Canonical Quantization
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1 Canonical Quantization

In this part of the lecture we will discuss how to quantize classical field theories using the traditional method of *canonical quantization*. We start by recalling how canonical quantization works for classical mechanics.

1.1 From Classical to Quantum Theory

In quantum mechanics (QM), canonical quantization is a recipe that takes us from the Hamiltonian $H = H(q_a, p^b)$ of classical dynamics to the quantum theory. The recipe tells us to take the generalized coordinates $q_a(t)$ and their conjugate momenta $p^a(t) = \partial\mathcal{L}/\partial\dot{q}_a(t)$ and promote them to operators.[1] The Poisson bracket structure of classical mechanics descends to the structure of commutation relations between operators, namely

$$[q_a(t), q_b(t)] = [p^a(t), p^b(t)] = 0\,, \qquad [q_a(t), p^b(t)] = i\delta_a{}^b\,, \tag{1.1}$$

where $[a, b] = ab - ba$ is the usual commutator. The dynamics of this system is governed by the operator version of Hamilton's equations

$$\frac{dq_a(t)}{dt} = i\,[H, q_a(t)]\ , \qquad \frac{dp^a(t)}{dt} = i\,[H, p^a(t)]\ , \tag{1.2}$$

Notice that we work in the *Heisenberg picture* in which the operators incorporate a dependence on time, but the state vectors are time independent. In contrast, in the *Schrödinger picture* the operators are constant while the states evolve in time. The two pictures differ only by a basis change with respect to time dependence, which is the difference between active and

[1] To avoid cluttering the notation, we will drop hats on operators throughout the presentation. It will usually be clear from the context whether we refer to a classical object or its operator version.

passive transformations.[2] We will use the Heisenberg picture here, since it serves to define a third, hybrid picture, the *interaction picture* to be introduced later.

If one wants to construct a quantum field theory (QFT), one can proceed in a similar fashion. The idea is to start with the classical field theory and then to quantize it, *i.e.*, reinterpret the dynamical variables as operators that obey *equal-time* commutation relations,[3]

$$[\phi_a(t,\boldsymbol{x}),\phi_b(t,\boldsymbol{y})] = [\pi^a(t,\boldsymbol{x}),\pi^b(t,\boldsymbol{y})] = 0\,, \qquad [\phi_a(t,\boldsymbol{x}),\pi^b(t,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_a{}^b\,. \quad (1.3)$$

Here $\phi_a(t,\boldsymbol{x})$ are field operators and the Kronecker delta in (1.1) has been replaced by a delta function since the momentum conjugates $\pi^a(t,\boldsymbol{x})$ are densities. The canonical commutation relations (1.3) together with the continuum version

$$\frac{d\phi_a(t,\boldsymbol{x})}{dt} = i\,[H,\phi_a(t,\boldsymbol{x})]\ , \qquad \frac{d\pi^a(t,\boldsymbol{x})}{dt} = i\,[H,\pi^a(t,\boldsymbol{x})]\ , \quad (1.4)$$

of the Hamilton's equations (1.2) provide the starting point for the canonical quantization of field theories. The Hamiltonian H, being a function of $\dot\phi_a$ and π^a, also becomes an operator in QFT. In order to solve the theory, one task is to find the spectrum, *i.e.*, the eigenvalues and eigenstates of H. This is usually very difficult, since there is an infinite number of degrees of freedom (dofs) within QFT, at least one for each point $\boldsymbol{x}$ in space. However, for certain theories, called *free theories*, one can find a way to write the dynamics such that each dof evolves independently from all the others. Free field theories typically have Lagrangians which are quadratic in the fields, so that the equations of motion (EOMs) are linear.

1.2 Quantization of Real Klein-Gordon Field

So far the discussion in this section was rather general. Let us be more specific and consider the simplest relativistic free theory as a practical example. It is provided by the classical real Klein-Gordon theory we have already discussed in length. Let us recall the main ingredients of this theory. The Lagrangian, the Hamiltonian, and the conjugate momentum for a free real scalar field ϕ with mass m are given by

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{1}{2}\,m^2\phi^2\,, \qquad \mathcal{H} = \frac{1}{2}\,\pi^2 + \frac{1}{2}\,(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\,m^2\phi^2\,, \qquad \pi = \dot\phi\,. \quad (1.5)$$

and the associated EOM is the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi = 0\,. \quad (1.6)$$

If we treat each Fourier mode of the field ϕ as an independent harmonic oscillator, we can apply canonical quantization to the real Klein-Gordon theory, and in this way find the spectrum of the corresponding Hamiltonian. In practice this can be done by starting with the

[2] A discussion of canonical quantization in the Schrödinger picture can be found in Chapter 3 of my script *Quantum Field Theory I* available at http://wwwthep.physik.uni-mainz.de/∼uhaisch/QFTI10/QFTI.pdf.

[3] This procedure is sometimes referred to as *second quantization*. We will not use this terminology here.

general classical solution we have already discussed, promoting the coefficients $a(p)$ and $a^*(p)$ to operators. In this way one obtains

$$\begin{aligned} \phi(t,\boldsymbol{x}) &= \int d^3\tilde{p}\, \left(a(p) e^{-ipx} + a^\dagger(p) e^{ipx} \right) , \\ \pi(t,\boldsymbol{x}) = \dot{\phi}(t,\boldsymbol{x}) &= -i \int d^3\tilde{p}\, \omega_{\boldsymbol{p}} \left(a(p) e^{-ipx} - a^\dagger(p) e^{ipx} \right) , \end{aligned} \tag{1.7}$$

where

$$d^3\tilde{p} = \frac{d^3p}{(2\pi)^3} \frac{1}{(2\omega_{\boldsymbol{p}})} , \qquad \omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2} , \qquad p_\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p}) . \tag{1.8}$$

In analogy to the annihilation and creation operators of a QM harmonic oscillator, the operators $a(p)$ and $a^\dagger(p)$ should satisfy canonical commutation relations. We make the ansatz

$$[a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0 , \qquad [a(p), a^\dagger(q)] = (2\pi)^3\, 2\omega_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) . \tag{1.9}$$

Recall that the combination $2\omega_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$ appearing in the second relation is Lorentz invariant and can thus be viewed as a covariant version of the three-dimensional delta function. Let us assume that the latter equations hold, it then follows that

$$\begin{aligned} [\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] &= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{-i}{4\omega_{\boldsymbol{p}}} \Big(-[a(p), a^\dagger(q)]\, e^{-i(px-qy)} + [a^\dagger(p), a(q)]\, e^{i(px-qy)} \Big) \\ &= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{-i}{4\omega_{\boldsymbol{p}}} (2\pi)^3\, 2\omega_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \Big(- e^{-i(px-qy)} - e^{i(px-qy)} \Big) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2} \Big(- e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \Big) = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) , \end{aligned} \tag{1.10}$$

where we have dropped terms $[a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0$ from the very beginning and in intermediate steps used the shorthand notations $x = (t, \boldsymbol{x})$ and $y = (t, \boldsymbol{y})$. The evaluation of the remaining two equal-time commutators $[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})]$ and $[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})]$ proceeds in a similar fashion and is left as an exercise. If done correctly, one finds in agreement with (1.3), the following expressions

$$[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})] = [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = 0 , \qquad [\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) . \tag{1.11}$$

Of course, it is also possible to derive the relations (1.9) starting from (1.11). To learn how this is done is part of a homework assignment.

