J. Fluid Mech. (1984), *vol.* 139, *pp.* 261–290
Printed in Great Britain

Calculation of the resistance and mobility functions for two unequal rigid spheres in low-Reynolds-number flow

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(Received 26 July 1982 and in revised form 15 August 1983)

Two unequal rigid spheres are immersed in unbounded fluid and are acted on by externally applied forces and couples. The Reynolds number of the flow around them is assumed to be small, with the consequence that the hydrodynamic interactions between the spheres can be described by a set of linear relations between, on the one hand, the forces and couples exerted by the spheres on the fluid and, on the other, the translational and rotational velocities of the spheres. These relations may be represented completely by either a set of 10 resistance functions or a set of 10 mobility functions. When non-dimensionalized, each function depends on two variables, the non-dimensionalized centre-to-centre separation s and the ratio of the spheres' radii λ. Two expressions are given for each function, one a power series in s^{-1} and the other an asymptotic expression valid when the spheres are close to touching.

1. Introduction

Comprehensive information about the interaction between two unequal spheres in low-Reynolds-number flow is needed in many studies of the properties of suspensions of small particles in fluid (Batchelor 1974; Jeffrey & Acrivos 1976). The term 'interaction' refers in this context to the relations that exist between the force, the couple and the stresslet that each sphere exerts on the fluid, the ambient flow and the velocity and angular velocity of each sphere. Although a large amount of data on these functions is available in the literature, it is not comprehensive enough to meet the demands of recent applications, nor is it in the most convenient form (Batchelor 1976, 1982). The insufficiency of the data is partly due simply to the fact that a large number of functions must be studied, but more significantly it is due to the amount of detail that is required in the study of each function. For example, the functions have singularities which are awkward to resolve from numerical tabulation, but which require special attention because of the critical roles they play in calculations of the properties of suspensions. Among the functions defined in the literature to describe the hydrodynamic interactions between particles, we study here the resistance functions defined by Brenner & O'Neill (1972) and the mobility functions defined by Batchelor (1976, 1982).

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The need for comprehensiveness, together with the availability of modern computers, influences our attitude toward numerical calculations. Our main consideration must be – aside from the fact that a complete tabulation of these functions of two variables would take many pages – that the interaction functions are required for use as components in further sizeable calculations, which will inevitably also be performed on a computer. Therefore, rather than directing our efforts entirely towards a tabulation of the functions, we have chosen to establish efficient ways in which computer subroutines can be written to evaluate them for any specified values of their arguments. We will produce tables of numbers, but the main aim of the investigation is the definition of efficient computational procedures. A review of the various methods for solving low-Reynolds-number flow around two spheres is given in §2.

A precise definition of the interactions we shall be studying is as follows. Two rigid spheres, labelled sphere 1 and sphere 2, are immersed in infinite fluid whose velocity in the absence of the spheres would be the ambient velocity $\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{U}_0 + \boldsymbol{\Omega} \wedge \boldsymbol{x}$. It is thus a superposition of a uniform stream and a rigid-body rotation. Sphere α has radius a_α and its centre is at $\boldsymbol{x}_\alpha$; it has angular velocity $\boldsymbol{\Omega}_\alpha$ and its centre has translational velocity $\boldsymbol{U}_\alpha$. The force $\boldsymbol{F}_\alpha$ that sphere α exerts on the fluid is given by

$$\boldsymbol{F}_\alpha = -\int_{A_\alpha} \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}A, \tag{1.1}$$

where A_α is the surface of the sphere, $\boldsymbol{n}$ is the outward normal to the surface and $\boldsymbol{\sigma}$ is the stress tensor. The couple exerted by the sphere on the fluid, calculated relative to the centre of the sphere, is

$$\boldsymbol{L}_\alpha = -\int_{A_\alpha} (\boldsymbol{x}-\boldsymbol{x}_\alpha) \wedge \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}A. \tag{1.2}$$

This is the antisymmetric part of the first moment of the surface stress expressed as a vector. Brenner & O'Neill (1972) work with the forces exerted by the fluid on the spheres, so their definitions are minus ours. The relations between the quantities $\boldsymbol{U}_\alpha$, $\boldsymbol{\Omega}_\alpha$, $\boldsymbol{F}_\alpha$, $\boldsymbol{L}_\alpha$, $\boldsymbol{U}_0$ and $\boldsymbol{\Omega}$ are the interactions we wish to study; they can be described using either a resistance matrix or a mobility matrix.

