APPLICATION I : THE IDEAL FERMI GAS

Consider an ideal gas of spin-1/2 fermions. The creation operators in the mtm representation are

$$c_\sigma^\dagger(\vec{p}) \qquad \sigma = \uparrow, \downarrow$$

where
$$\{ c_\sigma(\vec{p}), c_{\sigma'}^\dagger(\vec{k}) \} = \delta_{\sigma\sigma'} (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{p})$$

$$\{ c_\sigma(\vec{p}), c_{\sigma'}(\vec{k}) \} = 0.$$

The Hamiltonian in the grand canonical ensemble is

$$\boxed{\hat{H} - \mu \hat{N} = \int \frac{d^3\vec{p}}{(2\pi)^3} \underbrace{\left(\frac{\vec{p}^{\,2}}{2m} - \mu \right)}_{\varepsilon(\vec{p})} \sum_{\sigma=\uparrow,\downarrow} c_\sigma^\dagger(\vec{p}) c_\sigma(\vec{p})}$$

$\mu > 0$ is the chemical potential.

We have
$$(\hat{H}-\mu\hat{N}) |0\rangle = 0$$
$$(\hat{H}-\mu\hat{N}) c_\sigma^\dagger(\vec{k}) |0\rangle = \varepsilon(\vec{k}) |0\rangle$$
$$\vdots$$
$$(\hat{H}-\mu\hat{N}) \prod_{j=1}^{n} c_{\sigma_j}^\dagger(\vec{k}_j) |0\rangle = \left(\sum_{j=1}^{n} \varepsilon(\vec{k}_j) \right) \prod_{j=1}^{n} c_{\sigma_j}^\dagger(\vec{k}_j) |0\rangle$$

This is because $c_\sigma^\dagger(\vec{p}) c_\sigma(\vec{p}) \equiv \hat{n}_\sigma(\vec{p})$ is the number operator for fermions with spin σ and mtm $\vec{p}$.

The ground state is by definition the state with the lowest energy. As $\varepsilon(\vec{p})$ can be negative, the smallest energy is achieved by "filling all negative energy states", i.e.

$$\boxed{|GS\rangle = \prod_{\sigma=\uparrow,\downarrow} \prod_{\frac{\vec{k}^2}{2m} < \mu} c_\sigma^\dagger(\vec{k}) |0\rangle}$$

This is called a "FERMI SEA"

LET US BE MORE PRECISE and put the gas in a box of length L. Imposing periodic boundary conditions quantizes the momenta

$$(1) \qquad \vec{p} = \frac{2\pi}{L}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \qquad n_a \text{ integers}$$

Aside: to see this consider mtm eigenstates in the position representation

$$\hat{\vec{p}} = -i\hbar\vec{\nabla} \qquad \hat{\vec{p}}\, \psi_{\vec{k}}(\vec{r}) \overset{!}{=} \vec{k}\, \psi_{\vec{k}}(\vec{r})$$

Solve this $\longrightarrow \quad \psi_{\vec{k}}(\vec{r}) = \langle \vec{r} | \vec{k} \rangle = e^{i\vec{k}\cdot\vec{r}}$

Periodic bc's: $\quad \psi_{\vec{k}}(\vec{r} + L\vec{e}_x) \overset{!}{=} \psi_{\vec{k}}(\vec{r}) \quad \rightarrow \quad e^{ik_x L} = 1$

$$\Longrightarrow \quad k_x = \frac{2\pi n_x}{L}$$

and similarly for k_y, k_z.

So the mtm eigenstates in our box are $|\vec{p}\rangle$ with the $\vec{p}$ given by (1).

Using that $|\vec{p}\rangle = c^{+}_{\vec{p}}|0\rangle$ this applies to the creation ops as well.