In the Heisenberg picture the operators depend on time, so that we can study how they evolve when the clock starts ticking. For the field operator ϕ, we have

$$\begin{aligned} \dot{\phi}(t,\boldsymbol{x}) = i\, [H, \phi(t,\boldsymbol{x})] &= \frac{i}{2} \Big[\int d^3y\, \Big\{ \pi^2(t,\boldsymbol{y}) + \big(\boldsymbol{\nabla}\phi(t,\boldsymbol{y})\big)^2 + m^2\, \phi^2(t,\boldsymbol{y}) \Big\} , \phi(t,\boldsymbol{x}) \Big] \\ &= i \int d^3y\, \pi(t,\boldsymbol{y})\, (-i)\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) = \pi(t,\boldsymbol{x}) . \end{aligned} \tag{1.12}$$

Similarly, we get for the conjugate operator π,

$$\begin{aligned}
\dot{\pi}(t,\boldsymbol{x}) = i\,[H,\pi(t,\boldsymbol{x})] &= \frac{i}{2}\Big[\int d^3y\,\Big\{\pi^2(t,\boldsymbol{y}) + \big(\boldsymbol{\nabla}\phi(t,\boldsymbol{y})\big)^2 + m^2\,\phi^2(t,\boldsymbol{y})\Big\}\,,\pi(t,\boldsymbol{x})\Big] \\
&= \frac{i}{2}\int d^3y\,\Big\{\big(\boldsymbol{\nabla}_y\,[\phi(t,\boldsymbol{y}),\pi(t,\boldsymbol{x})]\big)\cdot\boldsymbol{\nabla}\phi(t,\boldsymbol{y}) + \big(\boldsymbol{\nabla}\phi(t,\boldsymbol{y})\big)\cdot\boldsymbol{\nabla}_y\,[\phi(t,\boldsymbol{y}),\pi(t,\boldsymbol{x})] \\
&\qquad\qquad + 2i\,m^2\,\phi(t,\boldsymbol{y})\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\Big\} \\
&= \big(\boldsymbol{\nabla}^2 - m^2\big)\,\phi(t,\boldsymbol{x})\,,
\end{aligned} \tag{1.13}$$

where we have included the subscript y on $\boldsymbol{\nabla}_y$ when there may be some confusion about which argument the derivative is acting on. To reach the last line, we have simply integrated by parts. Putting (1.12) and (1.13) together, we then find that ϕ indeed satisfies the Klein-Gordon equation (1.6). Of course, this is only a sanity check, since we constructed (1.7) precisely to fulfil this purpose.

In terms of the ladder operators $a(p)$ and $a^\dagger(p)$ the Hamiltonian of the real Klein-Gordon theory takes the form

$$\begin{aligned}
H &= \frac{1}{2}\int d^3x\,\big[\pi^2(t,\boldsymbol{x}) + (\boldsymbol{\nabla}\phi(t,\boldsymbol{x}))^2 + \,m^2\phi^2(t,\boldsymbol{x})\big] \\
&= \frac{1}{2}\int d^3x\,d^3\tilde{p}\,d^3\tilde{q}\,\big[-\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}\,\big(a(p)e^{-ipx} - a^\dagger(p)e^{ipx}\big)\big(a(q)e^{-iqx} - a^\dagger(q)e^{iqx}\big) \\
&\qquad\qquad - \boldsymbol{p}\cdot\boldsymbol{q}\,\big(a(p)e^{-ipx} - a^\dagger(p)e^{ipx}\big)\big(a(q)e^{-iqx} - a^\dagger(q)e^{iqx}\big) \\
&\qquad\qquad + m^2\,\big(a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\big)\big(a(q)e^{-iqx} + a^\dagger(q)e^{iqx}\big)\big] \\
&= \frac{1}{2}\int\frac{d^3\tilde{p}}{2\omega_{\boldsymbol{p}}}\,\Big[(-\omega_{\boldsymbol{p}}^2 + \boldsymbol{p}^2 + m^2)\,\big(a(p)a(-p) + a^\dagger(p)a^\dagger(-p)\big) \\
&\qquad\qquad + \big(\omega_{\boldsymbol{p}}^2 + \boldsymbol{p}^2 + m^2\big)\,\big(a(p)a^\dagger(p) + a^\dagger(p)a(p)\big)\Big]\,,
\end{aligned} \tag{1.14}$$

where we have first used the expressions for ϕ and π given in (1.7) and then integrated over d^3x to get delta functions $\delta^{(3)}(\boldsymbol{p}\pm\boldsymbol{q})$, which, in turn, allows us to perform the d^3q integral. Inserting finally the expression (1.8) for the frequency, the first term in (1.14) vanishes and we are left with

$$\begin{aligned}
H &= \frac{1}{2}\int d^3\tilde{p}\,\omega_{\boldsymbol{p}}\,\Big(a(p)a^\dagger(p) + a^\dagger(p)a(p)\Big) = \int d^3\tilde{p}\,\omega_{\boldsymbol{p}}\,\Big(a^\dagger(p)a(p) + \frac{1}{2}\,[a(p),a^\dagger(p)]\Big) \\
&= \int d^3\tilde{p}\,\omega_{\boldsymbol{p}}\,\Big(a^\dagger(p)a(p) + \frac{1}{2}\,(2\pi)^3\,2\omega_{\boldsymbol{p}}\,\delta^{(3)}(0)\Big)\,.
\end{aligned} \tag{1.15}$$

We see that the result contains a delta function, evaluated at zero where it has an infinite spike. This contribution arises from the infinite sum over all modes vibrating with the zero-point energy $\omega_{\boldsymbol{p}}/2$. Moreover, the integral over $\omega_{\boldsymbol{p}}$ diverges at large momenta $|\boldsymbol{p}|$. To better understand what is going on let us have a look at the ground state $|0\rangle$ where the former infinity first becomes apparent.

Structure of Vacuum

As in the case of the harmonic oscillator in QM, we define the vacuum state $|0\rangle$ through the condition that it is annihilated by the action of *all* $a(p)$,

$$a(p)\,|0\rangle = 0\,, \quad \forall\, p\,. \tag{1.16}$$

With this definition the energy E_0 of the vacuum comes entirely from the second term in the last line of (1.15),

$$H\,|0\rangle = E_0\,|0\rangle = \left(\int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_{\boldsymbol{p}}}{2}\,(2\pi)^3\,\delta^{(3)}(0)\right)|0\rangle = \infty\,|0\rangle\,. \tag{1.17}$$

In fact, the latter expression contains not only one but two infinities. The first arises because space is infinitely large. Infinities of this kind are often referred to as *infrared (IR) divergences*. In order to isolate this infinity, we put the theory into a box with sides of length L and impose periodic boundary conditions (BCs) on the field. Then, taking the limit $L \to \infty$, we arrive at

$$(2\pi)^3\,\delta^{(3)}(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x\;\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\big|_{\boldsymbol{p}=0} = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x = V\,, \tag{1.18}$$

where V denotes the volume of the box. This result tells us that the delta function singularity arises because we try to compute the total energy E_0 of the system rather than its energy density $\mathcal{E}_0$. The energy density is simply calculated from E_0 by dividing through the volume V. One finds

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_{\boldsymbol{p}}}{2}\,, \tag{1.19}$$

which is still divergent and resembles the sum of zero-point energies for each harmonic oscillator. Since $\mathcal{E}_0 \to \infty$ in the limit $|\boldsymbol{p}| \to \infty$, *i.e.*, high frequencies (or short distances), this singularity is an *ultraviolet (UV) divergence*. This divergence arises because we want too much. We have assumed that our theory is valid to arbitrarily short distance scales, corresponding to arbitrarily high energies. This assumption is clearly absurd. The integral should be cut off at high momentum, reflecting the fact that our theory presumably breaks down at some point.