1.1. *The resistance matrix*

If the specified quantities are the velocities of the particles and the ambient flow, we can invoke the linearity of the Stokes equations to write

$$\begin{pmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{pmatrix} = \mu \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \tilde{\boldsymbol{B}}_{11} & \tilde{\boldsymbol{B}}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \tilde{\boldsymbol{B}}_{21} & \tilde{\boldsymbol{B}}_{22} \\ \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_1 - \boldsymbol{U}(\boldsymbol{x}_1) \\ \boldsymbol{U}_2 - \boldsymbol{U}(\boldsymbol{x}_2) \\ \boldsymbol{\Omega}_1 - \boldsymbol{\Omega} \\ \boldsymbol{\Omega}_2 - \boldsymbol{\Omega} \end{pmatrix}. \tag{1.3}$$

The square matrix is the resistance matrix; it contains second-rank tensors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, and it is understood that, when multiplying out the matrix equation, the appropriate contractions between the various vectors and tensors will be made. It might be noted that the tensors have different dimensions.

The elements of the resistance matrix obey a number of symmetry conditions, some of which apply to particles of any shape and some of which are consequences of the geometry of the two-sphere system. These properties are expressed most easily by adopting suffix notation, in which we write an arbitrary tensor, $\boldsymbol{P}_{\alpha\beta}$ say, as $P_{ij}^{(\alpha\beta)}$. The reciprocal theorem of Lorentz (1906) shows that the resistance matrix is symmetric (Brenner & O'Neill 1972); explicitly

$$A_{ij}^{(\alpha\beta)} = A_{ji}^{(\beta\alpha)}, \quad \tilde{B}_{ij}^{(\alpha\beta)} = B_{ji}^{(\beta\alpha)}, \quad C_{ij}^{(\alpha\beta)} = C_{ji}^{(\beta\alpha)}, \tag{1.4a,b,c}$$

Next come the properties that depend on the two-sphere geometry, which is specified entirely by the three quantities $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$, a_1 and a_2. We observe first that the sphere labels 1 and 2 can be interchanged, i.e. sphere α can be relabelled sphere $3-\alpha$, without altering the physical situation, except for the definition of $\boldsymbol{r}$. This implies that any tensor $\boldsymbol{P}$ of the resistance matrix obeys

$$\boldsymbol{P}_{\alpha\beta}(\boldsymbol{r}, a_1, a_2) = \boldsymbol{P}_{(3-\alpha)(3-\beta)}(-\boldsymbol{r}, a_2, a_1). \tag{1.5}$$

Each tensor in the matrix is axisymmetric about $\boldsymbol{r}$ and can be reduced to an expression containing at most two scalar functions (Brenner 1963, 1964). We relate each scalar function to the tensor with which it is associated through the following notation. For any tensor $\boldsymbol{P}_{\alpha\beta}$, we denote the scalar functions upon which it depends by $X^P_{\alpha\beta}$ and $Y^P_{\alpha\beta}$, if both are needed; in addition the letter X is reserved for those functions that can be determined, when the time comes, by solving axisymmetric two-sphere problems. If, then, $\boldsymbol{e} = \boldsymbol{r}/r$ is the unit vector along the line of centres, we can write

$$A^{(\alpha\beta)}_{ij} = X^A_{\alpha\beta} e_i e_j + Y^A_{\alpha\beta}(\delta_{ij} - e_i e_j), \tag{1.6a}$$

$$B^{(\alpha\beta)}_{ij} = \tilde{B}^{(\beta\alpha)}_{ji} = Y^B_{\alpha\beta} \epsilon_{ijk} e_k, \tag{1.6b}$$

$$C^{(\alpha\beta)}_{ij} = X^C_{\alpha\beta} e_i e_j + Y^C_{\alpha\beta}(\delta_{ij} - e_i e_j). \tag{1.6c}$$

