1.) (Field theories with global $SO(n)$ symmetry)

a) The group $SO(n)$ consists of the real $n \times n$ matrices $R$ satisfying $R^T R = 1_n$ and $\det(R) = 1$. Consider infinitesimal transformations of the form $R = 1_n + iT$ and determine the allowed matrices $T$, i.e. the Lie algebra of $SO(n)$. Show that the matrices $T_{ab}$, labelled by an antisymmetric pair of indices $a$, $b$ and defined as $(T_{ab})_{cd} = i(\delta_{ac}\delta_d^b - \delta_{ad}\delta_c^b)$ (where $a, b, c, \ldots = 1, \ldots, n$) provide a basis for the Lie algebra. What is the dimension of this algebra?

b) A set of $n$ scalar fields $\phi_a(x)$, where $a = 1, \ldots, n$, has a Lagrangian density $L = \frac{1}{2} \sum_{a=1}^{n} \partial_\mu \phi_a \partial^\mu \phi_a - V$ with scalar potential $V = \frac{1}{2} m^2 \sum_{a=1}^{n} \phi_a^2 + \frac{\lambda}{4} (\sum_{a=1}^{n} \phi_a^2)^2$. Show that this Lagrangian density is invariant under $SO(n)$.

c) Compute the Noether currents of this $SO(n)$ symmetry.

d) For $m^2 < 0$ and $\lambda > 0$ find the minima of the scalar potential and show that the $SO(n)$ symmetry is broken to $SO(n-1)$. What is the number of Goldstone modes?

2.) ($SU(3)$ gauge symmetry with spontaneous symmetry breaking)

The Lie group $SU(3)$ is defined as the set of complex $3 \times 3$ matrices $U$ satisfying $U^\dagger U = 1_3$ and $\det(U) = 1$.

a) Show that the Lie algebra of $SU(3)$ consists of all hermitian $3 \times 3$ matrices with vanishing trace.

b) Write down a basis $t^a$ for the Lie algebra of $SU(3)$ which contains the matrices $(i = 1, 2, 3)$

$$ t^i = \begin{pmatrix} \tau^i & 0 \\ 0 & 0 \end{pmatrix} , $$

where $\tau^i \equiv \sigma_i/2$ with $\sigma^i$ the usual Pauli matrices. What is the dimension of this $SU(3)$ Lie algebra?

c) Consider now an $SU(3)$ gauge theory with a scalar field $\phi$ in the adjoint representation. The covariant derivative of $\phi$ takes the form

$$ D_\mu \phi_a = \partial_\mu \phi_a + g f_{abc} A^b_\mu \phi_c , $$

where $A^a_\mu$ denote the $SU(3)$ gauge fields and $f_{abc}$ are the corresponding fully antisymmetric structure constants, i.e. $[t^a, t^b] = i f^{abc} t^c$. Which term in the Lagrangian give rise to the masses of the gauge fields?

d) The mass term can be written more clearly by defining the quantity $\Phi \equiv \phi_a t^a$. Use this notation together with the definition of the $SU(3)$ structure constants to write the mass term as

$$ L \supset -g^2 \text{tr} \left( [t^a, \Phi] [t^b, \Phi] \right) A^a_\mu A^b_\mu . $$

e) Now let $\Phi$ acquire a vacuum expectation value (VEV). Consider the following two possibilities:

$$ \langle \Phi \rangle = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \langle \Phi \rangle = v \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} , $$

$$ \langle \Phi \rangle = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \langle \Phi \rangle = v \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} . $$

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Give the symmetry-breaking pattern induced by these VEVs and calculate the mass spectrum of the gauge bosons for both cases.

3.) (Interacting scalar field theory)
Two real scalar fields \( \phi_1 \) and \( \phi_2 \) with masses \( m_1^2 > 0 \) and \( m_2^2 > 0 \) transform as \( \phi_1 \to -\phi_1 \) and \( \phi_2 \to \phi_2 \) under a \( \mathbb{Z}_2 \) symmetry.

a) Write down the most general \( \mathbb{Z}_2 \) invariant Lagrangian (with standard kinetic terms and up to quartic terms in the fields) for \( \phi_1 \) and \( \phi_2 \).

b) Derive the Feynman rules for the vertices in this theory by computing the appropriate Green’s functions.

c) Based on the results in b) and assuming negligible masses \( m_1 \) and \( m_2 \), find the amplitudes and differential cross sections for \( \phi_1 \phi_1 \to \phi_1 \phi_1 \) and \( \phi_1 \phi_1 \to \phi_2 \phi_2 \) scattering.

4.) (Scalar Yukawa theory)
The Lagrangian density of the scalar Yukawa theory is given by

\[
\mathcal{L} = (\partial_\mu \varphi)^\dag (\partial^\mu \varphi) - M^2 \varphi^\dag \varphi + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - g \varphi^\dag \phi \phi,
\]

where \( \varphi \) and \( \phi \) is a complex and a real scalar, respectively. Assume that \( g \ll M, m \).

a) The given Lagrangian density has an internal global symmetry. How does this symmetry act on the fields \( \varphi \) and \( \phi \)? Calculate the associated Noether current \( J^\mu \) and Noether charge \( Q \).

b) Quantise the free theory \( (g = 0) \) and show that the normal-ordered Noether charge \( :Q: \) can be written in the form \( :Q: = N_+ - N_- \), where \( N_\pm = \int d^3 \tilde{p} a_\pm^\dag (p) a_\pm (p) \) are the number operators of the “+” quanta (anti-nucleons, \( \tilde{N} \)) and “−” quanta (nucleons, \( N \)) of the \( \varphi \) field.

c) Starting from the general result

\[
\sigma(p_1, p_2 \to q_1, q_2) = \frac{1}{4 \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \int \prod_{i=1,2} d^3 q_i (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \times |\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2,
\]
evaluate the differential cross section for a generic \( 2 \to 2 \) process. Work in the center-of-mass frame and assume for simplicity that the masses of all four external particles are equal to \( m \). You should obtain

\[
\frac{d\sigma(p_1, p_2 \to q_1, q_2)}{d\Omega} = \frac{1}{64\pi^2 E_{CM}^2} |\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2,
\]

where \( d\Omega = d\phi d\cos \theta \) with \( \theta \in [-\pi, \pi] \) and \( \phi \in [0, 2\pi] \), while \( E_{CM} \) denotes the total center-of-mass energy.

d) Draw the leading-order Feynman diagrams that describe the process \( N\tilde{N} \to MM \), where \( M \) denotes the quanta (mesons) of the \( \phi \) field. Write down the matrix element \( \mathcal{M}(N\tilde{N} \to MM) \) using the momentum-space Feynman rules.
\[ \tilde{D}_F(p,M) = \tilde{D}_F(p,m) = -ig \bar{N}^N_p, \bar{N}^M_p \]

with \( \tilde{D}_F(p,M) = i/(p^2 - M^2 + i\epsilon) \) and \( \tilde{D}_F(p,m) = i/(p^2 - m^2 + i\epsilon) \). Employ Mandelstam variables. Using the results of (c) derive the differential cross section for \( N \bar{N} \rightarrow MM \) production in the limit \( E_{CM} \gg M, m \) with \( m = M \). Consider only the leading term in the expansion in powers of \( m^2/E_{CM}^2 \).