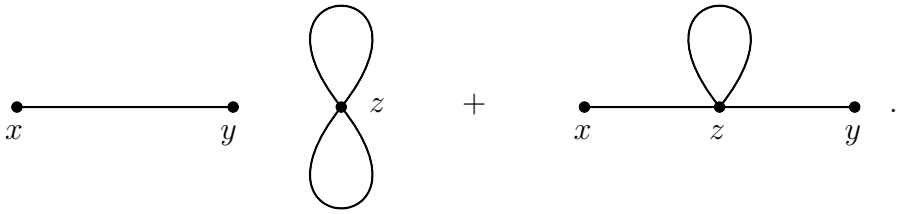
$$\begin{aligned}&\langle 0|T\left\{\phi(x)\phi(y)\,(-i)\int dt\int d^3z\,\frac{\lambda}{4!}\,\phi^4(z)\right\}|0\rangle\\ &\qquad = \langle 0|T\left\{\phi(x)\phi(y)\,\frac{-i\lambda}{4!}\int d^4z\,\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle\,.\end{aligned} \tag{1.50}$$

Now let's apply Wick's theorem (1.33) to (1.50). We get one term for each possible way to contract the six different ϕ's with each other in pairs. There are 15 such possibilities, but fortunately only two of these possibilities are really different. If we contract $\phi(x)$ and $\phi(y)$, there are 3 possible ways to contract the remaining $\phi(z)$'s. The other possibility is to contract $\phi(x)$ with $\phi(z)$ (four choices) and $\phi(y)$ with $\phi(z)$ (three choices), and $\phi(z)$ with $\phi(z)$ (one choice). There are 12 possible ways to do this, all giving the same result. In consequence, we have

$$\begin{aligned}&\langle 0|T\left\{\phi(x)\phi(y)\,(-i)\int dt\int d^3z\,\frac{\lambda}{4!}\,\phi^4(z)\right\}|0\rangle\\ &\quad = 3\,\frac{-i\lambda}{4!}\,D_F(x-y)\int d^4z\,D_F(z-z)\,D_F(z-z)\\ &\qquad + 12\,\frac{-i\lambda}{4!}\int d^4z\,D_F(x-z)\,D_F(y-z)\,D_F(z-z)\,.\end{aligned} \tag{1.51}$$

We can understand the latter expression better if we represent each term as a Feynman graph. Again we draw each propagator as a line and each point as a dot. This time we have

however to distinguish between the external points x and y and the internal point z, which is associated with a factor $\int d^4z\,(-i\lambda/(4!))$. Neglecting the overall factors, we see that the expression (1.51) is equal to the sum of the following two diagrams

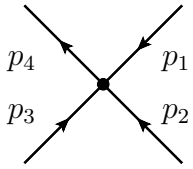


. (1.52)

We refer to the lines in these diagrams as *propagators*, since they represent the propagation amplitudes $D_F(x-y)$ *etc.* Internal points where four lines meet are called *vertices*. Since $D_F(x-y)$ is the amplitude for a free Klein-Gordon particle to propagate between x and y, the diagrams actually interpret the analytic formula as a process of creation, propagation, and annihilation which takes place in space-time.

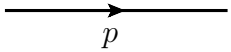
Momentum-Space Feynman Rules

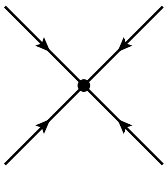
We are now ready to summarize our rules needed to find the analytic expression for each piece of a given Feynman diagram in the ϕ^4 theory. In fact, it turns out that in actual calculation it is more convenient to work in momentum-space rather than position-space by introducing the Fourier transformation of the Feynman propagator. To such a propagator one has to assign a 4-momentum p, indicating in general the direction of the momentum with an arrow (since $D_F(x-y) = D_F(y-x)$ the direction of p is arbitrary). The z-dependent factors of the vertices in a diagram are then given by

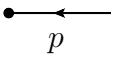


$$\Longleftrightarrow \int d^4z\, e^{-i\,(p_1+p_2+p_3-p_4)\cdot z} = (2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3-p_4)\,. \qquad (1.53)$$

In other words momentum is conserved at each vertex. The delta functions from the vertices can now be used to perform some of the momentum integrals from the propagators. We are left with the following *momentum-space Feynman rules*:

1. For each propagator one has  $= \dfrac{i}{p^2-m^2+i\epsilon}\,.$

2. For each vertex one has  $= -i\lambda\,.$

3. For each external point one has x  $= e^{-ipx}\,.$

4. Impose momentum conservation at each vertex.

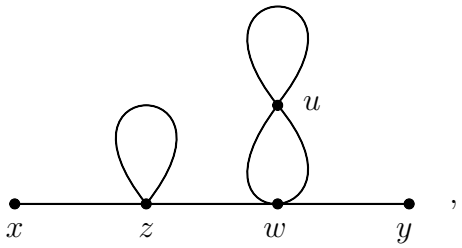
5. Integrate over each undetermined momentum $\int \frac{d^4l}{(2\pi)^4}$.

6. Divide by the symmetry factor.

To obtain the amplitude corresponding to each individual Feynman diagram one has to *multiply* all propagators and vertices and then one *adds* all possible amplitudes for a given process. In fact, one can think of the vertex Feynman rule $-i\lambda$ as the amplitude for the emission and/or absorption of particles at a vertex, while the exponential factor for an external point is just the amplitude for a particle at that point to have the needed momentum. The momentum integral $\int d^4l/(2\pi)^4$, on the other hand, is nothing but the *superposition principle* of QM and tells us that we have to "sum" over all possible momenta for which the process can occur.