The Hamiltonian then becomes

$$\boxed{\hat{H} - \mu\hat{N} = \sum_{\vec{p}} \left(\frac{\vec{p}^{\,2}}{2m} - \mu\right) \sum_{\sigma} n_\sigma(\vec{p})}$$

We define a Fermi momentum p_F by $\frac{p_F^2}{2m} = \mu$. Then

$$\boxed{|GS\rangle = \prod_{\sigma, |\vec{p}| < p_F} c^{+}_{\sigma}(\vec{p})\, |0\rangle}$$

The GS energy is $\boxed{E_{GS} = \sum_{\sigma=\uparrow\downarrow} \sum_{|\vec{p}|<p_F} \varepsilon(\vec{p})}$ prop. to the volume

mtm of $|GS\rangle$ $\vec{P}_{GS} = \sum_{\sigma=\uparrow\downarrow} \sum_{|\vec{p}|<p_F} \vec{p} = 0.$

EXCITATIONS :

(i) "PARTICLE" EXCITATIONS $c_\sigma^+(\vec{k})\,|GS\rangle$ with $|\vec{k}| > p_F$

energy: $E = E_{GS} + \varepsilon(\vec{k}) > E_{GS}$
mtm : $\vec{P} = \vec{k}$.

(ii) "HOLE" EXCITATIONS $c_\sigma(\vec{k})\,|GS\rangle$ with $|\vec{k}| < p_F$

energy: $E = E_{GS} - \varepsilon(\vec{k}) > E_{GS}$
$\vec{P} = \vec{P}_{GS} - \vec{k} = -\vec{k}$

(iii) "PARTICLE - HOLE" EXCITATIONS $c_\sigma^+(\vec{k})\, c_\tau(\vec{p})\,|GS\rangle$ $|\vec{k}| > p_F > |\vec{p}|$

$E = E_{GS} + \varepsilon(\vec{k}) - \varepsilon(\vec{p}) > E_{GS}$
$\vec{P} = \vec{P}_{GS} + \vec{k} - \vec{p} = \vec{k} - \vec{p}$.

DENSITY CORRELATIONS:

CONSIDER the 1-particle operator $|\vec{r}\rangle\langle\vec{r}|$.
It represents the particle density at position $|\vec{r}\rangle$. In 2^{nd} quantization it is

$$\rho(\vec{r}) = \sum_\sigma \int d^3\vec{r}'\, d^3\vec{r}'' \,\langle\vec{r}'|\vec{r}\rangle\langle\vec{r}|\vec{r}''\rangle\, c_\sigma^+(\vec{r}')\, c_\sigma(\vec{r}'')$$

$$= \sum_\sigma c_\sigma^+(\vec{r})\, c_\sigma(\vec{r})$$

The GS expectation value at zero temperature is

$$\langle GS|\ \rho(\vec{r})\ |GS\rangle = \sum_{\sigma} \langle GS|\ c^{+}_{\sigma}(\vec{r})\, c_{\sigma}(\vec{r})\ |GS\rangle$$

To evaluate this we go to mtm space

$$c_{\sigma}(\vec{r}) = \sum_{\vec{p}} \underbrace{\langle \vec{r}|\vec{p}\rangle}_{\frac{1}{L^{3/2}} e^{i\vec{p}\cdot\vec{r}}} c_{\sigma}(\vec{p})$$

according to our general formula for basis transformations

$$|GS\rangle = \prod_{|\vec{k}|<p_F} \prod_{\sigma=\uparrow,\downarrow} c^{+}_{\sigma}(\vec{k})\ |0\rangle$$

$$\Longrightarrow \quad \langle GS|\ \rho(\vec{r})\ |GS\rangle = \sum_{\sigma} \frac{1}{L^3} \sum_{\vec{p},\vec{p}'} e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}} \underbrace{\langle GS|\ c^{+}_{\sigma}(\vec{p}')\, c_{\sigma}(\vec{p})\ |GS\rangle}$$

