MANY-PARTICLE QUANTUM SYSTEMS VS QFT

So far you encountered QFTs from quantizing classical relativistic FTs :

$$\begin{tikzcd}
\boxed{\text{Classical FT}} \arrow[rr, "\text{CANONICAL}"', "\text{QUANTIZATION}"] & & \boxed{\text{QFT}} \arrow[d, "\text{MODE EXPANSION}"] \\
& & \boxed{\text{MANY PARTICLE SYSTEM}}
\end{tikzcd}$$

(harmonic oscillators at each point in space)

QFTs arise also in very different contexts :

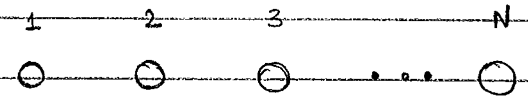
$$\begin{tikzcd}
\boxed{\text{MANY PARTICLE SYSTEM, NON-RELATIVISTIC}} \arrow[rr, "\text{``2}^{\text{nd}}\text{ QUANTIZATION''}"] & & \boxed{\text{FOCK SPACE}} \arrow[d, "\text{long length scales (low energies)}"] \\
& & \boxed{\text{RELATIVISTIC QFT}}
\end{tikzcd}$$

creation/annihilation operators, vacuum state

This provides a nice intuitive way of understanding QFT.

FIELD THEORY OF LATTICE VIBRATIONS (PHONONS)

CONSIDER A LINEAR CHAIN OF INTERACTING ATOMS OF MASS m

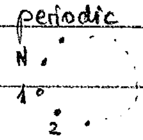


N even.

Classically we have $H = \sum_{l} \frac{p_l^2}{2m} + V(q_1, \ldots, q_N)$

where the q_j are the displacements from the equilibrium positions and $p_j = m\dot{q}_j$

Let us choose $V = \frac{k}{2} \sum_{l=1}^{N} (y_l - y_{l+1})^2$ and impose periodic boundary conditions $q_{N+1} \equiv q_1$. This corresponds to a ring geometry



Quantum mechanically we obtain

$$\boxed{H = \sum_{l=1}^{N} \frac{\hat{p}_l^2}{2m} + \frac{k}{2} \sum_{l=1}^{N} (\hat{y}_l - \hat{y}_{l+1})^2}$$

$$[\hat{p}_l, \hat{y}_j] = -i\hbar\, \delta_{lj}$$

This is a many-particle QM system. How to determine the energy eigenvalues etc?

Define new operators

$$\boxed{\begin{aligned} y(q_n) &\equiv \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \hat{y}_l \, e^{i q_n l} \\ p(q_n) &\equiv \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \hat{p}_l \, e^{-i q_n l} \end{aligned}} \qquad (1)$$

where $q_n = \frac{2\pi n}{N}$, $n = -\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, \frac{N}{2}$.

<u>Useful sums</u> :
$$\frac{1}{N}\sum_{l=1}^{N} e^{\frac{2\pi i n}{N} l} = \delta_{n,0} \qquad (n \text{ integer})$$
$$\frac{1}{N}\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} e^{\frac{2\pi i}{N} n l} = \delta_{l,N} \qquad (l \text{ integer})$$

<u>Proof</u> :
$$\frac{1}{N}\sum_{n} e^{\frac{2\pi i n}{N} l} = e^{\frac{2\pi i}{N} l \left(-\frac{N}{2}+1\right)} \frac{1}{N}\sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} n l}$$
$$= \text{—''—} \quad \frac{1}{N} \frac{1-e^{\frac{2\pi i l}{N} N}}{1-e^{\frac{2\pi i}{N} l}}$$
$$= \text{—''—} \quad \begin{cases} 0 & l = 1 \ldots N-1 \\ 1 & l = N \end{cases} = \delta_{l,N}$$

<u>Inverse transformation</u> :
$$\boxed{\begin{aligned} \hat{y}_l &= \frac{1}{\sqrt{N}} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} e^{-iq_n l} \, y(q_n) \\ \hat{p}_l &= \frac{1}{\sqrt{N}} \sum_{n} e^{+iq_n l} \, p(q_n) \end{aligned}} \qquad (2)$$

[Can be checked by substituting (1) into (2) and then carrying out the sums.]