Symmetry Factors

In practical applications one always draws the Feynman graphs first. *E.g.*, in the case of the propagator (or two-point function) of the ϕ^4 theory, one can draw the following "cactus" diagram

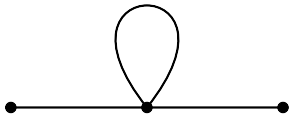


(1.54)

at $\mathcal{O}(\lambda^3)$ using it as a mnemonic device to write down the analytic expression. If this is done, one still has to figure out the multiplicative overall factor. The easiest way to find this factor consists in determining the so-called *symmetry factor* of the diagram. In the case of (1.54) the symmetry factor reads $S = 2 \cdot 2 \cdot 2 = 8$. Two factors of 2 come from lines that start and end on the same vertex, since the diagram is symmetric under the interchange of the ends of such lines (z and u in our case). The other factor of 2 comes from the two propagators connecting w and u, since the graph is symmetric under the interchange of these two lines. A third type of symmetry (not arising in the case at hand) is the equivalence of two vertices. In order to arrive at the correct overall factor, one has to divide by the symmetry factor, which is in general the number of possibilities to change parts of the diagrams without changing the result of the Feynman graph.

Most people never need to evaluate Feynman graphs with a symmetry factor larger than 2, so there is no need to worry too much about these technicalities. But for completeness let me give some examples of non-trivial symmetry factors. Here they are (dropping the labels x and

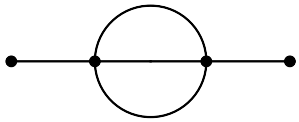
y at the external points):



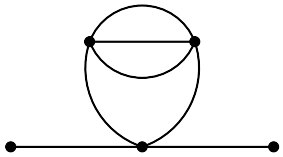
$S = 2\,,$

$S = 2 \cdot 2 \cdot 2 = 8\,,$

(1.55)



$S = 3! = 6\,,$



$S = 3! \cdot 2 = 12\,.$

Clearly, if you are in doubt about the symmetry factor you can always determine it by brute force, *i.e.*, counting equivalent contractions.

1.5 Scattering Processes

Let us now apply the things that we have learned to the case of $NN \to NN$ scattering. At order g^2 we have to consider the two diagrams shown in Figure 1.1. Employing the relevant momentum-space Feynman rules, it is readily seen that the analytic expression for the sum of the displayed graphs agrees with the final result (1.47) of the calculation that we performed earlier in Section 1.3. In fact, there is a nice physical interpretation of the graphs. We talk, rather loosely, of the nucleons exchanging a meson which, in the first diagram, has momentum $k = p_1 - q_1 = p_2 - q_2$. This meson does not satisfy the usual energy dispersion relation, because $k^2 \neq m^2$, where m is the mass of the meson. The meson is called a *virtual particle* and is said to be *off-shell* (or, sometimes, off mass-shell). Heuristically, it can't live long enough for its energy to be measured to great accuracy. In contrast, the momentum on the external, nucleon legs satisfy $p_1^2 = p_2^2 = q_1^2 = q_2^2 = M^2$, which means that the nucleons, having mass M, are *on-shell*. Similar considerations apply to the second diagram. It is important to notice that the appearance of the two diagrams above ensures that the particles satisfy *Bose statistics*.

The diagrams describing the scattering of a nucleon and an anti-nucleon, $N\bar{N} \to N\bar{N}$, are a little bit different than the ones for $NN \to NN$. At lowest order, the corresponding graphs are shown in Figure 1.2. Using the relevant Feynman rules, one finds the expression

$$i(-ig)^2 \left[\frac{1}{(p_1+p_2)^2 - m^2 + i\epsilon} + \frac{1}{(p_1-q_1)^2 - m^2 + i\epsilon} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)\,. \quad (1.56)$$

Notice that the delta function follows from the conservation of 4-momentum which, in turn, follows from space-time translational invariance. It is common to all S-matrix elements. We therefore define the amplitude $\mathcal{A}(f \to i)$ corresponding to a scattering process $i \to f$ as follows

$$\langle f\,|S-1|\,i\rangle = i\,\langle f\,|T|\,i\rangle = i\,(2\pi)^4\,\delta^{(4)}(p_f - p_i)\,\mathcal{A}(f \to i)\,, \quad (1.57)$$

where p_f (p_i) is the sum of the final (initial) 4-momenta, and the factor of i out front is a convention which is there to match non-relativistic QM.

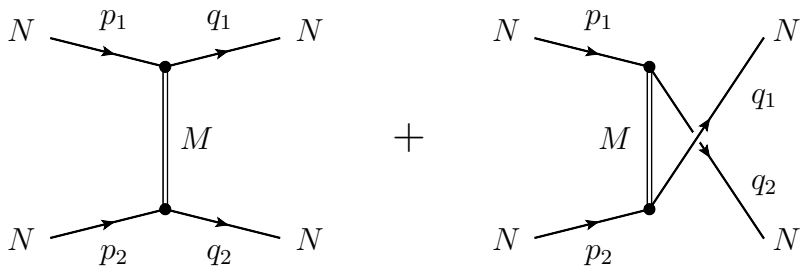


Figure 1.1: Feynman diagrams contributing to $NN \to NN$ scattering at order g^2.