Fortunately, the infinite energy shift in (1.15) is harmless if we want to measure the energy difference of the energy eigenstates from the vacuum. We can therefore "recalibrate" our energy levels (by an infinite constant) removing from the Hamiltonian operator the energy of the vacuum,

$$:H: \,= H - E_0 = H - \langle 0|H|0\rangle\,. \tag{1.20}$$

With this definition one has $:H:|0\rangle = 0$. In fact, the difference between the latter Hamiltonian and the previous one is merely an ordering ambiguity in moving from the classical theory to the quantum theory. This ordering ambiguity already arises in the case of the QM harmonic oscillator described by

$$H = \frac{1}{2}\,p^2 + \frac{1}{2}\,\omega^2 q^2\,. \tag{1.21}$$

Expressing q and p through the ladder operators a and $a^\dagger$ (with $[a, a^\dagger] = 1$) gives

$$q = \frac{1}{\sqrt{2\omega}} \left(a + a^\dagger\right) , \qquad p = -i\sqrt{\frac{\omega}{2}} \left(a - a^\dagger\right) , \tag{1.22}$$

and inserting this into (1.21) we end up with the familiar expression

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right) . \tag{1.23}$$

If, on the other hand, we would have defined our Hamiltonian to take the form

$$H = \frac{1}{2} \left(\omega q - ip\right) \left(\omega q + ip\right) , \tag{1.24}$$

which is classically the same as our original definition (1.21), then after quantization instead of (1.23), we would have gotten

$$H = \omega a^\dagger a . \tag{1.25}$$

This result obviously differs from (1.23) by the term $\omega/2$ which encodes the effects of the ground-state fluctuations.

This type of ordering ambiguity arises often in field theories. The method that we have used above to deal with it is called *normal ordering*. In practice, normal ordering works by placing all annihilation operators $a(p)$ in products of field operators to the right. Applied to the Hamiltonian of the real Klein-Gordon theory (1.15) this prescription leads to

$$:H: = \int d^3\tilde{p}\, \omega_{\boldsymbol{p}}\, a^\dagger(p) a(p) . \tag{1.26}$$

In the remainder of this section, we will normal order all operators in this manner (dropping the ": :" for simplicity).

Single-Particle States

After the discussion of the properties of the vacuum, we can now turn to the excitations of ϕ. It's easy to verify (and therefore left as an exercise) that, in full analogy to the case of the QM harmonic oscillator, the normal-ordered Hamiltonian (1.26) and the ladder operators of the real Klein-Gordon theory obey the following commutation relations

$$[H, a(p)] = -\omega_{\boldsymbol{p}}\, a(p) , \qquad [H, a^\dagger(p)] = \omega_{\boldsymbol{p}}\, a^\dagger(p) . \tag{1.27}$$

These relations imply that we can construct energy eigenstates by acting on the vacuum state $|0\rangle$ with $a^\dagger(p)$ (remember that they also imply that $a(p)\,|0\rangle = 0$, $\forall\, p$). We define

$$|p\rangle = a^\dagger(p)\,|0\rangle . \tag{1.28}$$

Using the commutation relations (1.9), one finds for the normalization of these states

$$\langle p|q\rangle = \langle 0|a(p) a^\dagger(q)|0\rangle = \langle 0|[a(p), a^\dagger(q)]|0\rangle = (2\pi)^3\, 2\omega_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) , \tag{1.29}$$

where we again meet the covariant version of the three-dimensional delta function.

The state (1.28) itself has energy

$$H\,|p\rangle = E_{\boldsymbol{p}}\,|p\rangle = \omega_{\boldsymbol{p}}\,|p\rangle\,, \tag{1.30}$$

with $\omega_{\boldsymbol{p}}$ given in (1.8), which is nothing but the *relativistic energy* of a particle with 3-momentum $\boldsymbol{p}$ and mass m. We thus interpret the state $|p\rangle$ as the *momentum eigenstate of a single particle* of mass m.

Let us check this interpretation by studying the other quantum numbers of $|p\rangle$. We begin with the *total momentum* $\boldsymbol{P}$. Turning this expression into an operator, we arrive, after normal ordering, at

$$\boldsymbol{P} = -\int d^3x\,\pi\,\boldsymbol{\nabla}\phi = \int d^3\tilde{p}\;\boldsymbol{p}\,a^\dagger(p)a(p)\,. \tag{1.31}$$

Acting with $\boldsymbol{P}$ on our state $|p\rangle$ gives

$$\begin{aligned}\boldsymbol{P}\,|p\rangle &= \int d^3\tilde{q}\;\boldsymbol{q}\,a^\dagger(q)a(q)a^\dagger(p)\,|0\rangle \\ &= \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{2w_{\boldsymbol{q}}}\;\boldsymbol{q}\,a^\dagger(q)\,\Big[(2\pi)^3\,2\omega_{\boldsymbol{p}}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) + a^\dagger(p)a(q)\Big]\,|0\rangle = \boldsymbol{p}\,|p\rangle\,,\end{aligned} \tag{1.32}$$

where we have employed the second line in (1.9) and used the fact that an annihilation operator acting on the vacuum is zero. The latter result tells us that the state $|p\rangle$ has momentum $\boldsymbol{p}$. Another property of $|p\rangle$ that we can study is its angular momentum. Again we take the classical expression for the total angular momentum

$$J^i = \epsilon^{ijk}\int d^3x\,\left(x^jT^{0k} - x^kT^{0j}\right)\,, \tag{1.33}$$

and turn it into an field operator. It is a good exercise to show that by acting with J^i on the one-particle state with zero 3-momentum, one gets

$$J^i\,|p = (m,\boldsymbol{0})\rangle = 0\,. \tag{1.34}$$

This result tells us that the particle carries no internal angular momentum. In other words, quantizing the real Klein-Gordon field gives rise to a spin-zero particle *aka* a scalar.

Multi-Particle States

Acting multiple times with the creation operators on the vacuum we can create multi-particle states. We interpret the state

$$|p_1, ..., p_n\rangle = a^\dagger(p_1)\dots a^\dagger(p_n)\,|0\rangle\,, \tag{1.35}$$

as an n-particle state. Since one has $[a^\dagger(p_i), a^\dagger(p_j)] = 0$, the state (1.35) is symmetric under exchange of any two particles. *E.g.*,

$$|p, q\rangle = a^\dagger(p)a^\dagger(q)\,|0\rangle = a^\dagger(q)a^\dagger(p)\,|0\rangle = |q, p\rangle\,. \tag{1.36}$$

This means that the particles corresponding to the real Klein-Gordon theory are *bosons*. We see that the relationship between spin and statistics is, unlike in QM, a consequence of the QFT framework, following, in the case at hand, from the commutation quantization conditions for boson fields (1.3).