<u>Commutation relations</u> :
$$[\, y(q_n), y(q_m) \,] = 0 = [\, p(q_n), p(q_m) \,]$$
$$[\, p(q_n), y(q_m) \,] = \frac{1}{N} \sum_{l,l'} e^{-i q_n l + i q_m l'} [\, \hat{p}_l, \hat{y}_{l'} \,]$$
$$= -i\hbar \, \delta_{q_n, q_m} .$$

In terms of the new operators we have
$$\sum_{l=1}^{N} \hat{p}_l^{\,2} = \frac{1}{\sqrt{N}} \sum_n \sum_{l=1}^{N} \hat{p}_l \, e^{i q_n l} \, p(q_n) = \sum_n p(-q_n)\, p(q_n)$$

Similarly $\frac{k}{2}\sum_{l=1}^{N}(\hat{y}_l-\hat{y}_{l+1})^2 = k\sum_n (1-\cos q_n)\; y(q_n)y(-q_n)$.

Now define new operators
$$\boxed{a(q_n) = \frac{1}{\sqrt{2m\hbar\omega(q_n)}}\left[m\omega(q_n)\,y(q_n) + i p(-q_n)\right]}$$
$$\boxed{\omega(q) = \sqrt{\frac{2k}{m}(1-\cos q)}}$$

$$(a(q_n))^+ = \frac{1}{\sqrt{2m\hbar\omega(q_n)}}\left[m\omega(q_n)\,y(-q_n) - i p(+q_n)\right] \quad \text{as } (y(q))^+ = y(-q) \text{ etc.}$$

$$\boxed{[a(q_n), a^+(q_m)] = \delta_{q_n,q_m} \qquad [a(q_n), a(q_m)] = 0}$$

So for every q_n we get harmonic oscillator creation / annihilation operators!

The Hamiltonian becomes

$$H = \frac{1}{2m}\sum_n p(-q_n)p(q_n) + k\sum_n (1-\cos q_n)\; y(q_n)y(-q_n)$$
$$= \boxed{\sum_n \hbar\omega(q_n)\left[a^+(q_n)a(q_n) + \frac{1}{2}\right]}$$ N independent harmonic oscillators.

Following the same steps as for a single harmonic oscillator we conclude that the ground state is

$$a(q_n)\,|0\rangle = 0 \quad \forall q_n \qquad E_0 = \sum_n \frac{\hbar\omega(q_n)}{2}$$

and excited states are

$$a^+(q_1)\cdots a^+(q_n)\,|0\rangle \qquad E = E_0 + \sum_{s=1}^{n}\omega(q_s).$$

<u>Now relate this to QFT:</u>

$$\mathcal{L} = \sum_{l=1}^{N} \frac{m}{2} \dot{y}_l^2 - \frac{k}{2}(y_l - y_{l+1})^2 = \sum_{l=1}^{N} p_l \dot{y}_l - H$$

(i) Introduce a lattice spacing a_0, which we want to take to zero

(ii) Introduce ϕ by $\qquad \phi(la_0) \equiv \frac{1}{\sqrt{a_0}} \hat{y}_l$

Then $\qquad \mathcal{L} = a_0 \sum_{l=1}^{N} \frac{m}{2}\left(\frac{\partial}{\partial t}\phi(la_0)\right)^2 - \frac{k}{2}\left[\phi((l+1)a_0) - \phi(la_0)\right]^2$

(iii) Now take the limit $\quad N \to \infty$, $a_0 \to 0$, $k \to \infty$ $\quad$ keeping $L = Na_0$ fixed, $\varkappa = ka_0^2$

$$la_0 \longrightarrow x \qquad\qquad \phi((l+1)a_0) - \phi(la_0) \longrightarrow a_0 \,\partial_x \phi(x)$$

$$a_0 \sum_l \longrightarrow \int dx$$

$$\phi(la_0) \longrightarrow \phi(x)$$

and we end up with $\quad \boxed{\mathcal{L} = \int_0^L dx \left[\frac{m}{2}(\partial_t\phi)^2 - \frac{\varkappa}{2}(\partial_x\phi)^2\right]}$

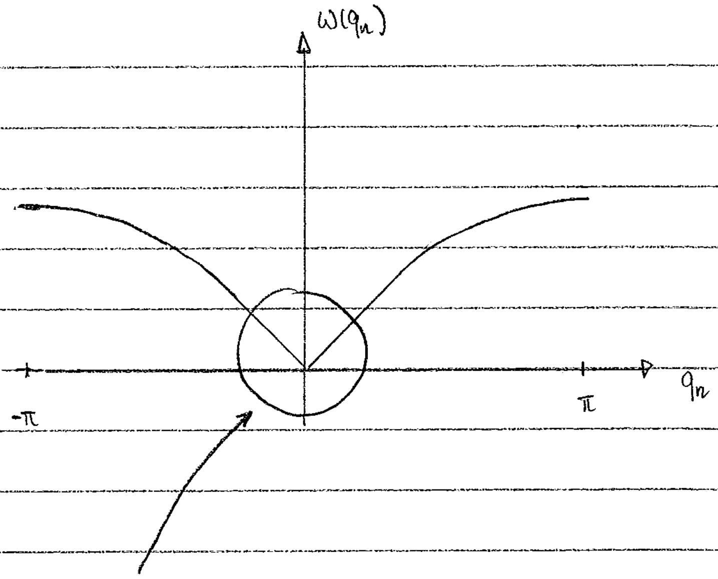
free massless scalar field

This corresponds to the low-energy limit of the lattice model:

$$\omega(q_n) = \sqrt{\frac{2k}{m}(1-\cos q_n)} \xrightarrow[q_n \to 0]{} \sqrt{\frac{k}{m} q_n^2} = \sqrt{\frac{ka_0^2}{m}\left(\frac{2\pi n}{Na_0}\right)^2} = \sqrt{\frac{\varkappa}{m}}\left|\frac{2\pi n}{L}\right| = v\,|\tilde{q}_n|$$

$\tilde{q}_n$ = momentum in the continuum limit

v = velocity —"—



scalar field theory description

8.1. SECOND QUANTIZATION

You already know from 2^{nd} year QM how to solve problems of independent particles:

$$H = \sum_{j=1}^{N} H_j$$ where H_j is the Hamiltonian of the j^{th} particle

e.g. $$H_j = -\frac{\hbar^2}{2m}\vec{\nabla}_j^2 + V(\vec{r}_j)$$

Step 1: Solve the single-particle problem

$$H_j |\phi_\ell\rangle = E_\ell |\phi_\ell\rangle$$

The corresponding wave-fns are $\phi_\ell(\vec{r}_j)$. They form an orthonormal basis

$$\delta_{\ell\ell'} = \langle\phi_\ell|\phi_{\ell'}\rangle = \int d^D\vec{r}_j \; \phi_\ell^*(\vec{r}_j)\,\phi_{\ell'}(\vec{r}_j)$$

Step 2: Form N-particle wave-functions as products

$$\left(\sum_{j=1}^{N} H_j\right)\phi_{\ell_1}(\vec{r}_1)\cdots\phi_{\ell_N}(\vec{r}_N) = \left(E_{\ell_1}+\ldots+E_{\ell_N}\right)\phi_{\ell_1}(\vec{r}_1)\cdots\phi_{\ell_N}(\vec{r}_N)$$

The corresponding states are $|\ell_1\rangle\otimes|\ell_2\rangle\otimes\ldots\otimes|\ell_N\rangle$

Step 3: Impose the appropriate exchange symmetry for identical particles

e.g. $$\frac{1}{\sqrt{2}}\left[\phi_\ell(\vec{r}_1)\,\phi_{\ell'}(\vec{r}_2) \pm \phi_\ell(\vec{r}_2)\,\phi_{\ell'}(\vec{r}_1)\right] = \Psi_\pm(\vec{r}_1,\vec{r}_2)$$

Generally we require $\Psi(\ldots \vec{r}_i \ldots \vec{r}_j \ldots) = \pm \Psi(\ldots \vec{r}_j \ldots \vec{r}_i \ldots)$

$+$ bosons

$-$ fermions

$$\Rightarrow \quad \boxed{\Psi_{l_1 \ldots l_N}(\vec{r}_1 \ldots \vec{r}_N) = \mathcal{N} \sum_{\substack{\text{all} \\ \text{permutations} \\ P=(P_1 \ldots P_N) \text{ of } (1 \ldots N)}} (\pm 1)^{|P|} \, \phi_{l_{P_1}}(\vec{r}_1) \cdots \phi_{l_{P_N}}(\vec{r}_N)}$$

$|P|$ = # of pair exchanges required to reduce $(P_1 \ldots P_N)$ to $(1 \ldots N)$.

and the normalization $\mathcal{N} = \dfrac{1}{\sqrt{N!\, n_1!\, n_2! \ldots}}$

where n_j = # of times j occurs in $\{l_1, \ldots, l_N\}$.