We see that the amplitudes (1.47) and (1.56) (and in general all processes that include the exchange of just a single particle) depend on the same combinations of momenta in the denominators. There are standard names for various sums and differences of momenta that are known as *Mandelstam variables*. They are

$$\begin{aligned} s &= (p_1 + p_2)^2 = (q_1 + q_2)^2 \,, \\ t &= (p_1 - q_1)^2 = (p_2 - q_2)^2 \,, \\ u &= (p_1 - q_2)^2 = (p_2 - q_1)^2 \,, \end{aligned} \tag{1.58}$$

where, as in the explicit examples above, p_1 and p_2 are the momenta of the two initial-state particles, and q_1 and q_2 are the momenta of the two final-state particles. In order to get a feel for what these variables mean, let us assume (for simplicity) that all four particles are the same. In the CM frame, the initial two particles have the following 4-momenta

$$p_1 = (E, 0, 0, p) \,, \qquad p_2 = (E, 0, 0, -p) \,, \tag{1.59}$$

The particles then scatter at some angle θ and leave with momenta

$$q_1 = (E, 0, p \sin\theta, p \cos\theta) \,, \qquad q_2 = (E, 0, -p \sin\theta, -p \cos\theta) \,. \tag{1.60}$$

Then from the definitions (1.58), we have that

$$s = 4E^2 \,, \qquad t = -2p^2 \left(1 - \cos\theta\right) \,, \qquad u = -2p^2 \left(1 + \cos\theta\right) \,. \tag{1.61}$$

We see that the variable s measures the total CM energy of the collision, while the variables t and u are measures of the energy exchanged between particles (they are basically equivalent, just with the outgoing particles swapped around). Now the amplitudes that involve exchange of a single particle can be written simply in terms of the Mandelstam variables. *E.g.*, for nucleon-nucleon scattering, the amplitude (1.47) is proportional to[6]

$$\mathcal{A}(NN \to NN) \;\propto\; \frac{1}{t - m^2} + \frac{1}{u - m^2} \,, \tag{1.62}$$

[6]Here and in the following we simply drop all $i\epsilon$ terms.

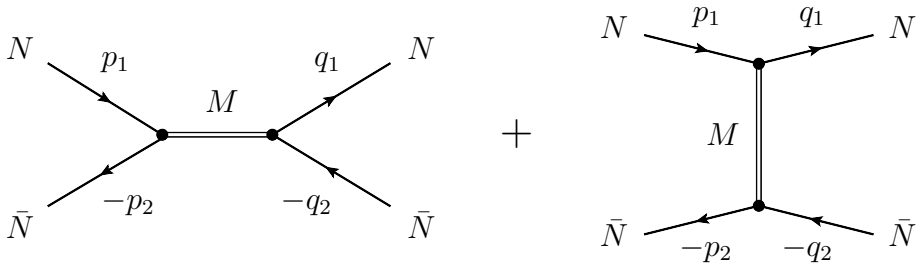


Figure 1.2: Feynman diagrams contributing to $N\bar{N} \to N\bar{N}$ scattering at order g^2.

while in the case of nucleon-anti-nucleon scattering one finds

$$\mathcal{A}(N\bar{N} \to N\bar{N}) \;\propto\; \frac{1}{s-m^2} + \frac{1}{t-m^2}\,. \tag{1.63}$$

We say that the first case involves t- and u-channel diagrams. On the other hand, the nucleon-anti-nucleon scattering is said to involve s- and t-channel exchange.

Note finally that there is a relationship between the Mandelstam variables. In the cases of $NN \to NN$ and $N\bar{N} \to N\bar{N}$ scattering, which involves external particles with the same mass, one has

$$s + t + u = 4M^2\,. \tag{1.64}$$

When the masses of the external particles are different this becomes $s + t + u = \sum_{i=1}^{4} m_i^2$, where m_i denotes the individual masses of the initial- and final-state particles.

1.6 Connected and Amputated Feynman Diagrams

I have explained in some detail how to compute scattering amplitudes by drawing all Feynman graphs and writing down the corresponding analytic expression for them using Feynman rules. In fact, there are a couple of caveats about what graphs one should draw and calculate. Both of these caveats are related to the assumption that the initial and final states are eigenstates of the free theory which, as we have mentioned before, is not correct.

The two caveats are as follows. First, we consider only *connected* graphs, where every part of the diagram is connected to at least one external point. This is related to the fact that the vacuum $|0\rangle$ of the free theory is not the true vacuum $|\Omega\rangle$ of the interacting theory. An example of a *disconnected* diagram is shown on the left-hand side in Figure 1.3. Second, we do not consider diagrams with loops on external lines so-called *unamputated* graphs. An example of such a diagram is depicted on the right-hand side of the latter figure. This omission is related to the fact that the one-particle states of the free and the interacting theory are not the same. We will refer to diagrams in which all loops on external legs have been removed as *amputated*.

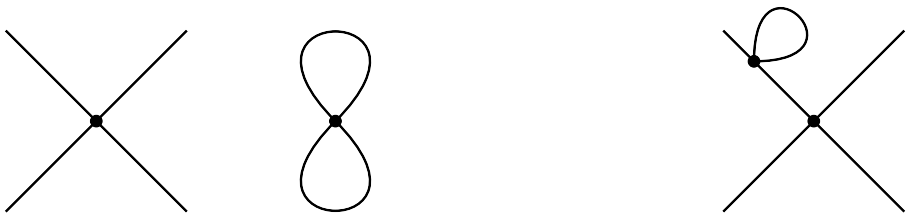


Figure 1.3: Example of a disconnected (left-hand side) and an unamputated (right-hand side) Feynman diagram in ϕ^4 theory.