The full Hilbert space of our theory is spanned by acting on the vacuum with all possible combinations of creation operators,

$$|0\rangle\,, \quad a^\dagger(p)\,|0\rangle\,, \quad a^\dagger(p)a^\dagger(q)\,|0\rangle\,, \quad a^\dagger(p)a^\dagger(q)a^\dagger(r)\,|0\rangle\,, \quad \ldots\,. \tag{1.37}$$

This space is known as the *Fock space* and is simply the sum of the n-particle Hilbert spaces, for all $n \geq 0$. Like in QM, there is also an operator which counts the number n of particles in a given state in the Fock space. It is the *number operator*

$$N = \int d^3\tilde{p}\; a^\dagger(p)a(p)\,, \tag{1.38}$$

which satisfies

$$\begin{aligned}[N, a^\dagger(q)] &= \int d^3\tilde{p}\,\left[a^\dagger(p)a(p), a^\dagger(q)\right] = \int d^3\tilde{p}\; a^\dagger(p)\,\left[a(p), a^\dagger(q)\right] \\ &= \int d^3\tilde{p}\; a^\dagger(p)\,(2\pi)^3\,2\omega_{\boldsymbol{p}}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) = a^\dagger(q)\,.\end{aligned} \tag{1.39}$$

On a n-particle state, the number operator N acts like

$$\begin{aligned}N\,|p_1, ..., p_n\rangle &= N a^\dagger(p_1)\ldots a^\dagger(p_n)\,|0\rangle = \left\{a^\dagger(p_1)N + \left[N, a^\dagger(p_1)\right]\right\} a^\dagger(p_2)\ldots a^\dagger(p_n)\,|0\rangle \\ &= a^\dagger(p_1)Na^\dagger(p_2)\ldots a^\dagger(p_n)\,|0\rangle + |p_1, ..., p_n\rangle\,.\end{aligned} \tag{1.40}$$

We can repeat this procedure and commute N with all creation operators, picking up at each step the term $|p_1, ..., p_n\rangle$. In the last step, we then use $N|0\rangle = 0$ and find in this way

$$N\,|p_1, \ldots, p_n\rangle = n\,|p_1, \ldots, p_n\rangle\,. \tag{1.41}$$

Hence, N indeed counts the number of particles in a state.

It is also important to realize that the number operator commutes with the Hamiltonian,

$$\begin{aligned}[N, H] &= \int d^3\tilde{p}\, d^3\tilde{q}\;\omega_{\boldsymbol{q}}\left[a^\dagger(p)a(p), a^\dagger(q)a(q)\right] \\ &= \int d^3\tilde{p}\, d^3\tilde{q}\;\omega_{\boldsymbol{q}}\left\{a^\dagger(p)\left[a(p), a^\dagger(q)a(q)\right] + \left[a^\dagger(p), a^\dagger(q)a(q)\right]a(p)\right\} \\ &= \int d^3\tilde{p}\, d^3\tilde{q}\;\omega_{\boldsymbol{q}}\left\{a^\dagger(p)a^\dagger(q)\left[a(p), a(q)\right] + a^\dagger(p)\left[a(p), a^\dagger(q)\right]a(q)\right. \\ &\qquad \left. + a^\dagger(q)\left[a^\dagger(p), a(q)\right]a(p) + \left[a^\dagger(p), a^\dagger(q)\right]a(q)a(p)\right\} = 0\,,\end{aligned} \tag{1.42}$$

ensuring that particle number is conserved. This means that we can place ourselves in the n-particle sector, and will remain there. This is a property of free theories, but will no longer be true when we consider interactions. Interactions create and destroy particles, taking us between the different sectors in the Fock space.

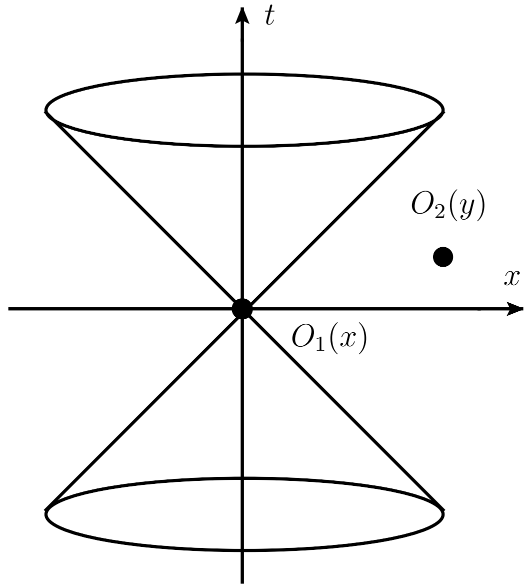

Figure 1.1: Picture of space-like separated operators $O_1(x)$ and $O_2(y)$.

Causality

The last thing we should check is that our scalar theory does not violate *causality* after quantization. For our QFT to be causal, we must require that all space-like separated operators O_1 and O_2 commute,

$$[O_1(x), O_2(y)] = 0\,, \quad \forall \;\; (x-y)^2 < 0\,. \tag{1.43}$$

This ensures that a measurement at $x = (x_0, \boldsymbol{x})$ cannot affect a measurement at $y = (y_0, \boldsymbol{y})$, when x and y are not causally connected (outside the light-cone). A graphical representation of the latter equation is given in Figure 1.1.

Does our theory satisfy the requirement (1.43)? In order to answer this question, we define the object

$$\Delta(x-y) = [\phi(x), \phi(y)]\,. \tag{1.44}$$

While the fields on the right-hand side are operators, it is seen (after a short calculation) that the left-hand side is simply a complex number,

$$\begin{aligned}
\Delta(x-y) &= \Big[\int d^3\tilde{p}\, \big(a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\big)\,, \int d^3\tilde{q}\, \big(a(q)e^{-iqy} + a^\dagger(q)e^{iqy}\big) \Big] \\
&= \int d^3\tilde{p}\, d^3\tilde{q}\, \Big([a(p), a^\dagger(q)]\, e^{-i(px-qy)} + [a^\dagger(p), a(q)]\, e^{i(px-qy)} \Big) \\
&= \int d^3\tilde{p}\, d^3\tilde{q}\, (2\pi)^3\, 2\omega_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\, \left[e^{-i(px-qy)} - e^{i(px-qy)} \right] \\
&= \int d^3\tilde{p}\, \left[e^{-ip\,(x-y)} - e^{ip\,(x-y)} \right]\,.
\end{aligned} \tag{1.45}$$

In order to further simplify this result, we note that

$$\delta(p^2 - m^2) = \frac{1}{2\omega_{\boldsymbol{p}}} \Big[\delta(p_0 + \omega_{\boldsymbol{p}}) + \delta(p_0 - \omega_{\boldsymbol{p}})\Big] . \tag{1.46}$$

It then follows that

$$\begin{aligned} \Delta(x-y) &= \int dp_0 \, \frac{d^3p}{(2\pi)^3} \, \frac{1}{2\omega_{\boldsymbol{p}}} \Big[\delta(p_0 + \omega_{\boldsymbol{p}}) + \delta(p_0 - \omega_{\boldsymbol{p}})\Big] \, \frac{p_0}{|p_0|} \, e^{-ip\,(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^3} \, \delta(p^2 - m^2) \, \frac{p_0}{|p_0|} \, e^{-ip\,(x-y)} . \end{aligned} \tag{1.47}$$

All elements in the final integral are Lorentz invariant apart from the term $p_0/|p_0|$. However, if we restrict ourselves to Lorentz transformations which preserve the sign of the time-component of four vectors, *i.e.*, orthochronous Lorentz transformations $\Lambda \in L^{\uparrow}$, then $p_0/|p_0|$ remains unchanged and we have that $\Delta(x-y)$ is Lorentz invariant.

But we also have that $\Delta(x-y) = 0$ at equal times for all $(x-y)^2 = -(\boldsymbol{x}-\boldsymbol{y})^2 < 0$, which can be seen explicitly by writing

$$[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = \int \frac{d^3p}{(2\pi)^3} \, \frac{1}{2\omega_{\boldsymbol{p}}} \left[e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right] = 0 . \tag{1.48}$$

Notice that in order to arrive at the final result, we have flipped the sign of $\boldsymbol{p}$ in the second exponent. This obviously does not change the result since $\boldsymbol{p}$ is an integration variable and $\omega_{\boldsymbol{p}} = (\boldsymbol{p}^2 + m^2)^{1/2}$ is invariant under such a change. But since $\Delta(x-y)$ is Lorentz invariant,[4] it can only be a function of $(x-y)^2$ and must hence vanish for all $(x-y)^2 < 0$.