Corresponding state: $\boxed{|l_1 \ldots l_N\rangle = \mathcal{N} \sum_P (\pm 1)^{|P|} \, |l_{P_1}\rangle \otimes \ldots \otimes |l_{P_N}\rangle}$

For fermions the wave-fns are "Slater determinants"

$$\Psi_{l_1 \ldots l_N}(\vec{r}_1 \ldots \vec{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_{l_1}(\vec{r}_1) & \cdots & \phi_{l_1}(\vec{r}_N) \\ \vdots & & \\ \phi_{l_N}(\vec{r}_1) & \cdots & \phi_{l_N}(\vec{r}_N) \end{vmatrix}$$

By definition we have $|l_{Q_1} \ldots l_{Q_N}\rangle = \pm |l_1 \ldots l_N\rangle$ for any permutation Q of $(1 \ldots N)$. As the overall sign of the state does not matter, we can therefore oBdA choose them as

$$|\underbrace{1 \ldots 1}_{n_1} \underbrace{2 \ldots 2}_{n_2} \underbrace{3 \ldots 3}_{n_3} 4 \ldots\rangle \equiv |n_1 n_2 n_3 \ldots\rangle$$

For fermions $n_j = 0, 1$ only (Pauli principle)

This is called the <u>occupation number representation</u>.
<u>By construction</u> $\{|n_1 n_2 n_3 \ldots\rangle \mid \sum_j n_j = N\}$ form a <u>basis</u> of our N-particle problem.

We now want to allow the particle # N to <u>vary</u>. The motivation is that experimental probes like photoemission change particle number and we want to be able to describe these.

$\longrightarrow$ The resulting space of states is called <u>Fock space</u>

① the state with zero particles is denoted by $|0\rangle$

② N-particle states are $|n_1 n_2 \ldots\rangle$ with $\sum_{j=1}^{N} n_j = N$

Having a space of states we can now <u>define</u> <u>operators</u>

- creation operators of particles with quantum numbers l c_l^+:

$$\boxed{c_l^+ |n_1 n_2 \ldots n_l \ldots\rangle \equiv \begin{cases} \sqrt{n_l+1}\ (\pm 1)^{S_l}\ |n_1 n_2 \ldots n_l+1 \ldots\rangle \\ 0 \quad \text{if } n_l = 1 \text{ for fermions} \end{cases}} \qquad (*)$$

$$S_l = \sum_{j=1}^{l-1} n_j$$

- the corresponding annihilation op. is

$$\boxed{c_l |n_1 \ldots\rangle = \sqrt{n_l}\ (\pm 1)^{S_l}\ |n_1 \ldots n_l - 1 \ldots\rangle}$$

Follows from $(*)$ by $\langle n_1' n_2' \ldots | c_l^+ | n_1 \ldots\rangle^* = \langle n_1 n_2 \ldots | c_l | n_1' n_2' \ldots\rangle$

Canonical (anti) commutation relations:

$$[c_\ell, c_{\ell'}] = 0 = [c_\ell^+, c_{\ell'}^+] \qquad [c_\ell, c_{\ell'}^+] = \delta_{\ell\ell'} \qquad \underline{\text{BOSONS}}$$

$$\{c_\ell, c_{\ell'}\} = 0 = \{c_\ell^+, c_{\ell'}^+\} \qquad \{c_\ell, c_{\ell'}^+\} = \delta_{\ell\ell'} \qquad \underline{\text{FERMIONS}}$$

Let us see how to prove these: take fermions and $\ell' > \ell$

$$c_\ell^+ c_{\ell'} |n_1 \dots\rangle = c_\ell^+ \sqrt{n_{\ell'}}\, (-1)^{\sum_{j=1}^{\ell'-1} n_j} |\dots n_{\ell'}-1 \dots\rangle$$

$$= \sqrt{n_\ell+1}\sqrt{n_{\ell'}}\, (-1)^{\sum_{j=\ell+1}^{\ell'-1} n_j} |\dots n_\ell+1 \dots n_{\ell'}-1 \dots\rangle$$

$$c_{\ell'} c_\ell^+ |n_1 \dots\rangle = \sqrt{n_{\ell'}}\sqrt{n_\ell+1}\, (-1)^{1+\sum_{j=\ell+1}^{\ell'-1} n_j} |\dots n_\ell+1 \dots n_{\ell'}-1 \dots\rangle$$

$$\Rightarrow (c_\ell^+ c_{\ell'} + c_{\ell'} c_\ell^+) |\text{any state}\rangle = 0 \qquad \ell' > \ell$$

$$\Rightarrow \{c_\ell^+, c_{\ell'}\} = 0 \quad \text{if } \ell' > \ell \qquad (\ell < \ell' \text{ similarly})$$