Green's Functions in Interacting Theory

In order to keep the following discussion[7] as simple as possible, we will work in the real Klein-Gordon theory. We start by defining the n-point correlation (or Green's) function

$$G^{(n)}(x_1, \ldots, x_n) = \langle \Omega | T \left(\phi_H(x_1) \ldots \phi_H(x_n) \right) | \Omega \rangle \,, \tag{1.65}$$

where ϕ_H denotes the ϕ field in the Heisenberg picture of the full theory, rather than the interaction picture that we have been dealing with so far. The first question that one can ask, is how to compute $G^{(n)}$ in terms of matrix elements evaluated on $|0\rangle$, the vacuum of the free theory. I only state the result here. It reads

$$G^{(n)}(x_1, \ldots, x_n) = \lim_{t \to \infty} \frac{\langle 0 | T \left\{ \phi_I(x_1) \ldots \phi_I(x_n) \, \exp \left[-i \int_{-t}^{t} dt' \, H_I(t') \right] \right\} | 0 \rangle}{\langle 0 | T \left\{ \exp \left[-i \int_{-t}^{t} dt' \, H_I(t') \right] \right\} | 0 \rangle} \,. \tag{1.66}$$

Notice that both the numerator and denominator appearing on the right-hand side of the latter equation can be calculated using the methods developed for S-matrix elements, namely Feynman diagrams (or alternatively Dyson's formula and Wick's theorem) after expanding the exponentials into a Taylor series.

Exponentiation of Bubble Diagrams

By means of the above equality we can now calculate any n-point correlation function. Before you now start doing this using brute force, let me tell you that there is a theorem with the fancy name *linked-cluster theorem* that allows you to save a lot of time. This theorem states that the contributions to $G^{(n)}$ from disconnected diagrams *exponentiate*. I do not prove this theorem here, but let me illustrate the meaning of exponentiation (and its consequences) for the simplest case, *i.e.*, the two-point correlation function $G^{(2)}$.

The exponentiation of disconnected diagrams is relevant both for the denominator and numerator of the right-hand side of (1.66). Employing the linked-cluster theorem to the

[7] In this subsection we will state some important QFT results without proof. The actual proofs can be found in Chapter 4.9 of my script *Quantum Field Theory I* available at http://wwwthep.physik.uni-mainz.de/~uhaisch/QFTI10/QFTI.pdf.

denominator leads to the following pictorial identity

$$\lim_{t\to\infty} \langle 0|T \exp\left[-i\int_{-t}^{t} dt'\, H_I(t')\right]|0\rangle$$

$$= 1 \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig1.pdf}} \; + \; \left[\; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig2.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig3.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig4.pdf}} \; \right] \; + \; \cdots \tag{1.67}$$

$$= \exp\left[\raisebox{-0.5em}{\includegraphics[height=2.2em]{fig5.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig6.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig7.pdf}} \; + \; \cdots \right].$$

The disconnected Feynman diagrams appearing in this relation are called *vacuum bubbles*. In the case of $G^{(2)}$, the numerator of (1.66) takes the form

$$\lim_{t\to\infty} \langle 0|T\left\{\phi_I(x)\,\phi_I(y)\,\exp\left[-i\int_{-t}^{t} dt'\, H_I(t')\right]\right\}|0\rangle =$$

$$\left(\raisebox{-0.5em}{\includegraphics[height=2.2em]{fig8.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig9.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig10.pdf}} \; + \; \cdots \right) \tag{1.68}$$

$$\times \exp\left[\raisebox{-0.5em}{\includegraphics[height=2.2em]{fig11.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig12.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig13.pdf}} \; + \; \cdots \right].$$

Combining now (1.67) and (1.68), it follows that the exponentials involving the sum of disconnected diagrams cancel between the numerator and denominator in the formula for the correlation functions. In the case of the two-point function, the final form of (1.66) is thus

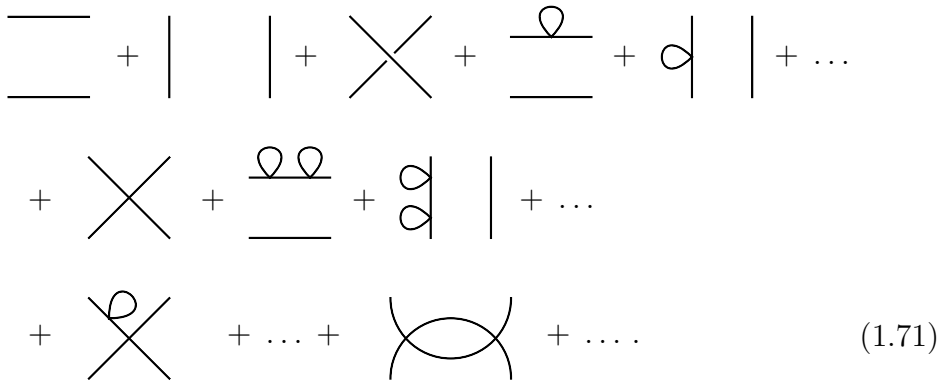
$$G^{(2)}(x,y) \; = \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig14.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig15.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig16.pdf}} \; + \; \raisebox{-0.5em}{\includegraphics[height=2.2em]{fig17.pdf}} \; + \; \ldots . \tag{1.69}$$