ZERO UNLESS $|\vec{p}|<p_F$ AND $\vec{p}'=\vec{p}$
because $c^{+}_{\sigma}(\vec{p}')\, c_{\sigma}(\vec{p})\ |GS\rangle$ must reproduce $|GS\rangle$ itself

$$= \sum_{\sigma} \frac{1}{L^3} \sum_{\vec{p},\vec{p}'} e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}}\ \Theta(p_F-|\vec{p}|)\ \delta_{\vec{p},\vec{p}'}$$

$$= \underset{\substack{\uparrow \\ \text{spin}}}{2}\ \frac{1}{L^3} \sum_{\vec{p}} \Theta(p_F-|\vec{p}|) = \frac{N}{V}$$

particle density

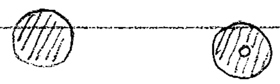
2-POINT FUNCTION AT T = 0

$$\langle GS|\, \rho(\vec r)\, \rho(\vec r\,')\, |GS\rangle = \frac{1}{L^6} \sum_{\vec p,\vec p\,'} \sum_{\vec k,\vec k'} e^{-i(\vec p-\vec p\,')\cdot\vec r}\, e^{-i(\vec k-\vec k')\cdot\vec r\,'}$$

$$\times \sum_{\sigma\sigma'} \langle GS|\, c^+_\sigma(\vec p)\, c_\sigma(\vec p\,')\, c^+_{\sigma'}(\vec k)\, c_{\sigma'}(\vec k')\, |GS\rangle$$

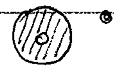
Can calculate this by Wick's thm or <u>directly</u> :

$|GS\rangle$ $\quad$ $c_{\sigma'}(\vec k')\,|GS\rangle$



hole at $\vec k' \rightsquigarrow \Theta(p_F - |\vec k'|)$

Acting with $c^+_{\sigma'}(\vec k)$ gives <u>either</u> $\delta_{\vec k,\vec k'}$  case (i)

<u>or</u> $\Theta(|\vec k| - p_F)$  case (ii)

case (i) : get $\langle GS|\, c^+_\sigma(\vec p)\, c_\sigma(\vec p\,')\, |GS\rangle = \delta_{\vec p,\vec p\,'}\, \Theta(p_F - |\vec p\,'|)$

case (ii) : must have $\sigma = \sigma'$, $\vec p\,' = \vec k$, $\vec p = \vec k'$ to reproduce 
when acting with $c^+_\sigma(\vec p)\, c_\sigma(\vec p\,')$

$$\Longrightarrow \quad \langle GS|\, c^+_\sigma(\vec p)\, c_\sigma(\vec p\,')\, c^+_{\sigma'}(\vec k)\, c_{\sigma'}(\vec k')\, |GS\rangle = \delta_{\vec k,\vec k'}\, \delta_{\vec p,\vec p\,'}\, \Theta(p_F - |\vec p\,'|)\, \Theta(p_F - |\vec k'|)$$
$$+ \delta_{\sigma\sigma'}\, \delta_{\vec k,\vec p\,'}\, \delta_{\vec k',\vec p}\, \Theta(|\vec k| - p_F)\, \Theta(p_F - |\vec k'|)$$

$$\Longrightarrow \quad \langle GS|\, \rho(\vec r)\, \rho(\vec r\,')\, |GS\rangle = \frac{1}{L^6} \sum_{\vec k,\vec p} \sum_{\sigma,\sigma'} \Theta(p_F - |\vec p|)\, \Theta(p_F - |\vec k|)$$
$$+ \frac{1}{L^6} \sum_{\vec k,\vec k'} \sum_{\sigma} \Theta(|\vec k| - p_F)\, \Theta(p_F - |\vec k'|)\, e^{-i\vec k\cdot(\vec r-\vec r\,')}\, e^{+i\vec k'\cdot(\vec r-\vec r\,')}$$

$$= \langle GS|\, \rho(\vec r)\, |GS\rangle\, \langle GS|\, \rho(\vec r\,')\, |GS\rangle$$
$$+ 2\, \frac{1}{L^3} \sum_{|\vec k| > p_F} e^{-i\vec k\cdot(\vec r-\vec r\,')}\, \frac{1}{L^3} \sum_{|\vec k| < p_F} e^{i\vec k'\cdot(\vec r-\vec r\,')}$$