$\underline{\ell = \ell'}$:

$$c_\ell^+ c_\ell |n_1 \dots\rangle = \sqrt{n_\ell}\, c_\ell^+ (-1)^{\sum_{j=1}^{\ell-1} n_j} |\dots n_\ell-1 \dots\rangle$$

$$= n_\ell |\dots n_\ell \dots\rangle$$

$$c_\ell c_\ell^+ |n_1 \dots\rangle = \begin{cases} \sqrt{n_\ell+1}\, c_\ell\, (-1)^{\sum_{j=1}^{\ell-1} n_j} |\dots n_\ell+1 \dots\rangle & n_\ell = 0 \\ 0 & n_\ell = 1 \end{cases}$$

$$= \begin{cases} |\dots n_\ell \dots\rangle & n_\ell = 0 \\ 0 & n_\ell = 1 \end{cases}$$

$$\Rightarrow \{c_\ell^+, c_\ell\} |\text{state}\rangle = |\text{state}\rangle$$

BASIS OF THE FOCK SPACE

STATE WITHOUT PARTICLES : $|0\rangle$

SINGLE-PARTICLE STATES : $c_\ell^+|0\rangle = |0\ldots 0\underset{\substack{\uparrow\\ \ell}}{1}0\ldots\rangle$

TWO-PARTICLE STATES : $\underset{\substack{\uparrow\\ \text{normalization}}}{N_2}\, c_{\ell_1}^+ c_{\ell_2}^+ |0\rangle$ with $\begin{cases} \ell_1 < \ell_2 & \text{for fermions} \\ \ell_1 \le \ell_2 & \text{for bosons} \end{cases}$

N-PARTICLE STATES : $$\boxed{\mathcal{N}_N \prod_{j=1}^{N} c_{\ell_j}^+ |0\rangle \quad \text{with} \begin{cases} \ell_1 < \ell_2 < \ldots < \ell_N & \text{fermions} \\ \ell_1 \le \ell_2 \le \ldots \le \ell_N & \text{bosons} \end{cases}}$$

This provides a basis of states for the Fock Space.

CHANGE OF BASIS

The Fock-Space is built from a given basis of single-particle states :

$$\boxed{\begin{matrix}\text{single-particle}\\ \text{states } |\ell\rangle\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{N-particle states}\\ |n_1 n_2 \ldots\rangle \quad \sum_{j=1} n_j = N\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{Fock}\\ \text{Space}\end{matrix}}$$

You know from QM that it is often useful to switch from one basis to another, e.g. from energy eigenstates to momentum eigenstates

This is achieved by a unitary transformation

$$\{|\ell\rangle\} \longrightarrow \{|\alpha\rangle\}$$

$$|\alpha\rangle = \sum_{\ell} \underbrace{\langle \ell|\alpha\rangle}_{U_{\ell\alpha}} |\ell\rangle$$

$$\sum_{\alpha} U_{\ell\alpha} U^{+}_{\alpha m} = \sum_{\alpha} \langle \ell|\alpha\rangle \langle \alpha|m\rangle = \delta_{\ell m} .$$

How does this work for the Fock Space ?

We know that $|\ell\rangle = c^{+}_{\ell} |0\rangle$
$|\alpha\rangle = d^{+}_{\alpha} |0\rangle$

But $|\alpha\rangle = \sum_{\ell} U_{\ell\alpha} |\ell\rangle = \sum_{\ell} U_{\ell\alpha} c^{+}_{\ell} |0\rangle$

This suggests to take $\boxed{d^{+}_{\alpha} = \sum_{\ell} U_{\ell\alpha} c^{+}_{\ell}}$, and indeed this gives the correct transformation for N-particle states.

Taking the h.c. we obtain $\boxed{d_{\alpha} = \sum_{\ell} U^{+}_{\alpha\ell} c_{\ell}}$

(Anti) Commutation Relations :

fermions: $$\{ d_{\alpha}, d^{+}_{\beta} \} = \sum_{\ell,\ell'} U^{+}_{\alpha\ell} U_{\ell'\beta} \overbrace{\{ c_{\ell}, c^{+}_{\ell'} \}}^{=\delta_{\ell\ell'}}$$

$$= \sum_{\ell} U^{+}_{\alpha\ell} U_{\ell\beta} = (U^{+}U)_{\alpha\beta}$$

$$= \delta_{\alpha\beta}$$

→ So the d's have the same (anti) commutation relations as the c's !

2^{nd} QUANTIZED FORM OF OPERATORS

So far we only know how creation/annihilation operators act on the Fock Space.
How about other operators such as H, P, L etc ?