The above findings imply that the real Klein-Gordon theory is indeed causal with commutators vanishing outside the light-cone. This property will continue to hold in the interacting theory. Indeed, it is usually given as one of the *axioms* of local QFTs. Let me mention, however, that the fact that $[\phi(x), \phi(y)]$ is a complex function, rather than an operator, is a property of free fields only and does not hold in an interacting theory.

1.3 Quantization of Complex Klein-Gordon Field

We have already discussed the classical complex scalar field theory with a global $U(1)$ symmetry. The goal is now to quantize the free version of this theory following the same steps as for the free real scalar. We again begin by recalling the main ingredients of the classic theory. In terms of the field $\varphi = 1/\sqrt{2}\,(\phi_1 + i\phi_2)$, the Lagrangian, Hamiltonian, and the conjugate momentum are given by

$$\mathcal{L} = (\partial_\mu \varphi^*)\,(\partial^\mu \varphi) - m^2 \varphi^* \varphi \,, \qquad \mathcal{H} = \pi^* \pi + (\boldsymbol{\nabla}\varphi^*) \cdot (\boldsymbol{\nabla}\varphi) + m^2 \varphi^* \varphi \,, \qquad \pi = \dot{\varphi}^* \,, \tag{1.49}$$

while the EOM for φ reads

$$\left(\Box + m^2\right) \varphi = 0 . \tag{1.50}$$

[4]Because one can always find a orthochronous Lorentz transformation such that $(\Lambda^\mu{}_\nu (x-y)^\nu)_0 = 0$ if x and y are space-like separated, the invariance under $\Lambda \in L^{\uparrow}$ is enough to show that $[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = 0$ outside the light-cone implies $\Delta(x-y) = 0$ for all $(x-y)^2 < 0$.

After replacing the classic fields by operators ($\varphi \to \varphi$, $\varphi^* \to \varphi^\dagger$, *etc.*), canonical quantization of this system can be performed by simply imposing the general commutation relations (1.3) on the real fields ϕ_1 and ϕ_2 and conjugate momenta $\pi_1 = \dot{\phi}_1$ and $\pi_2 = \dot{\phi}_2$. Since both φ and π are linear combinations of the real components with the same coefficients, it is not difficult to understand that the only non-trivial commutators take the form

$$[\varphi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = [\varphi^\dagger(t,\boldsymbol{x}), \pi^\dagger(t,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \,. \tag{1.51}$$

The operator versions of the solutions to the complex Klein-Gordon equation (1.50) and its hermitian conjugate read

$$\begin{aligned} \varphi(t,\boldsymbol{x}) &= \int d^3\tilde{p}\, \left(a_+(p)e^{-ipx} + a_-^\dagger(p)e^{ipx}\right) \,, \\ \varphi^\dagger(t,\boldsymbol{x}) &= \int d^3\tilde{p}\, \left(a_-(p)e^{-ipx} + a_+^\dagger(p)e^{ipx}\right) \,. \end{aligned} \tag{1.52}$$

As in the case of the real scalar field, one can invert these relations and compute the commutators of $a_\pm(p)$ and $a_\pm^\dagger(p)$. The only non-zero commutators one finds in this way are

$$[a_\pm(p), a_\pm^\dagger(q)] = (2\pi)^3\, 2\omega_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \,. \tag{1.53}$$

The latter relations imply that there are two sets of annihilation and creation operators, namely $a_+(p), a_+^\dagger(p)$ and $a_-(p), a_-^\dagger(p)$. In consequence, we will also have two types of states,

$$|p,+\rangle = a_+^\dagger(p)|0\rangle \,, \qquad |p,-\rangle = a_-^\dagger(p)|0\rangle \,, \tag{1.54}$$

where the vacuum state $|0\rangle$ is defined in the usual way $a_\pm(p)|0\rangle = 0$ for all p and normalized such that $\langle 0|0\rangle = 1$. Multi-particle states are then given by

$$|p_1,\varepsilon_1;\ldots;p_n,\varepsilon_n\rangle = a_{\varepsilon_1}^\dagger(p_1)\ldots a_{\varepsilon_n}^\dagger(p_n)|0\rangle \,, \tag{1.55}$$

and labelled both by the momenta p_i and $\varepsilon_i = \pm 1$ to distinguish the two types of quanta. For each type of excitations, we can also introduce a number operator

$$N_\pm = \int d^3\tilde{p}\; a_\pm^\dagger(p)a_\pm(p) \,, \tag{1.56}$$

which, in analogy to (1.39), satisfies

$$[N_\pm, a_\pm^\dagger(q)] = a_\pm^\dagger(q) \,, \qquad [N_\pm, a_\mp^\dagger(q)] = 0 \,. \tag{1.57}$$

It follows that N_+ (N_-) acting on a multi-particle state counts the number of quanta of $+$ type ($-$ type). The (normal-ordered) conserved 4-momentum can be computed as for the real scalar field. One obtains

$$P_\mu = \int d^3\tilde{p}\; p_\mu \left(a_+^\dagger(p)a_+(p) + a_-^\dagger(p)a_-(p)\right) \,, \tag{1.58}$$

where as usual $p_0 = \omega_{\boldsymbol{p}}$. It is straightforward to show that

$$[P_\mu, a_\pm^\dagger(q)] = q_\mu a_\pm^\dagger(q)\,, \tag{1.59}$$

which tells us that the state $|q, \varepsilon\rangle$ can be interpreted as a single particle with 4-momentum q_μ, irrespectively of its type ε. Furthermore, it is easy to see that the n-particle state $|p_1, \varepsilon_1; \ldots; p_n, \varepsilon_n\rangle$ has total momentum $\sum_{i=1}^n (p_i)_\mu$.

So far, our discussion has been in complete analogy with the one for the real scalar field. However, the free complex scalar theory has one additional feature, namely the conserved $U(1)$ current. The operator version of the associated charge can be written as

$$Q = i \int d^3x \, \left(\varphi^\dagger \pi^\dagger - \pi \varphi\right) \,. \tag{1.60}$$

Using (1.52) one can express this charge in terms of annihlation and creation operators. I spare you the details of this computation and simply quote the final result after normal ordering. One finds

$$Q = \int d^3\tilde{p} \, \left(a_+^\dagger(p) a_+(p) - a_-^\dagger(p) a_-(p)\right) = N_+ - N_- \,. \tag{1.61}$$

This result shows that the states $|p, +\rangle$ have charge $+1$ and the states $|p, -\rangle$ have charge -1. It is hence sensible to identify the $+$ states with particles and $-$ states with anti-particles. With this terminology, we see from (1.52) that φ annihilates a particle or creates an anti-particle while $\varphi^\dagger$ creates a particle or annihilates an anti-particle. It is also important to notice that the charge Q is a conserved quantity in our quantum theory, since $[Q, H] = 0$. Of course, in a free field theory this isn't such a big deal because both N_+ and N_-, *i.e.*, the numbers of positively and negatively charged states, are separately conserved (*i.e.*, $[N_\pm, H] = 0$). However, we will see soon that in the interacting version of our complex Klein-Gordon theory, Q survives as a conserved quantity, while $N_\pm$ individually do not.

Is the quantum theory of our complex scalar causal? Of course it is (and is not too difficult to check this explicitly)! In fact, a neat way to think about this is in terms of amplitudes of particles and anti-particles. If we consider the commutator $[\varphi(x), \varphi^\dagger(y)] = 0$ outside the light-cone, the physical interpretation of our finding that (1.44) (or better its analog in the complex case) vanishes for space-like separated points, is that the amplitude for the particle to propagate from x to y cancels the amplitude for the anti-particle to travel from y and x. This interpretation also applies (maybe in a less obvious way) to the case of the real scalar field, because the particle is then its own anti-particle.