The generalization to higher correlation function is straightforward and reads

$$G^{(n)}(x_1,\ldots,x_n) = \langle\Omega|T\left(\phi_H(x_1)\ldots\phi_H(x_n)\right)|\Omega\rangle = \begin{pmatrix}\text{sum of all connected graphs}\\ \text{with } n \text{ external points}\end{pmatrix}. \tag{1.70}$$

The disconnected diagrams exponentiate, factor, and cancel as before. It is important to remember that by "disconnected" we mean "disconnected from all external points". In higher

correlations functions, diagrams can also be disconnected in another sense. Consider, *e.g.*, the four-point function

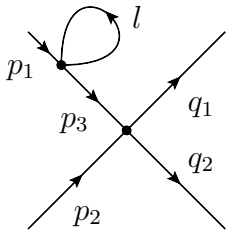
$G^{(4)}(x_1, x_2, x_3, x_4) =$  (1.71)

In many of the displayed diagrams, external points are disconnected from each other. Such diagrams do neither exponentiate nor factor, they contribute to the amplitude just as do the fully connected diagrams in which any point can be reached from any other by traveling along the lines.

One-Particle States in Interacting Theory

We now have an extremely beautiful formula (1.70) for computing an extremely abstract quantity the n-point correlation function. Our next task is to relate these objects back to S-matrix elements (1.20) $\big($or equivalent T-matrix elements (1.40)$\big)$, which will allow us to compute quantities that can actually be measured, namely decay rates and cross sections.

In order to achieve this goal, we still have to learn how to deal with diagrams involving loops on the external lines. Let us first try to understand the problem with such graphs, looking at a specific example. We consider the following Feynman diagram



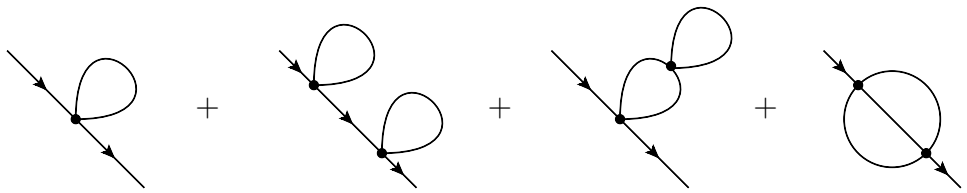
$$\begin{aligned} &= \frac{1}{2}\int d^4p_3\,\frac{i}{p_3^2-m^2}\int d^4l\,\frac{i}{l^2-m^2} \\ &\quad\times(-i\lambda)\,(2\pi)^4\delta^{(4)}(p_2+p_3-q_1-q_2) \\ &\quad\times(-i\lambda)\,(2\pi)^4\delta^{(4)}(p_1-p_3)\,, \end{aligned} \tag{1.72}$$

appearing in ϕ^4 theory. We can integrate over p_3 using the second delta function. It tells us to evaluate

$$\frac{1}{p_3^2-m^2}\bigg|_{p_3=p_1} = \frac{1}{p_1^2-m^2} = \frac{1}{0}\,. \tag{1.73}$$

We get an infinity, since p_1, being the momentum of an external particle, is on-shell, *i.e.*, $p_1^2 = m^2$. This is not good! Clearly, diagrams like (1.73) should not contribute to the S-

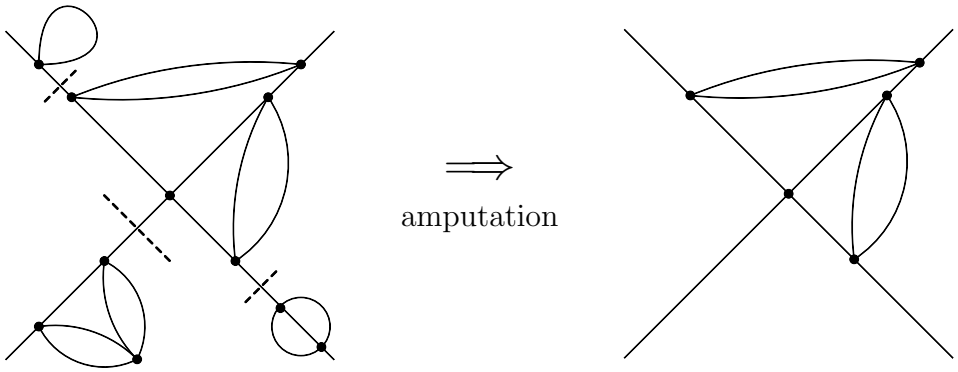
matrix elements. In fact, this is physically reasonable, since the external leg corrections,



(1.74)

represent the evolution of one-particle state of the free theory into the one-particle state of the interacting theory, in the same way that the vacuum-bubble diagrams represent the evolution of $|0\rangle$ into $|\Omega\rangle$. Since these corrections have nothing to do with the scattering process itself, it is somehow clear that one should exclude them from the calculation of the S-matrix.

This procedure goes by the name of *amputation*. For a generic Feynman diagram with external legs, we define amputation in the following way. Starting from the tip of each external leg, find the last point at which the diagram can be cut by removing a single propagator, such that this operation separates the leg from the rest of the diagram. Cut there. Let me give an non-trivial example of a diagram that appears at $\mathcal{O}(\lambda^{10})$, if one wants to compute $\phi\phi \to \phi\phi$ scattering in ϕ^4 theory. Here it is:



(1.75)

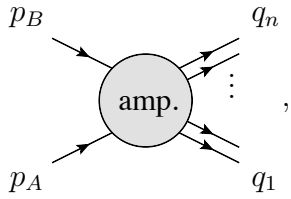
From Correlation Functions to Scattering Matrix Elements

We have argued that the problem with external-leg corrections is that they become singular for on-shell external states as implied by (1.73). In fact, it turns out that the structure of singularities of Feynman diagrams viewed as analytic functions of their external momenta (*i.e.*, Fourier-transformed correlation functions) plays a crucial role in the derivation of a general relation between Green's functions and S-matrix elements. This relation has first been derived by Harry Lehmann, Kurt Symanzik, and Wolfhart Zimmermann and is today known as the *LSZ reduction formula*. Combining the LSZ reduction formula with our Feynman rules for computing correlation functions will then lead to a master formula for S-matrix elements in terms of Feynman diagrams. In the following, I will simply state this relation without derivation for the case of the $(n+2)$-point correlator in ϕ^4 theory.