(i) The simplest hermitian operators we can form from $c_\ell, c_\ell^\dagger$ are the occupation number operators

$$\hat{n}_\ell \equiv c_\ell^\dagger c_\ell$$

From the definition of $c_\ell^\dagger, c_\ell$ it follows immediately that

$$\hat{n}_\ell |n_1 n_2 \ldots\rangle = n_\ell |n_1 n_2 \ldots\rangle$$

(ii) General single-particle operators are of the form

$$\hat{O} = \sum_j \hat{O}_j$$

where $\hat{O}_j$ acts only on the j^{th} particle

Examples:
$$\hat{T} = \sum_{j=1}^{N} \frac{\hat{p}_j^2}{2m} \quad \text{kinetic energy op}$$
$$\hat{V} = \sum_{j=1}^{N} V(\hat{x}_j) \quad \text{potential energy op}$$

Trick: Consider the Fock Space built from eigenstates of $\hat{O}_j$

$$\hat{O}_j |\ell\rangle = \lambda_\ell |\ell\rangle$$

Then
$$\hat{O}\, c_\ell^\dagger |0\rangle = \lambda_\ell\, c_\ell^\dagger |0\rangle$$
$$\hat{O}\, c_{\ell_1}^\dagger c_{\ell_2}^\dagger |0\rangle = (\lambda_{\ell_1} + \lambda_{\ell_2})\, c_{\ell_1}^\dagger c_{\ell_2}^\dagger |0\rangle$$

$$\hat{O} \prod_{j=1}^{N} c_{\ell_j}^{+} |0\rangle = \underbrace{\left(\sum_{j=1}^{N} \lambda_{\ell_j} \right)}_{} \prod_{k=1}^{N} c_{\ell_k}^{+} |0\rangle$$

$\equiv \sum_{\ell} \lambda_\ell n_\ell$ where n_ℓ are the occupation numbers

This implies that $\boxed{\hat{O} = \sum_{\ell} \lambda_\ell \hat{n}_\ell}$ because $\{|n_1 n_2 \ldots\rangle\}$ is a basis.

Rewrite this as $\hat{O} = \sum_{\ell} \langle \ell | \hat{O} | \ell \rangle c_\ell^{+} c_\ell = \sum_{\ell,\ell'} \langle \ell' | \hat{O} | \ell \rangle c_{\ell'}^{+} c_\ell$.

Now we can express $\hat{O}$ is other bases $\{|\alpha\rangle\}$ using $|\ell\rangle = \sum_{\alpha} U_{\alpha\ell}^{+} |\alpha\rangle$

$$c_\ell^{+} = \sum_{\alpha} U_{\alpha\ell}^{+} d_\alpha^{+} \quad :$$

$$\hat{O} = \sum_{\ell,\ell'} \sum_{\alpha,\beta} \langle \ell' | \hat{O} | \ell \rangle U_{\alpha\ell'}^{+} d_\alpha^{+} U_{\ell\beta} d_\beta$$

$$= \sum_{\alpha,\beta} \Big(\underbrace{\langle \ell' | U_{\alpha\ell'}^{+}}_{\langle\alpha|} \Big) \hat{O} \Big(\underbrace{U_{\ell\beta} |\ell\rangle}_{|\beta\rangle} \Big) d_\alpha^{+} d_\beta$$

i.e. $\boxed{\hat{O} = \sum_{\alpha,\beta} \langle \alpha | \hat{O} | \beta \rangle d_\alpha^{+} d_\beta}$

EXAMPLES OF SINGLE-PARTICLE OPERATORS

① MOMENTUM OPERATOR

(i) IN SINGLE-PARTICLE BASIS OF MTM EIGENSTATES

$$\hat{\vec{P}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle \qquad \langle\vec{k}|\vec{p}\rangle = (2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{p})$$

$$\boxed{\hat{\vec{P}} = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\vec{p}\,c^{\dagger}(\vec{p})\,c(\vec{p})}$$

WHERE $\boxed{[\,c(\vec{p}),\,c^{\dagger}(\vec{k})\,\} = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{k})}$

~~COMMUTATOR FOR BOSONS~~
ANTICOMM. FOR FERMIONS

(ii) IN SINGLE-PARTICLE BASIS OF POSITION EIGENSTATES

$$\hat{\vec{X}}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}\rangle \qquad \langle\vec{x}|\vec{x}'\rangle = \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\hat{\vec{P}} = \int d^3\vec{x}\,d^3\vec{x}'\,\langle\vec{x}'|\hat{\vec{P}}|\vec{x}\rangle\,c^{\dagger}(\vec{x}')\,c(\vec{x})$$

$$\langle\vec{x}'|\hat{\vec{P}}|\vec{x}\rangle = -i\,\vec{\nabla}_{x'}\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$\Rightarrow\ \boxed{\hat{\vec{P}} = \int d^3\vec{x}\,d^3\vec{x}'\,(-i)\vec{\nabla}_{x'}\delta^{(3)}(\vec{x}-\vec{x}')\,c^{\dagger}(\vec{x}')\,c(\vec{x}) = \int d^3\vec{x}\;c^{\dagger}(\vec{x})(-i\vec{\nabla})c(\vec{x})}$$