1.4 Klein-Gordon Correlators

Just as in the case of QM, we can now also determine correlation functions (or simply correlators) which are also of primary importance in QFT. Maybe the most famous correlator in QFT is the so-called *Feynman propagator*. In the case of our real scalar field ϕ, it is defined as follows

$$\begin{aligned} D_F\,(x-y) &= \langle 0\,| T\,(\phi(x)\phi(y))|\,0\rangle \\ &= \theta(x_0 - y_0)\,\langle 0\,|\phi(x)\phi(y)|\,0\rangle + \theta(y_0 - x_0)\,\langle 0\,|\phi(y)\phi(x)|\,0\rangle \,. \end{aligned} \tag{1.62}$$

Here θ with $\theta(x) = 0$ for $x < 0$, $\theta(x) = 1/2$ for $x = 0$,[5] and $\theta(x) = 1$ for $x > 0$ is the Heaviside step function. Furthermore, T stands for *time ordering*, *i.e.*, placing all operators evaluated at later times to the left,

$$T(\phi(x_1)\dots\phi(x_n)) = \phi(t_{i_n}, \boldsymbol{x}_{i_n})\dots\phi(t_{i_1}, \boldsymbol{x}_{i_1}), \qquad t_{i_1} \leq \dots \leq t_{i_n}. \tag{1.63}$$

In order to compute the Feynman propagator we first consider the case $x_0 > y_0$. Inserting the field expansion (1.52), we have

$$\begin{aligned} D_F(x-y) &= \langle 0\,|\phi(x)\phi(y)|\,0\rangle \\ &= \langle 0|\int d^3\tilde{p}\,d^3\tilde{q}\left(a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right)\left(a(q)e^{-iqy} + a^\dagger(q)e^{iqy}\right)|0\rangle \\ &= \langle 0|\int d^3\tilde{p}\,d^3\tilde{q}\,\left[a(p), a^\dagger(q)\right]e^{-i(px-qy)}|0\rangle = \int d^3\tilde{p}\,e^{-ip(x-y)} = D(x-y)\,. \end{aligned} \tag{1.64}$$

Analogously, one finds for the case $y_0 > x_0$, the relation $D_F(x-y) = D(y-x)$. Combining these latter two results, the Feynman propagator can be written as

$$\begin{aligned} D_F(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(\theta(x_0-y_0)e^{-i\omega_{\boldsymbol{p}}(x_0-y_0)} + \theta(y_0-x_0)e^{i\omega_{\boldsymbol{p}}(x_0-y_0)}\right)e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{(p_0-\omega_{\boldsymbol{p}}+i\epsilon)(p_0+\omega_{\boldsymbol{p}}-i\epsilon)}\,e^{-ip(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip(x-y)}\,, \end{aligned} \tag{1.65}$$

where the terms $i\epsilon$ (with $\epsilon > 0$ and infinitesimal) have the effect of shifting the poles in the denominator

$$p^2 - m^2 = (p_0)^2 - \boldsymbol{p}^2 - m^2 = (p_0)^2 - \omega_{\boldsymbol{p}}^2 = (p_0 - \omega_{\boldsymbol{p}})(p_0 + \omega_{\boldsymbol{p}})\,, \tag{1.66}$$

slightly off the real p_0-axis. This way of writing the Feynman propagator is, for obvious reasons, hence called the *"$i\epsilon$" prescription.* The equality of the first and second line in (1.65) follows by an appropriate contour integration (using Cauchy's residue theorem). Again we distinguish the cases $x_0 > y_0$ and $y_0 > x_0$. In the former case, we perform the p_0 integration following the contour shown in Figure 1.2 which encloses the pole at $p_0 = +\omega_{\boldsymbol{p}} - i\epsilon$ with residuum $-2\pi i/(2\omega_{\boldsymbol{p}})$, where the minus sign arises since the path has a clockwise orientation. In the latter case, the path is closed in the upper-half plane, surrounding the pole $p_0 = -\omega_{\boldsymbol{p}} + i\epsilon$ with residuum $+2\pi i/(-2\omega_{\boldsymbol{p}})$ (due to the counter-clockwise orientation of the half-circle the residuum does not pick up a minus sign now). This proofs the correctness of (1.65).

[5]It seldom matters what particular value is chosen for $\theta(0)$, since the Heaviside function usually appears in integrations, and the value of a function at a single point does not affect its integral. We employ the half-maximum convention here.

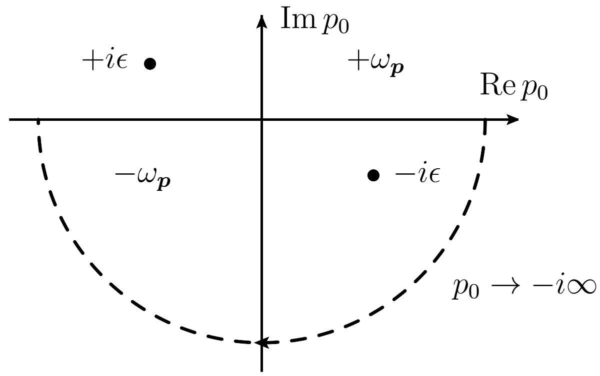

Figure 1.2: Schematic picture of the "$i\epsilon$" prescription for $x_0 > y_0$. In the case $y_0 > x_0$, the integration contour is closed in the upper-half plane.

An important property of the Feynman propagator is that it solves the Klein-Gordon equation,

$$\begin{aligned}\left(\Box_x + m^2\right) D_F\,(x-y) &= \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\epsilon}\,\left(-p^2 + m^2\right) e^{-ip(x-y)} \\ &= -i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} = -i\delta^{(4)}(x-y)\,.\end{aligned} \tag{1.67}$$

with a delta-function source on the right-hand side. Functions with this property are called *Green's functions.* There are other Green's functions of the Klein-Gordon equation which are given by an integral such as in (1.65) but with the poles in a different position relative to the real p_0-axis. *E.g.*, the case where both poles are above the real p_0-axis leads to the so-called *advanced Green's function.* It vanishes for $x_0 > y_0$ since the contour in Figure 1.2 contains no poles in this case. Similarly, one can define an *retarded Green's function* which vanishes for $y_0 > x_0$. You will get more familiar with Green's functions in an exercise.

Let us briefly also discuss the simplest time-ordered product for a free complex scalar field $\varphi = 1/\sqrt{2}\,(\phi_1 + i\phi_2)$. In terms of the real fields ϕ_1 and ϕ_2 the Lagrangian in (1.49) splits into a sum of Lagrangians for two free real scalars with the same mass m. In consequence, each of ϕ_1 and ϕ_2 has an oscillator expansion as in (1.7) and it immediately follows that $\langle 0|T(\phi_1(x)\phi_1(y))|0\rangle = \langle 0|T(\phi_2(x)\phi_2(y))|0\rangle = D_F(x-y)$ while $\langle 0|T(\phi_1(x)\phi_2(y))|0\rangle = 0$. It is then easy to see that this implies for the complex scalar that

$$\langle 0|T(\varphi(x)\varphi^\dagger(y))|0\rangle = D_F(x-y)\,, \qquad \langle 0|T(\varphi(x)\varphi(y))|0\rangle = \langle 0|T(\varphi^\dagger(x)\varphi^\dagger(y))|0\rangle = 0\,. \tag{1.68}$$

Enough words spent on correlators involving scalar field operators. Let's move on.