For the process $\phi(p_A)\phi(p_B) \to \phi(q_1)\dots\phi(q_n)$ the relation between the S-matrix element

and the Feynman graphs reads[8]

$$\langle q_1,\ldots,q_n|S|p_A,p_B\rangle = \left(\sqrt{Z}\right)^{n+2}$$

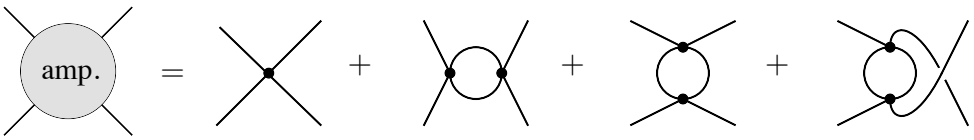


, (1.76)

where Z is called the *field-strength renormalization factor*, while the light gray blob represents the *sum of fully connected, amputated* $(n+2)$*-point Feynman diagrams in momentum space with on-shell external momenta*. With this master formula at hand you can now in principle calculate any S-matrix element!

In physical terms, the field-strength renormalization factor corresponds to the probability to create a one-particle state out of the vacuum $|\Omega\rangle$. It is a universal factor and can be calculated in perturbation theory. At leading order, *i.e.*, to $\mathcal{O}(\lambda^0)$ in the ϕ^4 theory, one has $Z=1$ which implies that the factors $Z^{1/2}$ in (1.76) are irrelevant for the tree-level calculations that we are dealing with in this lecture.

The Fourier-transform $\tilde{G}^{(n+2)}_{\text{amp.}}$ of the fully connected, amputated $(n+2)$-point on-shell correlation function represented by the light gray blob on the right-hand side in (1.76) is, on the other hand, process dependent. *E.g.*, in the case of $2\to 2$ scattering in the ϕ^4 theory, it has the following perturbative expansion



(1.77)

The discussion in this subsection looks all quite terrible, but at the end it is not, if one restricts oneself to the leading quantum effects. *E.g.*, the order λ contribution to $\tilde{G}^{(4)}_{\text{amp.}}$ as described by the first Feynman diagram on the right-hand side of (1.77) is easy to write down. Dropping disconnected graphs, (1.70) leads to

$$\begin{aligned}
G^{(4)}(x_1,x_2,x_3,x_4) &= -\frac{i\lambda}{4!}\int d^4z\,\big\langle 0\,\big|T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(z)\right)\big|\,0\big\rangle \\
&= -i\lambda\int d^4z\,D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z) \\
&= -i\lambda\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}\int d^4z\,e^{iz(p_1+\ldots+p_4)}\,e^{-i(p_1x_1+\ldots+p_4x_4)} \\
&\qquad\qquad \times\tilde{D}_F(p_1)\tilde{D}_F(p_2)\tilde{D}_F(p_3)\tilde{D}_F(p_4) \\
&= -i\lambda\,(2\pi)^4\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}\,\delta^{(4)}(p_1+\ldots+p_4)\,e^{-i(p_1x_1+\ldots+p_4x_4)} \\
&\qquad\qquad \times\tilde{D}_F(p_1)\tilde{D}_F(p_2)\tilde{D}_F(p_3)\tilde{D}_F(p_4)\,.
\end{aligned} \tag{1.78}$$

[8]If the external particles are of different species, each has its own renormalization factor $Z^{1/2}$. Furthermore, if the particles have spin, there will be additional polarization factors on the right-hand side of the equation.

From the first to the second line we have used Wick's theorem and then employed the representation

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \tilde{D}_F(p)\, e^{-ip(x-y)}\,, \tag{1.79}$$

with

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}\,, \tag{1.80}$$

of the Feynman propagator to rewrite the expression in momentum space. We now Fourier transform the final result in (1.78), stripping off $(2\pi)^4\,\delta^{(4)}(p_1 + \ldots + p_4)$ and the four external propagators to obtain the amputated 4-point Green's function:

$$G^{(4)}_{\text{amp.}}(p_1, p_2, p_3, p_4) = -i\lambda\,. \tag{1.81}$$

1.7 Decay Rates and Cross Sections

As in usual QM, also in QFT the probabilities for things to happen are the (modulus) square of the quantum amplitudes. In this subsection we will compute these probabilities, known as decay widths and cross sections. One small subtlety here is that any T-matrix element (1.57) comes with a factor of $(2\pi)^4\,\delta^{(4)}(p_f - p_i)$, so that we end up with the square of a delta function. As we will see in a moment, this subtlety is a result of the fact that we are working in an infinite space.

