M.Phys Option in Theoretical Physics: C6. Problem Sheet 5

1.) Consider the set $\mathrm{Sl}(2,\mathbb{C})$ of complex 2×2 matrices with determinant one.
a) Show that this set forms a group.
b) Compute the Lie-algebra of this group, show that the dimension of this algebra is 6 and write down a basis of generators using the Pauli matrices.
c) By making an appropriate choice for the generators and computing their commutation relations, show that the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ is a representation of the Lorentz group Lie algebra.
d) As explained in the lecture, representations of the Lorentz group Lie algebra are classified by two spins (j_+, j_-). Which pair of spins does the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ corresponds to and why?

2.) The Lagrangian density $\mathcal{L} = \mathcal{L}(\partial_\mu\phi_a(x), \phi_a(x))$ for a set of fields $\phi_a = \phi_a(x)$ is assumed to be invariant under the infinitesimal transformation

$$\phi_a \to \phi_a - it^i (T_i)_a{}^b \phi_b \ ,$$

where T_i are the generators of a Lie algebra with commutators $[T_i, T_j] = if_{ij}{}^k T_k$ and t^i are the symmetry parameters.
a) Find the conserved currents $j_{i\mu}$ and the associated conserved charges Q_i for this symmetry.

3.) A model with a real scalar field σ and three other real scalar fields $\boldsymbol{\phi} = (\phi_a)$ is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\phi_a\partial^\mu\phi_a\right) - V(\sigma,\boldsymbol{\phi}) \ , \quad V = \frac{m^2}{2}(\sigma^2 + \boldsymbol{\phi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\phi}^2)^2 \ .$$

a) Show that this Lagrangian is invariant under SO(4) acting on the four-dimensional vectors $(\sigma, \boldsymbol{\phi})$.
b) Show that infinitesimal SO(4) transformations of the fields can be written as

$$\sigma \to \sigma + \boldsymbol{\beta}\cdot\boldsymbol{\phi} \ , \quad \boldsymbol{\phi} \to \boldsymbol{\phi} + \boldsymbol{\alpha}\times\boldsymbol{\phi} - \boldsymbol{\beta}\sigma \ ,$$

for suitable defined small symmetry parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
c) Find the six conserved currents and charges.
d) Analyze spontaneous breaking of the SO(4) symmetry in the case where $m^2 < 0$. In particular, find the vacua, determine the unbroken sub-group and the Goldstone modes. (Hint: For the two final tasks, choose a minimum of the potential for which $\boldsymbol{\phi} = 0$ and $\sigma \neq 0$.)

4.) a) Derive the energy-momentum tensor $T^{\mu\nu}$ from the Lagrangian formulation of the free Maxwell theory, using the general procedure explained in the lecture.
b) Given that $T^{\mu\nu}$ is conserved show that

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$$

is still conserved provided the tensor $K^{\rho\mu\nu}$ is anti-symmetric in its first two indices.
c) Choosing $K^{\rho\mu\nu} = F^{\mu\rho}A^\nu$ show that $\tilde{T}^{\mu\nu}$ is a symmetric tensor.

d) Write down the conserved charges associated to $\tilde{T}^{\mu\nu}$ and (by writing them in terms of $\mathbf{E}$ and $\mathbf{B}$) show that they yield the standard expressions of the electromagnetic energy and momentum densities.

5.) Consider a real scalar $\phi(t,x)$ field living on a two-dimensional space-time and defined on an interval $x \in [0,L]$ with *Dirichlet boundary conditions* $\phi(t,0) = \phi(t,L) = 0$.
a) Show that the (classical) positive- and negative-frequency solutions to the Klein-Gordon equation that also satisfy the boundary conditions have the form

$$\phi_n^{(\pm)}(t,x) = \frac{1}{\sqrt{\omega_n L}}\, e^{\pm i\omega_n t}\, \sin(k_n x) \ .$$

Give the expression for k_n in terms of L. How is ω_n related to k_n?
b) Now quantise the field $\phi(t,x)$, keeping in mind that momentum is discrete

$$\phi(t,x) = \sum_{n=1}^{\infty} \left(\phi_n^{(-)}(t,x)\, a_n + \phi_n^{(+)}(t,x)\, a_n^\dagger \right) \ ,$$

with the annihilation/creation operators satisfying $[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0$ and $[a_n, a_m^\dagger] = \delta_{mn}$. Compute the vacuum expectation value $\langle 0|\mathcal{H}|0\rangle$ of the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left[\dot{\phi}^2 + (\partial_x\phi)^2 + m^2\phi^2\right] \ .$$

Integrating your result over the interval $[0,L]$ and show that the total vacuum energy is

$$E_0(L) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n \ .$$

c) Since this quantity is infinite, we need some form of regularisation in order to handle the divergence. Let us introduce an exponentially damping function $\exp(-\delta\omega_n)$ with $\delta > 0$ in the sum, and consider for simplicity the case of a massless field. Prove that in this case the vacuum energy can be written as

$$E_0(L,\delta) = \frac{\pi}{8L}\, \sinh^{-2}\left(\frac{\delta\pi}{2L}\right) \ .$$

Take the limit $\delta \to 0$ and determine the vacuum energy for the case when no boundary conditions are imposed. With all this at hand calculate the *Casimir force*, that is, the attractive force associated to the mismatch between the vacuum energy of the unbounded space and that of the theory on the interval.