EVALUATING THE SUMS :

$$\frac{1}{L^3}\sum_{|\vec{k}|>p_F} e^{-i\vec{k}\cdot\vec{R}} = \underbrace{\frac{1}{L^3}\sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{R}}}_{=\,\delta^{(3)}(\vec{R})} - \frac{1}{L^3}\sum_{|\vec{k}|<p_F} e^{-i\vec{k}\cdot\vec{R}}$$

To see this calculate $\int d^3\vec{R}\, f(\vec{R})\, \frac{1}{L^3}\sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{R}}$
for periodic fns $f(\vec{R})$

$$= \sum_{\vec{k}} \frac{1}{L^3}\int d^3\vec{R}\, f(\vec{R})\, e^{-i\vec{k}\cdot\vec{R}}$$

$$= \sum_{\vec{k}} f_{\vec{k}} = f(0)$$

cf Riley/Hobson/Bence 12.7 Fourier series

The other sum is turned into an integral for $L\to\infty$

$$\frac{1}{L^3}\sum_{|\vec{k}|<p_F} e^{-i\vec{k}\cdot\vec{R}} \to \int \frac{d^3\vec{k}}{(2\pi)^3}\, \Theta(p_F-|\vec{k}|)\, e^{-i\vec{k}\cdot\vec{R}}$$

$$= \int_0^\infty dp\, p^2 \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi\, \frac{\Theta(p_F-p)}{(2\pi)^3}\, e^{-ip|\vec{R}|\cos\vartheta}$$

$$= \int_0^{p_F} \frac{dp}{(2\pi)^2}\, p^2\, J_0(p|\vec{R}|)$$

Putting everything together :

$$\boxed{\begin{aligned}\langle GS|\,\rho(\vec{r})\,\rho(\vec{r}\,')\,|GS\rangle &= \left(\langle GS|\rho(\vec{r})|GS\rangle\right)^2 + \langle GS|\rho(\vec{r})|GS\rangle\,\delta^{(3)}(\vec{r}-\vec{r}\,') \\ &\quad - 2\left(\int_0^{p_F} \frac{dp}{(2\pi)^2}\, p^2\, J_0(p|\vec{r}-\vec{r}\,'|)\right)^2\end{aligned}}$$

1st line: result for <u>classical</u> ideal gas
2nd line: negative contribution due to <u>Fermionic statistics</u> ("Pauli exclusion")

APPLICATION 2 : WEAKLY INTERACTING BOSONS

Consider a gas of bosons with kinetic energy

$$\hat{T} = \sum_{\vec{p}} \frac{\vec{p}^{\,2}}{2m} c^{+}(\vec{p})\, c(\vec{p})$$

and interactions

$$\hat{V} = \frac{1}{2} \int d^3\vec{r}\, d^3\vec{r}\,' \; c^{+}(\vec{r})\, c^{+}(\vec{r}\,')\; U\, \delta^{(3)}(\vec{r}-\vec{r}\,')\; c(\vec{r}\,')\, c(\vec{r})$$

$$= \frac{U}{2L^3} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}_3} c^{+}(\vec{p}_1)\, c^{+}(\vec{p}_2)\, c(\vec{p}_3)\, c(\vec{p}_1+\vec{p}_2-\vec{p}_3)$$

where annihilation ops in position and mtm space are related by

$$c(\vec{p}) = \int d^3\vec{r}\; \langle \vec{p} | \vec{r} \rangle\, c(\vec{r}) = \int d^3\vec{r}\; e^{-i\vec{p}\cdot\vec{r}}\, c(\vec{r})$$