Our final task is to non-dimensionalize the tensors (and thus the scalar functions) of the matrix, and then present the relations between the scalar functions that can be deduced by combining (1.4) with (1.5) and (1.6). The non-dimensionalization is an obvious generalization of the scheme used by Batchelor (1976) and is temporarily indicated by placing a circumflex over the non-dimensional quantity. We define

$$\hat{\boldsymbol{A}}_{\alpha\beta} = \frac{\boldsymbol{A}_{\alpha\beta}}{3\pi(a_\alpha + a_\beta)}, \quad \hat{\boldsymbol{B}}_{\alpha\beta} = \frac{\boldsymbol{B}_{\alpha\beta}}{\pi(a_\alpha + a_\beta)^2}, \tag{1.7a, b}$$

$$\hat{\boldsymbol{C}}_{\alpha\beta} = \frac{\boldsymbol{C}_{\alpha\beta}}{\pi(a_\alpha + a_\beta)^3}. \tag{1.7c}$$

The numerical factor 3 appears in (1.7*a*) so that the scalar functions derived from $\hat{\boldsymbol{A}}$ tend to 1 when the spheres are far apart; by coincidence no similar factor is needed in (1.7*c*). The non-dimensional functions depend on two non-dimensional variables derived from r, a_1 and a_2. We follow Batchelor (1982) and choose these to be

$$s = \frac{2r}{a_1 + a_2}, \quad \lambda = \frac{a_2}{a_1}. \tag{1.8a, b}$$

From now on we shall use only the non-dimensionalized functions and therefore not persist with the circumflex notation. When (1.4)–(1.7) are combined, we can show that

$$X^A_{\alpha\beta}(s, \lambda) = X^A_{\beta\alpha}(s, \lambda) = X^A_{(3-\alpha)(3-\beta)}(s, \lambda^{-1}), \tag{1.9a}$$

$$Y^A_{\alpha\beta}(s, \lambda) = Y^A_{\beta\alpha}(s, \lambda) = Y^A_{(3-\alpha)(3-\beta)}(s, \lambda^{-1}), \tag{1.9b}$$

$$Y^B_{\alpha\beta}(s, \lambda) = -Y^B_{(3-\alpha)(3-\beta)}(s, \lambda^{-1}), \tag{1.9c}$$

$$X^C_{\alpha\beta}(s, \lambda) = X^C_{\beta\alpha}(s, \lambda) = X^C_{(3-\alpha)(3-\beta)}(s, \lambda^{-1}), \tag{1.9d}$$

$$Y^C_{\alpha\beta}(s, \lambda) = Y^C_{\beta\alpha}(s, \lambda) = Y^C_{(3-\alpha)(3-\beta)}(s, \lambda^{-1}). \tag{1.9e}$$

Thus we have 10 independent non-dimensional scalar functions to tabulate for $2 \leqslant s \leqslant \infty$ and $0 \leqslant \lambda \leqslant \infty$. Alternatively we could tabulate 16 scalar functions in the range $2 \leqslant s \leqslant \infty$ and $0 \leqslant \lambda \leqslant 1$.

1.2. *The mobility matrix*

We now regard as the given quantities the forces and couples exerted by the particles on the fluid and write

$$\begin{pmatrix} \boldsymbol{U}_1-\boldsymbol{U}(\boldsymbol{x}_1) \\ \boldsymbol{U}_2-\boldsymbol{U}(\boldsymbol{x}_2) \\ \boldsymbol{\Omega}_1-\boldsymbol{\Omega} \\ \boldsymbol{\Omega}_2-\boldsymbol{\Omega} \end{pmatrix} = \mu^{-1} \begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \tilde{\boldsymbol{b}}_{11} & \tilde{\boldsymbol{b}}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \tilde{\boldsymbol{b}}_{21} & \tilde{\boldsymbol{b}}_{22} \\ \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{c}_{11} & \boldsymbol{c}_{12} \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{c}_{21} & \boldsymbol{c}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{pmatrix}. \tag{1.10}$$