1.5 Quantization of Massless Vector Field

After having understood how to quantize scalar field theories, let us turn to a more complicated example, *i.e.*, the free massless vector field. Like before we start by recalling the main ingredients of the classic theory. The Lagrangian for a vector field A_μ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{1.69}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field-strength tensor. This Lagrangian is invariant under the gauge transformations

$$A_\mu \to A_\mu + \partial_\mu f \,. \tag{1.70}$$

The canonical momenta π^μ are given by

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} = -F^{0\mu} \,. \tag{1.71}$$

Finally, the EOM for A_μ reads

$$\Box A_\mu - \partial_\mu \left(\partial_\nu A^\nu\right) = 0 \,. \tag{1.72}$$

In order to quantize the theory we now think of the field A_μ as a collection of four scalar fields, that happen to be labelled by the space-time index μ, and naively impose canonical commutation relations of the form (for brevity we only write the non-trivial commutators)

$$[A_\mu(t, \boldsymbol{x}), \pi^\nu(t, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \delta_\mu{}^\nu \,. \tag{1.73}$$

These relations, in particular, imply that $[A_0(t, \boldsymbol{x}), \pi^0(t, \boldsymbol{y})]$ is non-vanishing. Unfortunately, this result is inconsistent with (1.76), which tells us that $\pi^0 = 0$. We see that viewing A_μ as a collection of four scalar fields is too simple. In fact, while typical kinetic terms for four scalar fields A_μ would be of the form $\sum_\nu (\partial_\mu A_\nu)(\partial^\mu A^\nu)$ and, hence, depend on both the symmetric and antisymmetric parts of $\partial_\mu A_\nu$, the Lagrangian (1.69) involves only the antisymmetric part, *i.e.*, the field-strength tensor $F_{\mu\nu}$. With this in mind it is then not difficult to understand that the gauge symmetry (1.70) of the vector field is in essence the crucial difference to the scalar field theory. There are various viable methods to quantize a gauge theory, but here we will follow the most obvious approach of fixing a gauge before quantization.

Gauge Fixing

Since we would like to preserve covariance we use the Lorenz gauge condition

$$\partial_\mu A^\mu = 0 \,, \tag{1.74}$$

which we impose on the theory by means of a Lagrange multiplier ξ.[6] Our starting point to quantize our massless vector field theory is thus not (1.69), but

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\xi}{2} \left(\partial_\mu A^\mu\right)^2 \,. \tag{1.75}$$

It follows that the expression for the canonical momenta (1.76) receives an additional term, namely

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \xi \eta^{\mu 0} \partial_\nu A^\nu \,. \tag{1.76}$$

[6]The gauges defined by the possible values of ξ are known as *renormalizability*, or simply R_ξ, gauges.

This means that π^0 is now no longer zero, but given by $\pi^0 = -\xi \partial_\nu A^\nu$. Similarly, the EOM (1.77) is modified to

$$\Box A_\mu - (1-\xi)\,\partial_\mu\,(\partial_\nu A^\nu) = 0\,. \tag{1.77}$$

On the solution of this new EOM we should then impose the gauge condition (1.74), which formally arises from (1.75) as the EOM for ξ, if the gauge parameter is treated as a independent dof (*i.e.*, $\partial\mathcal{L}/\partial\xi = 0$). Yet we should not impose this condition as an operator equation since this would lead us back to a situation where $\pi^0 = 0$. Instead, the Lorenz gauge condition (1.74) will later be imposed as a condition that defines physical states. After the gauge fixing, the obstruction to imposing canonical quantisation conditions has been removed and we require the canonical commutation relations (1.73) for A_μ and the conjugate momenta (1.76).

Oscillator Expansion

In order to work out the properties of the creation and annihilation operators, we now have to solve (1.77). This can be done for any ξ, but to simplify matters we will adopt the so-called *Feynman gauge* in what follows. This special gauge corresponds to the choice $\xi = 1$, which in turn leads to the modified EOM

$$\Box A_\mu = 0\,. \tag{1.78}$$

We have already discussed the general solution to this equation in the classic case. The quantum version is obtained by replacing the coefficients $a^{(\alpha)}(p)$ and $a^{(\alpha)*}(p)$, entering the classic oscillator expansion, simply by operators. One obtains

$$A_\mu(x) = \sum_{\alpha=0}^{3} \int d^3\tilde{p}\; \epsilon_\mu^{(\alpha)}(p)\left[a^{(\alpha)}(p)e^{-ipx} + a^{(\alpha)\dagger}(p)e^{ipx}\right]\,, \tag{1.79}$$

where the polarization vectors $\epsilon_\mu^{(\alpha)}(p)$ are defined as in my script "Elements of Classic Field Theory" (see (1.124) to (1.133)) and $p_\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$ with $\omega_{\boldsymbol{p}} = |\boldsymbol{p}|$. Recall that classically $\epsilon_\mu^{(\alpha)}(p)$ with $\alpha = 1, 2$ are the two transversal, physical polarizations while the other two polarizations can be gauged away, meaning that they are unphysical dofs.

Inserting (1.79) into (1.76) gives us the expansion for the conjugate momenta. Like in the case of the scalar field, we can now invert these two relations to express $a^{(\alpha)}(p)$ and $a^{(\alpha)\dagger}(p)$ through A_μ and π^μ, which then allows to determine the commutation relations of the annihilation and creation operators. After a straightforward (but somewhat lengthy) calculation, one arrives at

$$\begin{aligned} [a^{(\alpha)}(p), a^{(\beta)}(q)] &= [a^{(\alpha)\dagger}(p), a^{(\beta)\dagger}(q)] = 0\,, \\ [a^{(\alpha)}(p), a^{(\beta)\dagger}(q)] &= -\eta^{\alpha\beta}\,(2\pi)^3\,2\omega_{\boldsymbol{p}}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\,. \end{aligned} \tag{1.80}$$

These relations look pretty much like one would have naively expected from glancing at (1.9). Only the overall minus sign in the second expression looks kind of weird (and potentially troublesome). In order to understand its meaning we have to investigate the structure of the Fock space of our theory.

Fock Space

Naively, the Fock space of our quantum gauge field theory is spanned by states created from the vacuum $|0\rangle$ by acting with any combination and any number of creation operators $a^{(\alpha)\dagger}(p)$, where $\alpha = 0, 1, 2, 3$. Let us call this space $\mathcal{F}_0$. By definition, $\mathcal{F}_0$ contains all the states $|p, 0\rangle = a^{(0)\dagger}(p)|0\rangle$. Using (1.80) it is readily shown that these states satisfy

$$\langle p, 0|p, 0\rangle = \langle 0|a^{(0)}(p)a^{(0)\dagger}(p)|0\rangle = \langle 0|[a^{(0)}(p), a^{(0)\dagger}(p)]|0\rangle < 0\,, \tag{1.81}$$

which tells us that the states $|p, 0\rangle$ have negative norm. Such a property is physically unacceptable since it compromise the probabilistic interpretation of QM and signals that the space $\mathcal{F}_0$ contains unphysical states and therefore cannot be the proper Fock space of the theory. A related problem emerges when looking at the conserved 4-momentum P_μ. Proceeding in the usual way (including normal ordering), one finds after some work that

$$P_\mu = \int d^3\tilde{p}\; p_\mu \left[\sum_{\alpha=1}^{3} a^{(\alpha)^\dagger}(p)a^{(\alpha)}(p) - a^{(0)^\dagger}(p)a^{(0)}(p)\right]. \tag{1.82}$$

We see that the Hamiltonian counts states with spatial polarizations and subtracts the number of states with time-like polarizations. It is therefore indefinite.