② SINGLE PARTICLE HAMILTONIAN

$$H = \sum_{j=1}^{N} \frac{\hat{\vec{p}}_j^{\,2}}{2m} + V(\hat{\vec{x}}_j)$$

(i) POSITION REPRESENTATION

$$H = \int d^3\vec{x}\,d^3\vec{x}'\,\langle\vec{x}'|H|\vec{x}\rangle\,c^{\dagger}(\vec{x}')\,c(\vec{x})$$

$$\langle\vec{x}'|\hat{V}|\vec{x}\rangle = V(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{x}') \qquad \langle\vec{x}'|\hat{\vec{p}}^{\,2}|x\rangle = -\nabla^2\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$\rightarrow \quad \boxed{H = \int d^3\vec{x} \;\; c^{+}(\vec{x}) \left[\frac{-\vec{\nabla}^2}{2m} + V(\vec{x}) \right] c(\vec{x})}$$

(ii) MTM REPRESENTATION

$$\langle \vec{p}' | \hat{V} | \vec{p} \rangle = \int d^3\vec{x} \, d^3\vec{x}' \;\; \langle \vec{p}' | \vec{x}' \rangle \langle \vec{x}' | \hat{V} | \vec{x} \rangle \langle \vec{x} | \vec{p} \rangle$$

But $\langle \vec{x} | \vec{p} \rangle = e^{i\vec{p}\cdot\vec{x}}$ and so

$$\langle \vec{p}' | \hat{V} | \vec{p} \rangle = \int d^3\vec{x} \, d^3\vec{x}' \;\; \underbrace{V(\vec{x}) \; \delta^{(3)}(\vec{x}-\vec{x}')}_{\langle \vec{x}' | \hat{V} | \vec{x} \rangle} \;\; e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{x}'}$$

$$= \int d^3\vec{x} \;\; V(\vec{x}) \, e^{i(\vec{p}-\vec{p}')\cdot\vec{x}} \;\; \equiv \;\; \tilde{V}(\vec{p}-\vec{p}')$$

$$\rightarrow \quad \boxed{H = \int \frac{d^3\vec{p}}{(2\pi)^3} \; \frac{\vec{p}^{\,2}}{2m} \; a^{+}(\vec{p}) \, a(\vec{p}) + \int \frac{d^3\vec{p} \, d^3\vec{p}'}{(2\pi)^6} \; \tilde{V}(\vec{p}-\vec{p}') \; a^{+}(\vec{p}') \, a(\vec{p})}$$

(iii) BASIS OF HAMILTONIAN EIGENSTATES

$$H |\ell\rangle = E_\ell |\ell\rangle \qquad\qquad |\ell\rangle = C_\ell^{+} |0\rangle$$

$$\boxed{\hat{H} = \sum_\ell E_\ell \; C_\ell^{+} C_\ell}$$

TWO-PARTICLE OPERATORS

THESE ARE OPERATORS that act on two particles at a time.
A good example is the interaction potential between 2 particles at positions $\vec{r}_1$ and $\vec{r}_2$:

$$V(\hat{\vec{r}}_1, \hat{\vec{r}}_2)$$

FOR N PARTICLES we want to consider $\hat{V} = \sum_{i<j}^{N} V(\hat{\vec{r}}_i, \hat{\vec{r}}_j)$

In 2nd Quantization is a basis of position eigenstates this is

$$\boxed{\hat{V} = \frac{1}{2} \int d^3\vec{r} \int d^3\vec{r}\,' \; c^\dagger(\vec{r})\, c^\dagger(\vec{r}\,')\; V(\vec{r},\vec{r}\,')\; c(\vec{r}\,')\, c(\vec{r})} \qquad (1)$$

Using our formula for basis transformations this gives in a general basis

$$\boxed{\hat{V} = \sum_{\ell\ell' m m'} \langle \ell \ell' | V | m m' \rangle \; c^\dagger_\ell c^\dagger_{\ell'} c_m c_{m'}} \qquad (2)$$

where $\quad |\ell\ell'\rangle = \frac{1}{\sqrt{2}} \Big[|\ell\rangle \otimes |\ell'\rangle \pm |\ell'\rangle \otimes |\ell\rangle \Big] \quad$ (for $\ell \neq \ell'$)

Eqn (2) generalizes to arbitrary 2-particle operators $\hat{O}$.