The reciprocal theorem shows that this matrix is also symmetric, i.e.

$$a_{ij}^{(\alpha\beta)} = a_{ji}^{(\beta\alpha)}, \quad \tilde{b}_{ij}^{(\alpha\beta)} = b_{ji}^{(\beta\alpha)}, \quad c_{ij}^{(\alpha\beta)} = c_{ji}^{(\beta\alpha)}. \tag{1.11a,b,c}$$

The axisymmetry allows us to use the same decompositions of the tensors into scalar functions. Note that the scalar functions for the mobility matrix are indicated by lower-case letters and superscripts, in contrast to the capital letters used for the resistance functions:

$$a_{ij}^{(\alpha\beta)} = x_{\alpha\beta}^{a} e_i e_j + y_{\alpha\beta}^{a}(\delta_{ij} - e_i e_j), \tag{1.12a}$$

$$b_{ij}^{(\alpha\beta)} = y_{\alpha\beta}^{b} \epsilon_{ijk} e_k, \tag{1.12b}$$

$$c_{ij}^{(\alpha\beta)} = x_{\alpha\beta}^{c} e_i e_j + y_{\alpha\beta}^{c}(\delta_{ij} - e_i e_j). \tag{1.12c}$$

We non-dimensionalize the tensors in the mobility matrix as follows, again temporarily using circumflex notation:

$$\hat{\boldsymbol{a}}_{\alpha\beta} = 3\pi(a_\alpha + a_\beta)\, \boldsymbol{a}_{\alpha\beta}, \quad \hat{\boldsymbol{b}}_{\alpha\beta} = \pi(a_\alpha + a_\beta)^2\, \boldsymbol{b}_{\alpha\beta}, \tag{1.13a,b}$$

$$\hat{\boldsymbol{c}}_{\alpha\beta} = \pi(a_\alpha + a_\beta)^3\, \boldsymbol{c}_{\alpha\beta}. \tag{1.13c}$$

As before, the non-dimensional functions will be taken to depend on s and λ as defined in (1.8) and the circumflex will be taken as understood. Finally, relations between the scalar mobility functions that are analogous to (1.9) can be written down, showing that there are an additional 10 independent functions to be tabulated for $2 \leqslant s \leqslant \infty$ and $0 \leqslant \lambda \leqslant \infty$.

1.3. *Relations between the resistance and mobility functions*

It is obvious that the dimensional resistance and mobility matrices obey the equation

$$\begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \tilde{\boldsymbol{b}}_{11} & \tilde{\boldsymbol{b}}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \tilde{\boldsymbol{b}}_{21} & \tilde{\boldsymbol{b}}_{22} \\ \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{c}_{11} & \boldsymbol{c}_{12} \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{c}_{21} & \boldsymbol{c}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \tilde{\boldsymbol{B}}_{11} & \tilde{\boldsymbol{B}}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \tilde{\boldsymbol{B}}_{21} & \tilde{\boldsymbol{B}}_{22} \\ \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.14}$$

Using the decompositions into scalar functions, we separate this tensor matrix equation into three scalar matrix equations. This is possible because axisymmetric motions are not coupled to non-axisymmetric ones and because axisymmetric translation is not coupled to axisymmetric rotation. Thus the non-dimensionalized scalar functions obey

$$\begin{pmatrix} x_{11}^{a} & \dfrac{2}{1+\lambda} x_{12}^{a} \\ \dfrac{2}{1+\lambda} x_{12}^{a} & \dfrac{1}{\lambda} x_{22}^{a} \end{pmatrix} = \begin{pmatrix} X_{11}^{A} & \tfrac{1}{2}(1+\lambda)\, X_{12}^{A} \\ \tfrac{1}{2}(1+\lambda)\, X_{12}^{A} & \lambda X_{22}^{A} \end{pmatrix}^{-1}. \tag{1.15}$$