$U = 0 \Rightarrow$ ideal Bose gas $\boxed{|GS\rangle_0 = \left(c^{+}(\vec{p}=0)\right)^N |0\rangle}$ BOSE-EINSTEIN CONDENSATE

$\vec{p}=0$ is special. For small $U>0$ we expect the BE condensate to persist, i.e.

$$\langle GS |\, c^{+}(\vec{p}=0)\, c(\vec{p}=0)\, | GS \rangle = N_0 \gg 1$$

HOWEVER $[\, c^{+}(0)\, c(0), \hat{V}\,] \neq 0 \Rightarrow$ # of $\vec{p}=0$ bosons is not conserved &

$|GS\rangle$ = superposition of states with different numbers of $\vec{p}=0$ bosons

BOGOLIUBOV APPROXIMATION : $c^{+}(0)\, c(0)\, |\psi\rangle \simeq N_0 |\psi\rangle \qquad N_0 \gg 1$

where $|\psi\rangle$ is the ground state or a low-energy excited state.

Then for such states $\begin{matrix} c^{+}(0) \simeq \sqrt{N_0} \\ c(0) \simeq \sqrt{N_0} \end{matrix}$ are approximately diagonal.

As $N_0 \gg 1$ we can then expand H in powers of $\frac{1}{N_0}$:

$$\hat{H} \simeq \sum_{\vec{p}} \frac{\vec{p}^2}{2m} c^{+}(\vec{p}) c(\vec{p}) + \frac{u}{2L^3} N_0^2 + \frac{u N_0}{2L^3} \sum_{\vec{k}\neq 0} \begin{matrix} 2 c^{+}(\vec{k}) c(\vec{k}) + 2 c^{+}(-\vec{k}) c(-\vec{k}) \\ + c^{+}(\vec{k}) c^{+}(-\vec{k}) + c(-\vec{k}) c(\vec{k}) \end{matrix}$$

$$+ \ldots$$

Now use $N_0 = c^{+}(0) c(0) = N - \sum_{\vec{p}\neq 0} c^{+}(\vec{p}) c(\vec{p})$

$$\Rightarrow \boxed{\hat{H} = \frac{u\rho}{2} N + \sum_{\vec{p}\neq 0} \left(\frac{\vec{p}^2}{2m} + u\rho \right) c^{+}(\vec{p}) c(\vec{p}) + \frac{u\rho}{2} \left[c^{+}(\vec{p}) c^{+}(-\vec{p}) + c(-\vec{p}) c(\vec{p}) \right] + \ldots}$$

where $\rho = \frac{N}{L^3}$ = density of particles

BOGOLIUBOV TRANSFORMATION

Consider the creation/annihilation ops defined by

$$\begin{pmatrix} b(\vec{p}) \\ b^{+}(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \cosh\theta_{\vec{p}} & +\sinh\theta_{\vec{p}} \\ +\sinh\theta_{\vec{p}} & \cosh\theta_{\vec{p}} \end{pmatrix} \begin{pmatrix} c(\vec{p}) \\ c^{+}(-\vec{p}) \end{pmatrix}$$

Then

$$[b(\vec{p}), b^{+}(\vec{q})] = \cosh\theta_{\vec{p}} \cosh\theta_{\vec{q}} \underbrace{[c(\vec{p}), c^{+}(\vec{q})]}_{\delta_{\vec{p},\vec{q}}} + \sinh\theta_{\vec{p}} \sinh\theta_{\vec{q}} \underbrace{[c^{+}(\vec{p}), c(\vec{q})]}_{-\delta_{\vec{p},\vec{q}}}$$

$$= (\cosh^2\theta_{\vec{p}} - \sinh^2\theta_{\vec{p}}) \, \delta_{\vec{p},\vec{q}} = \delta_{\vec{p},\vec{q}}$$