DERIVATION OF (1) :

go back to our original representation of N-particle states :

$$|\vec{r}_1 \ldots \vec{r}_N\rangle = \frac{1}{\sqrt{N! n_1! n_2! \ldots}} \sum_P (-1)^{|P|} |\vec{r}_1\rangle \otimes |\vec{r}_2\rangle \otimes \ldots \otimes |\vec{r}_N\rangle$$

Then $\hat{V} |\vec{r}_1 \ldots \vec{r}_N\rangle = \sum_{i<j} V(\vec{r}_i, \vec{r}_j) |\vec{r}_1 \ldots \vec{r}_N\rangle = \frac{1}{2} \sum_{i \neq j} V(\vec{r}_i, \vec{r}_j) |\vec{r}_1 \ldots \vec{r}_N\rangle$

On the other hand we know that

$$|\vec{r}_1 \ldots \vec{r}_N\rangle = \prod_{j=1}^{N} c^+(\vec{r}_j) |0\rangle$$

Now consider $c(\vec{r}) |\vec{r}_1 \ldots \vec{r}_N\rangle = c(\vec{r}) \prod_{j=1}^{N} c^+(\vec{r}_j) |0\rangle$

Because $c(\vec{r})|0\rangle = 0$ this is equal to $[c(\vec{r}), \prod_{j=1}^{N} c^+(\vec{r}_j)\} |0\rangle$

Let us evaluate $\mathcal{C} = \{c(\vec{r}), \prod_{j=1}^{N} c^+(\vec{r}_j)\}$ in the fermionic case :

$$\begin{aligned} \mathcal{C} = & \{c(\vec{r}), c^+(\vec{r}_1)\} c^+(\vec{r}_2) \cdots c^+(\vec{r}_N) - c^+(\vec{r}_1) \{c(\vec{r}), c^+(\vec{r}_2)\} c^+(\vec{r}_3) \cdots c^+(\vec{r}_N) \\ & + c^+(\vec{r}_1) c^+(\vec{r}_2) \{c(\vec{r}), c^+(\vec{r}_3)\} c^+(\vec{r}_4) \cdots c^+(\vec{r}_N) - \ldots + \prod_{j=1}^{N-1} c^+(\vec{r}_j) \{c(\vec{r}), c^+(\vec{r}_N)\} \end{aligned}$$

$$= \sum_{n=1}^{N} (-1)^{n-1} \delta^{(3)}(\vec{r} - \vec{r}_n) \prod_{\substack{j=1 \\ j \neq n}}^{N} c^+(\vec{r}_j) |0\rangle$$

So $c(\vec{r}) |\vec{r}_1 \ldots \vec{r}_N\rangle = \sum_{n=1}^{N} (-1)^{n-1} \delta^{(3)}(\vec{r} - \vec{r}_n) |\vec{r}_1 \ldots \overset{\text{missing}}{\widehat{\vec{r}_n}} \ldots \vec{r}_N\rangle$

Hence $\underbrace{c^+(\vec{r}\,') c(\vec{r}\,')}_{\text{number op.}} c(\vec{r}) |\vec{r}_1 \ldots \vec{r}_N\rangle = \sum_{n=1}^{N} (-1)^{n-1} \delta^{(3)}(\vec{r} - \vec{r}_n) \sum_{\substack{m=1 \\ m \neq n}}^{N} \delta(\vec{r}\,' - \vec{r}_m) |\vec{r}_1 \ldots \widehat{\vec{r}_n} \ldots \vec{r}_N\rangle$

and finally $\boxed{c^+(\vec{r}) c^+(\vec{r}\,') c(\vec{r}\,') c(\vec{r}) |\vec{r}_1 \ldots \vec{r}_N\rangle = \sum_{n \neq m = 1}^{N} \delta^{(3)}(\vec{r} - \vec{r}_n) \delta^{(3)}(\vec{r}\,' - \vec{r}_m) |\vec{r}_1 \ldots \vec{r}_N\rangle}$

This implies that

$$\frac{1}{2}\int d^3\vec{r}\, d^3\vec{r}\,'\; V(\vec{r},\vec{r}\,')\; c^+(\vec{r})\, c^+(\vec{r}\,')\, c(\vec{r}\,')\, c(\vec{r})\quad |\vec{r}_1 \dots \vec{r}_N\rangle$$

$$= \frac{1}{2}\sum_{n\neq m} V(\vec{r}_n,\vec{r}_m)\; |\vec{r}_1 \dots \vec{r}_N\rangle .$$

As $\{|\vec{r}_1 \dots \vec{r}_N\rangle\}$ forms a basis this establishes that

$$\boxed{\hat{V} = \frac{1}{2}\int d^3\vec{r}\, d^3\vec{r}\,'\; V(\vec{r},\vec{r}\,')\; c^+(\vec{r})\, c^+(\vec{r}\,')\, c(\vec{r}\,')\, c(\vec{r})}$$