Decay Rates

We would now like to calculate the probability for a single-particle initial state $|i\rangle$ of momentum p_i and rest mass m to decay into the final state $|f\rangle$ consisting of n particles with total momentum $p_f = \sum_{j=1}^n q_j$. This quantity is given by the ratio

$$P_n = \frac{|\langle f|S|i\rangle|^2}{\langle i|i\rangle\langle f|f\rangle}\,. \tag{1.82}$$

The states $|i\rangle$ and $|f\rangle$ obey the relativistic normalization formula

$$\langle i|i\rangle = (2\pi)^3\, 2\omega_{\boldsymbol{p}_i}\, \delta^{(3)}(0) = 2\omega_{\boldsymbol{p}_i} V\,, \tag{1.83}$$

where we have replaced the delta function $\delta^{(3)}(0)$ by the volume V of space (remember that we have already seen this when we discussed the energy density of the vacuum in the "Canonical Quantization" part of the lecture). Similarly, one has for the final state

$$\langle f|f\rangle = \prod_{j=1}^n 2\omega_{\boldsymbol{q}_j} V\,. \tag{1.84}$$

If the initial-state particle is at rest, *i.e.*, $\omega_{\boldsymbol{p}_i} = m$ and $\boldsymbol{p}_i = 0$, we get using (1.57) for the $i \to f$ decay probability

$$\begin{aligned} P_n &= \frac{1}{2mV} \prod_{j=1}^{n} \frac{1}{2\omega_{\boldsymbol{q}_j} V} \, \left[(2\pi)^4 \delta^{(4)}(p_f - p_i)\right]^2 \, |\mathcal{A}(i \to f)|^2 \\ &= \frac{1}{2mV} \, (2\pi)^4 \delta^{(4)}(p_f - p_i) \, |\mathcal{A}(i \to f)|^2 \, Vt \, \prod_{j=1}^{n} \frac{1}{2\omega_{\boldsymbol{q}_j} V} \, . \end{aligned} \tag{1.85}$$

Notice that in order to arrive at the second line we have replaced one of the delta functions $(2\pi)^4 \delta^{(4)}(0)$ by the space-time volume Vt.

We can now divide out t to get the transition function per unit time. After integrating over all possible momenta of the final-state particles, *i.e.*, $V \int d^3q_j/(2\pi)^3$, we then obtain in terms of the relativistically-invariant n-body phase-space element[9]

$$d\Pi_n = (2\pi)^4 \, \delta^{(4)}(p_f - p_i) \, \prod_{j=1}^{n} \frac{d^3 q_j}{(2\pi)^3} \, \frac{1}{2\omega_{\boldsymbol{q}_j}} = (2\pi)^4 \, \delta^{(4)}(p_f - p_i) \, \prod_{j=1}^{n} d^3\tilde{q}_j \, , \tag{1.86}$$

the following expression for the *partial decay width* into the considered n-particle final state

$$\Gamma_n = \frac{1}{2m} \int d\Pi_n \, |\mathcal{A}(i \to f)|^2 \, . \tag{1.87}$$

Notice that the factors of the spatial volume V in the measure $V \int d^3q_j/(2\pi)^3$ have cancelled those in (1.85), while the factors $1/(2\omega_{\boldsymbol{q}_j})$ in (1.85) have conspired with the 3-momentum integrals in $V \int d^3q_j/(2\pi)^3$ to produce Lorentz-invariant measures $d^3\tilde{q}_j$. In consequence, the density of final states (1.86) is a Lorentz-invariant quantity.

After summation over all possible n-particle final states, one finally finds the so-called *total decay width*

$$\Gamma = \frac{1}{2m} \sum_n \int d\Pi_n \, |\mathcal{A}(i \to f)|^2 \, , \tag{1.88}$$

with $d\Pi_n$ corresponding to a given final state. The total decay width is equal to the reciprocal of the *half-life* $\tau = 1/\Gamma$ of the decaying particle. If the decaying particle is not at rest, the decay rate becomes $m\Gamma/\omega_{\boldsymbol{p}_i}$. This leads to an increased half-life $\omega_{\boldsymbol{p}_i}\tau/m = \tau/\sqrt{1 - \boldsymbol{v}^2} = \gamma\tau$, where $\boldsymbol{v}$ is the velocity of the decaying particle. Of course, this is a well-known effect related to *time dilation*. *E.g.* taking the muon lifetime at rest as the laboratory value of $2.22\,\mu$s, the lifetime of a cosmic ray produced muon traveling at 98% of the speed of light is about five times longer.

In terms of the partial and total decay width, (1.87) and (1.88), the *branching ratio* (or *branching fraction*) for the n-particle decay $i \to f$ reads

$$\mathcal{B}(i \to f) = \frac{\Gamma_n}{\Gamma} \, . \tag{1.89}$$

Needless to say that $\mathcal{B}(i \to f) \in [0, 1]$ and $\sum_n \mathcal{B}(i \to f) = 1$.

[9] This object is in some textbooks denoted by dPS_n.