i.e. the Bogoliubov transformation preserves the commutation relns

In terms of the new bosons

$$\hat{H} = \text{const} + \frac{1}{2}\sum_{p\neq 0}\left[\left(\frac{\vec{p}^2}{2m} + Ug\right)\cosh 2\theta_{\vec{p}} - Ug \sinh 2\theta_{\vec{p}}\right]\left[b^+(\vec{p})b(\vec{p}) + b^+(-\vec{p})b(-\vec{p})\right]$$

$$- \left[\left(\frac{p^2}{2m} + Ug\right)\sinh 2\theta_{\vec{p}} - \cosh 2\theta_{\vec{p}}\, Ug\right]\left[b^+(\vec{p})b^+(-\vec{p}) + b(\vec{p})b(-\vec{p})\right]$$

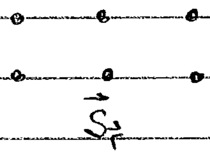
$+ \ldots$

Now choose $\boxed{\tanh 2\theta_{\vec{p}} = \dfrac{Ug}{\frac{\vec{p}^2}{2m} + Ug}}$ This removes the $b^+b^+ + bb$ terms!

$$\Rightarrow \boxed{\begin{aligned} \hat{H} &= \text{const} + \sum_{\vec{p}\neq 0} \varepsilon(\vec{p})\, b^+(\vec{p})\, b(\vec{p}) + \ldots \\ \varepsilon(\vec{p}) &= \sqrt{\left(\frac{\vec{p}^2}{2m} + Ug\right)^2 - (Ug)^2} \longrightarrow \begin{cases} \sqrt{\frac{Ug}{m}}\; |\vec{p}| & \text{for } |\vec{p}| \to 0 \\ \frac{\vec{p}^2}{2m} & \text{for } |\vec{p}| \to \infty \end{cases} \end{aligned}}$$

APPLICATION 3 : SPINWAVES IN A FERROMAGNET

Consider the following model of a magnetic insulator



at each site $\vec{r}$ we have a magnetic moment $\rightarrow$ $S^{\alpha}_{\vec{r}}$ $\alpha = x, y, z$

$$[S^{\alpha}_{\vec{r}}, S^{\beta}_{\vec{r}\,'}] = \delta_{\vec{r}\vec{r}\,'}\; i\,\varepsilon_{\alpha\beta\gamma}\, S^{\gamma}_{\vec{r}}$$

$$\vec{S}^{\,2}_{\vec{r}} = S(S+1) \gg 1$$

An appropriate Hamiltonian is $\boxed{\hat{H} = -J \sum_{\langle \vec{r}, \vec{r}\,' \rangle} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}\,'}}$

"Heisenberg ferromagnet" $J > 0$, $\langle \vec{r}, \vec{r}\,' \rangle$ nearest neighbour pairs

A ground state of $\hat{H}$ is $\boxed{|GS\rangle = \prod_{\vec{r}} |\uparrow\rangle_{\vec{r}}}$

where $S^{z}_{\vec{r}} |\uparrow\rangle_{\vec{r}} = S\, |\uparrow\rangle_{\vec{r}}$

$$\hat{H}\,|GS\rangle = \Big(-J \sum_{\langle \vec{r}, \vec{r}\,' \rangle} S^2\Big)\,|GS\rangle = -JS^2 zN\;|GS\rangle$$

zN = total # of bonds

Proof that $|GS\rangle$ is a ground state :

$$2\,\vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}\,'} = (\vec{S}_{\vec{r}} + \vec{S}_{\vec{r}\,'})^2 - (\vec{S}^{\,2}_{\vec{r}} + \vec{S}^{\,2}_{\vec{r}\,'}) = \vec{J}^{\,2} - 2S(S+1)$$

$\vec{J}^{\,2}$ = total angular mtm $\Rightarrow$ $\vec{J}^{\,2}\,|j, m\rangle = j(j+1)\,|j, m\rangle$ $j = 2S, 2S-1, \ldots, 0$

