1) A Green’s function

\[ \Delta(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\Delta}(k), \]  

(1)
of the Klein-Gordon equation is defined as a function satisfying \((\Box + m^2)\Delta(x) = -i\delta^{(4)}(x)\).

a) Show that the Fourier transform \(\tilde{\Delta}\) of \(\Delta\) is given by \(\tilde{\Delta}(k) = i/(k^2 - m^2)\).

b) Insert the result from a) into (1) and discuss the various paths in the complex \(k_0\) plane in relation to the two poles at \(k_0 = \pm \sqrt{k^2 + m^2}\) which can be chosen to carry out the \(k_0\) integral.

c) Perform the \(k_0\) integral for the paths in b) and show that for one choice of path \(\Delta(x) = 0\) whenever \(x_0 > 0\) and for another choice \(\Delta(x) = 0\) whenever \(x_0 < 0\). Interpret the physical relevance of these two Green’s functions, as solutions to the Klein-Gordon equation with a delta function source.

2) Consider a real, free scalar field \(\phi\) with mass \(m\), evaluate the following time-order product of field operators \(\int d^4x \langle 0 | T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) | 0 \rangle\), using Wick’s theorem and draw the associated Feynman diagrams. Focus on the part without loops, Fourier transform \(x_1, \ldots, x_4\) to \(k_1, \ldots, k_4\) and express the result in terms of momentum-space Feynman propagators.

3) Consider a complex, free scalar field \(\varphi\). Apply Wick’s theorem to \(T(\varphi(x_1)\varphi^\dagger(x_2))\) and prove the so-obtained relation explicitly. Do the same for \(T(\varphi(x_1)\varphi(x_2))\).

4) Consider the decay of a particle with mass \(M\) into two particles with mass \(m\).

a) Use the general formulas (1.86) and (1.87) of the script “Interacting Quantum Fields” to show that

\[ \Gamma = \frac{1}{16\pi M} \sqrt{1 - \left(\frac{2m}{M}\right)^2} |A|^2, \]

if the matrix element \(A\) describes a isotropic decay.

b) Two particle species \(A_1\) and \(A_2\) with masses \(m_1\) and \(m_2\) scatter in a process of the type \(A_1 + A_2 \rightarrow A_1 + A_2\). For incoming momenta \(k_1, k_2\) and outgoing momenta \(q_1, q_2\) define the Mandelstam variables \(s, t, u\) by \(s = (k_1 + k_2)^2, t = (k_1 - q_1)^2,\) and \(u = (k_1 - q_2)^2\). Show that \(s + t + u = 2m_1^2 + 2m_2^2\).

c) Set \(m = m_1 = m_2\) for simplicity. In the centre of mass frame, express the Mandelstam variables in terms of the total centre-of-mass (CM) energy \(E_{CM}\) and the scattering angle \(\theta\) (and the mass \(m\)).

d) Show that the matrix element \(A\) for the scattering process in b) can always be written as a function of \(s\) and \(t\).

5) A model with two real scalar fields \(\phi_1\) and \(\phi_2\) is described by the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \left( (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 - M^2 \phi_1^2 - m^2 \phi_2^2 - \mu \phi_1 \phi_2^* \right). \]
a) Derive the Feynman rule for the triplet vertex (i.e. the vertex involving one $\phi_1$ and two $\phi_2$ fields) in this theory by computing the appropriate *amputated* Green’s function as defined in Section 1.6 of the script "Interacting Quantum Fields".

b) Assume that $M > 2m$ and compute the rate $\Gamma$ for the decay of a $\phi_1$ particle into two $\phi_2$ particles at leading order.

c) Compute the matrix element for scattering of two $\phi_2$ particles into two $\phi_2$ particles at leading order and express it in terms of Mandelstam variables.

d) Write the matrix element in c) in terms of the total CM energy $E_{CM}$ and the scattering angle $\theta$. Compute the differential cross section $(d\sigma/d\Omega)_{CM}$ for the process in c), assuming, for simplicity, that $E_{CM} \gg m$. Without derivation you can use (1.98) of the script “Interacting Quantum Fields”.