In fact, it is not a big surprise that we have unphysical dofs in the spectrum, because we have not used the Lorenz gauge condition so far. So let us do this now. Bearing in mind that requiring the operator equation $\partial_\mu A^\mu = 0$ is a too strong condition and leads to problems with quantization, we define a subspace $\mathcal{F}_1 \subset \mathcal{F}_0$ of physical states such that

$$\langle\tilde{\Phi}|\partial_\mu A^\mu|\Phi\rangle = 0\,, \tag{1.83}$$

for all $|\Phi\rangle$, $|\tilde{\Phi}\rangle \in \mathcal{F}_1$. In order to guarantee that this condition is satisfied it is sufficient that the annihilation part of the operator $\partial_\mu A^\mu$ acting on $|\Phi\rangle$ gives zero. We calculate

$$\partial_\mu A^\mu = -i\sum_{\alpha=0}^{3}\int d^3\tilde{p}\;\epsilon^{(\alpha)}(p)\cdot p\left[a^{(\alpha)}(p)e^{-ipx} - a^{(\alpha)\dagger}(p)e^{ipx}\right]. \tag{1.84}$$

Using now that

$$\epsilon^{(1)}\cdot p = \epsilon^{(2)}\cdot p = 0\,, \qquad \epsilon^{(0)}\cdot p = -\epsilon^{(3)}\cdot p = p_0\,, \tag{1.85}$$

one finds that the part of $\partial_\mu A^\mu$ involving annihilation operators is proportional to the following expression

$$\int d^3\tilde{p}\; p_0\left[a^{(0)}(p) - a^{(3)}(p)\right]e^{-ipx}\,. \tag{1.86}$$

This implies that the physical states $|\Phi\rangle \in \mathcal{F}_1$ can be defined by the condition

$$b_-(p)|\Phi\rangle = 0\,, \tag{1.87}$$

where we have introduced the new annihilators

$$b_\mp(p) = \frac{1}{\sqrt{2}}\left(a^{(0)}(p) \mp a^{(3)}(p)\right)\,, \tag{1.88}$$

Notice that $b_\mp(p)$ are linear combinations of the original non-transversal operators. Whether one uses $b_\mp(p)$ or $a^{(0)}(p)$ and $a^{(3)}(p)$ is hence just a choice of basis, which obviously leaves physics unaltered.

It is now important to realize that transversal states, *i.e.*, states created from the vacuum by acting only with transversal creation operators $a^{(\alpha)\dagger}(p)$, where $\alpha = 1, 2$, satisfy the condition (1.87), since

$$b_-(q)a^{(1)\dagger}(p)|0\rangle = a^{(1)\dagger}(p)b_-(q)|0\rangle = 0\,, \tag{1.89}$$

and analog for $a^{(2)\dagger}(p)$. Transversal states are hence elements of $\mathcal{F}_1$. If these states where the only ones in $\mathcal{F}_1$ things would be simple, but unfortunately this is not the whole story. In order to analyze the condition (1.87) for non-transversal states we first note that

$$[b_-(p), b_-^\dagger(q)] = \frac{1}{2}\left([a^{(0)}(p), a^{(0)\dagger}(q)] + [a^{(3)}(p), a^{(3)\dagger}(q)]\right) = 0\,, \tag{1.90}$$

where we have used (1.80). A similar calculation shows that $[b_-(p), b_+^\dagger(q)] \neq 0$ and in consequence one has

$$b_-(q)b_+^\dagger(q)|0\rangle \neq 0\,. \tag{1.91}$$

This means that, unlike $a^{(1)\dagger}(p)|0\rangle$, $a^{(2)^\dagger}(p)|0\rangle$, and $b_-^\dagger(p)|0\rangle$, the states $b_+^\dagger(p)|0\rangle$ are not physical and hence not part of $\mathcal{F}_1$. In this context, one should also realize that one has

$$[a^{(1)}(p), a^{(1)\dagger}(p)] > 0\,, \qquad [a^{(2)}(p), a^{(2)\dagger}(p)] > 0\,, \tag{1.92}$$

as well as (1.90) with $q = p$, which shows that $\mathcal{F}_1$ does not contain negative norm states. Yet, if a state contains at least one operator $b_-^\dagger(p)$ its norm vanishes. From a physical point of view it is kind of clear that we should discard such zero-norm states and the formal way of doing this is to identify each two states in $\mathcal{F}_1$ if their difference has zero norm. In this way, we obtain the proper Fock space $\mathcal{F}_2$, whose elements are the classes of states obtained from this identification. In particular, in each class there is a "representative" with only transverse modes. In conclusion, we see that the proper Fock space $\mathcal{F}_2$ can be thought of as spanned by states of the form

$$|p_1, \alpha_1; \ldots; p_n, \alpha_n\rangle = a^{(\alpha_1)\dagger}(p_1)\ldots a^{(\alpha_n)\dagger}(p_n)\,|0\rangle\,, \tag{1.93}$$

with $\alpha_i = 1, 2$. This resembles the physical picture we obtained in the classical theory of electromagnetism.

One remaining point which needs checking is that physical quantities ought to be independent of which representative for a class in $\mathcal{F}_2$ is picked. Let us verify this for the case of the 4-momentum (1.82). Changing to the new basis (1.88) of non-transversal modes, P_μ can be shown to take the form

$$P_\mu = \int d^3\tilde{p}\; p_\mu \left[\sum_{\alpha=1}^{2} a^{(\alpha)\dagger}(p)a^{(\alpha)}(p) + b_+^\dagger(p)b_-(p) + b_-^\dagger(p)b_+(p)\right]. \tag{1.94}$$

Hence one has

$$\langle\tilde{\Phi}|P_\mu|\Phi\rangle = \langle\tilde{\Phi}|\int d^3\tilde{p}\; p_\mu \left[\sum_{\alpha=1}^{2} a^{(\alpha)^\dagger}(p)a^{(\alpha)}(p)\right]|\Phi\rangle\,, \tag{1.95}$$

where we have taken into account that the last two terms in (1.94) do not contribute to the matrix element given that $b_-(p)|\Phi\rangle = 0$ and $\langle\tilde{\Phi}|b_-^\dagger(p) = 0$ for $|\Phi\rangle$, $|\tilde{\Phi}\rangle \in \mathcal{F}_1$. The result (1.95) shows that the 4-momentum only depends on the transverse modes and, since every class in $\mathcal{F}_2$ has (by definition) exactly one representative with transverse modes only, the desired independence on the choice of representative follows.

Feynman Propagator

We close this part of the lecture by looking at the Feynman propagator for a massless vector field, defined, as usual, as the vacuum expectation value of the time-ordered product of two field operators, *i.e.*, $D_F^{\mu\nu}(x-y) = \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle$. Inserting the field expansion (1.79) and using the commutation relations (1.80) we can perform a calculation completely analogous to the one we did before for the real scalar field to obtain $D_F^{\mu\nu}(x-y)$ in Feynman gauge, $\xi = 1$. I simply quote the final result

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta^{\mu\nu}}{p^2+i\epsilon} \, e^{-ip(x-y)} \,. \tag{1.96}$$

Working in an arbitrary R_ξ gauge, the Feynman propagator for our massless vector field will also involve a part proportional to $(1-\xi)\, p^\mu p^\nu/(p^2+i\epsilon)^2$. If one calculates a physical quantity it however does not matter whether this more complicated expression or (1.96) is used, because if the computation is done correctly the final result will not depend on the parameter ξ. So using $D_F^{\mu\nu}(x-y)$ in the form (1.96) in actual calculations from the very beginning is perfectly fine (of course, using (1.96) one misses out on the rush that one gets after computing 891 Feynman diagrams and seeing that all ξ's magically disappear in the sum of graphs).