$\Rightarrow$ maximal eigenvalue of $\vec{J}^{\,2}$ is $2S(2S+1)$ $\Rightarrow$ $\langle \Psi | \vec{J}^{\,2} | \Psi \rangle \le 2S(2S+1)$

$\Rightarrow$ $\langle \Psi |\, 2\vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}\,'}\, | \Psi \rangle \le 2S(2S+1) - 2S(S+1) = 2S^2$

$$\Rightarrow \quad \langle\Psi|\hat{H}|\Psi\rangle \geq -J \sum_{\langle \vec{r},\vec{r}'\rangle} S^2 = -JS^2 Nz \, .$$

So $|GS\rangle$ has the lowest possible energy.

Note that $|GS\rangle$ breaks the spin-rotational symmetry of $\hat{H}$

$$\left[\sum_{\vec{r}} S^\alpha_{\vec{r}}, \hat{H}\right] = 0 \qquad \alpha = x,y,z$$

But $\boxed{\langle GS| S^\alpha_{\vec{r}} |GS\rangle = S\,\delta_{\alpha,z} \neq 0}$ Spontaneous Symmetry Breaking

HOLSTEIN - PRIMAKOFF TRANSFORMATION:

$$\text{(HP)}\quad \boxed{\begin{aligned} S^z_{\vec{r}} &= S - a^\dagger_{\vec{r}} a_{\vec{r}} \\ S^+_{\vec{r}} &= S^x_{\vec{r}} + i S^y_{\vec{r}} = \sqrt{2S}\sqrt{1 - \frac{a^\dagger_{\vec{r}} a_{\vec{r}}}{2S}}\; a_{\vec{r}} \end{aligned}} \qquad [a_{\vec{r}}, a^\dagger_{\vec{r}'}] = \delta_{\vec{r}\vec{r}'}$$

Can check that this gives $[S^\alpha_{\vec{r}}, S^\beta_{\vec{r}'}] = \delta_{\vec{r}\vec{r}'}\, i\,\varepsilon_{\alpha\beta\gamma}\, S^\gamma_{\vec{r}}$

CAVEATS: the spaces of QM states are different

$$(S^-_{\vec{r}})^n |\uparrow\rangle_{\vec{r}} \qquad n = 0,\dots,2S$$

$$\text{vs} \quad (a^\dagger_{\vec{r}})^n |0\rangle_{\vec{r}} \qquad n = 0,1,\dots \infty$$

$\Rightarrow$ must impose a constraint that we have at most $2S$ bosons per site

large S: $|GS\rangle$ has 0 bosons
low-lying excitations have few bosons

$\rightarrow$ can ignore the constraint

Use (HP) to rewrite $\hat{H}$:

$$\boxed{\hat{H} = - J \sum_{\langle \vec{r}, \vec{r}' \rangle} S^2 - S \left[a^+_{\vec{r}} a_{\vec{r}} + a^+_{\vec{r}'} a_{\vec{r}'} - a^+_{\vec{r}} a_{\vec{r}'} - a^+_{\vec{r}'} a_{\vec{r}} \right] + \dots}$$

terms $\propto S^0, S^{-1}, \dots$

Drop the higher order in S^{-1} terms & go to mtm space

$$a(\vec{k}) = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i \vec{k} \cdot \vec{r}} a_{\vec{r}} \qquad [a(\vec{k}), a^+(\vec{p})] = \delta_{\vec{k}, \vec{p}}$$

$$\Rightarrow \boxed{\hat{H} = - J S^2 N z - J S \sum_{\vec{q}} \varepsilon(\vec{q})\, a^+(\vec{q}) a(\vec{q}) + \dots}$$

For a simple cubic lattice: $\boxed{\varepsilon(\vec{q}) = 2JS\,(3 - \cos q_x - \cos q_y - \cos q_z)}$

$$\simeq JS^2 |\vec{q}|^2 \quad \text{for } |\vec{q}| \to 0$$

GOLDSTONE MODES