Cross Sections

Consider an initial state consisting of one target and one beam particle in the momentum state $|i\rangle = |p_A, p_B\rangle$ scattering into a final state $|f\rangle = |q_1, \ldots, q_n\rangle$. In analogy with the calculation that lead to (1.88), the corresponding differential transition probability (1.85) per unit time and flux is given by

$$d\sigma = \frac{1}{F}\,\frac{d\Pi_n}{4E_A E_B V}\,|\mathcal{A}(i \to f)|^2\,, \tag{1.90}$$

which is usually referred to as the *differential cross section*. Here we have written E_A and E_B instead of $w_{\boldsymbol{p}_A}$ and $w_{\boldsymbol{p}_B}$. In the latter expression F stands for the flux associated with the incoming beam of particles. The flux can be written as

$$F = \frac{|\boldsymbol{v}_{\rm rel}|}{V} = \frac{|\boldsymbol{v}_A - \boldsymbol{v}_B|}{V} = \frac{|\boldsymbol{p}_A/E_A - \boldsymbol{p}_B/E_B|}{V} = \frac{\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}}{E_A E_B V}\,. \tag{1.91}$$

The equality of the next-to-last and last expression can be simply shown by using that $p_A^2 = m_A^2 = E_A^2 - \boldsymbol{p}_A^2$ and $p_B^2 = m_B^2 = E_B^2 - \boldsymbol{p}_B^2$. Combining (1.90) and (1.91) we thus find the expression

$$\sigma = \frac{1}{4\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}} \int d\Pi_n\,|\mathcal{A}(i \to f)|^2\,, \tag{1.92}$$

for the *total cross section*. The *flux factor* $1/4\left((p_A p_B)^2 - m_A^2 m_B^2\right)^{-1/2}$ is not Lorentz invariant, but invariant under boosts along the beam direction, as expected for a cross-sectional area perpendicular to the beam.

Differential Cross Section for $2 \to 2$ Scattering

To close this chapter we would now like to specialize the result (1.92) to the easiest case of $2 \to 2$ scattering. We carry out the discussion in the CM frame, the frame in which the total initial 3-momentum vanishes, *i.e.*, $\boldsymbol{p}_A + \boldsymbol{p}_B = 0$. Momentum conservation of course then implies that $\boldsymbol{q}_1 + \boldsymbol{q}_2 = 0$. For simplicity, we also assume that the masses of all four particles are equal to m, so in particular we have $p_A p_B = q_1 q_2$. Under this assumptions one has

$$\begin{aligned}\sqrt{(p_A p_B)^2 - m^4} &= \sqrt{(q_1 q_2)^2 - m^4} = \sqrt{(E_1 E_2 - \boldsymbol{q}_1 \cdot \boldsymbol{q}_2)^2 - (E_1^2 - \boldsymbol{q}_1^2)(E_2^2 - \boldsymbol{q}_2^2)} \\ &= \sqrt{(E_1 E_2 + \boldsymbol{q}_1^2)^2 - (E_1^2 - \boldsymbol{q}_1^2)(E_2^2 - \boldsymbol{q}_1^2)} = \sqrt{(E_1 + E_2)^2\,\boldsymbol{q}_1^2} = E|\boldsymbol{q}_1|\,,\end{aligned} \tag{1.93}$$

where in the third step we have used $\boldsymbol{q}_2 = -\boldsymbol{q}_1$ and in the final step identified $E = E_1 + E_2$ as the total CM energy.

Working in the CM frame we can also partially evaluate the integrals appearing in the two-body phase space

$$\int d\Pi_2 = \int \frac{d^3 q_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3 q_2}{(2\pi)^3}\frac{1}{2E_2}\,(2\pi)^4 \delta^{(4)}(p_A + p_B - q_1 - q_2)\,. \tag{1.94}$$

We first chose to integrate all three components of $\mathbf{q_2}$ over the delta function enforcing 3-momentum conservation. This sets $\mathbf{q_2} = -\mathbf{q_1}$ and converts the integral over the two-body phase space into

$$\int d\Pi_2 = \int \frac{d\Omega\, d|\mathbf{q}_1|\, |\mathbf{q}_1|^2}{(2\pi)^3 2E_1 2E_2}\, (2\pi)\, \delta(E - E_1 - E_2)\ . \tag{1.95}$$

Integrating over the final delta function gives

$$\int d\Pi_2 = \int d\Omega\, \frac{|\mathbf{q}_1|^2}{16\pi^2 E_1 E_2} \left(\frac{|\mathbf{q}_1|}{E_1} + \frac{|\mathbf{q}_1|}{E_2} \right)^{-1} = \frac{1}{16\pi^2} \int d\Omega\, \frac{|\mathbf{q}_1|}{E}\, . \tag{1.96}$$

Here we have used the relation

$$\frac{d}{d|\mathbf{q}_1|}\, (E_1 + E_2) = \frac{d}{d|\mathbf{q}_1|} \left(\sqrt{|\mathbf{q}_1|^2 + m^2} + \sqrt{|\mathbf{q}_1|^2 + m^2} \right) = \frac{|\mathbf{q}_1|}{E_1} + \frac{|\mathbf{q}_1|}{E_2}\ . \tag{1.97}$$

From (1.93) and (1.96) we then find the expression

$$\frac{d\sigma(p_A, p_B \to q_1, q_2)}{d\Omega} = \frac{1}{64\pi^2} \frac{|\mathcal{A}(p_A, p_B \to q_1, q_2)|^2}{s}\, . \tag{1.98}$$

which provides the cross section differential in the *solid angle* Ω for a $2 \to 2$ scattering. Notice that we have used the relation $s = E^2$ here to express the total CM energy through the Mandelstam variable s introduced in (1.58).

Notice that the expression for $d\sigma$ as given in (1.90) is also valid for identical particles in the final state. Finding a set of particles in the required momentum bin effectively identifies the particles. However, when integrating $d\sigma$ to obtain the total cross section σ for the scattering into the n particles one has to restrict this integration to inequivalent configurations. *E.g.*, the total cross section for the $2 \to 2$ reaction $N\bar{N} \to MM$ in the scalar Yukawa theory is obtained as $\sigma = 1/2 \int d\sigma$.