Next, $x^c_{\alpha\beta}$ and $X^C_{\alpha\beta}$ obey a similar equation with $\frac{1}{2}(1+\lambda)$ cubed. Finally

$$\begin{pmatrix} y^a_{11} & \dfrac{2}{1+\lambda}y^a_{12} & \frac{3}{2}y^b_{11} & \dfrac{6}{(1+\lambda)^2}y^b_{21} \\ \dfrac{2}{1+\lambda}y^a_{12} & \dfrac{1}{\lambda}y^a_{22} & \dfrac{6}{(1+\lambda)^2}y^b_{12} & \dfrac{3}{2\lambda^2}y^b_{22} \\ \frac{3}{2}y^b_{11} & \dfrac{6}{(1+\lambda)^2}y^b_{12} & \frac{3}{4}y^c_{11} & \dfrac{6}{(1+\lambda)^3}y^c_{12} \\ \dfrac{6}{(1+\lambda)^2}y^b_{21} & \dfrac{3}{2\lambda^2}y^b_{22} & \dfrac{6}{(1+\lambda)^3}y^c_{12} & \dfrac{3}{4\lambda^3}y^c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} Y^A_{11} & \frac{1}{2}(1+\lambda)\,Y^A_{12} & \frac{2}{3}Y^B_{11} & \frac{1}{6}(1+\lambda)^2\,Y^B_{21} \\ \frac{1}{2}(1+\lambda)\,Y^A_{12} & \lambda\,Y^A_{22} & \frac{1}{6}(1+\lambda)^2\,Y^B_{12} & \frac{2}{3}\lambda^2\,Y^B_{22} \\ \frac{2}{3}Y^B_{11} & \frac{1}{6}(1+\lambda)^2\,Y^B_{12} & \frac{4}{3}Y^C_{11} & \frac{1}{6}(1+\lambda)^3\,Y^C_{12} \\ \frac{1}{6}(1+\lambda)^2\,Y^B_{21} & \frac{2}{3}\lambda^2\,Y^B_{22} & \frac{1}{6}(1+\lambda)^3\,Y^C_{12} & \frac{4}{3}\lambda^3\,Y^C_{22} \end{pmatrix}^{-1}. \tag{1.16}$$

2. The solution of two-sphere problems

Several methods for solving the equations of low-Reynolds-number flow around two spheres have been developed over the years, and each has advantages and disadvantages. The methods include those using reflections (Happel & Brenner 1965), bispherical coordinates (O'Neill & Majumdar 1970*a*), tangent-sphere coordinates (Cooley & O'Neill 1969*b*), collocation methods (Ganatos, Pfeffer & Weinbaum 1978) and asymptotic methods (O'Neill & Stewartson 1967; Jeffrey 1982). For this paper, a development of the method of reflections is used; it is called twin multipole expansions and has been applied already to finding solutions of Laplace's equation (Jeffrey 1973). It is chosen because it is accurate, although not as inherently accurate as bispherical coordinates, and it produces results in a convenient form for further use in applications; also it can be combined easily with the results of the asymptotic methods.

We shall describe the method of twin multipole expansions using a general set of boundary conditions that will encompass all the special cases that must be considered later in this paper and in subsequent papers. Following Happel & Brenner (1965, figure 6–3.1), we take two sets of spherical polar coordinates $(\rho_\alpha, \theta_\alpha, \phi)$ with the difference that θ_2 here equals their $\pi - \Theta$; let the unit vectors in the coordinate directions be $\hat{\boldsymbol{\rho}}_\alpha$, $\hat{\boldsymbol{\theta}}_\alpha$ and $\hat{\boldsymbol{\phi}}$. Our choice has the advantage that the transformation rule for spherical harmonics becomes (Hobson 1931)

$$\left(\frac{a_\alpha}{\rho_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha, \phi) = \left(\frac{a_\alpha}{r}\right)^{n+1} \sum_{s=m}^{\infty} \binom{n+s}{s+m} \left(\frac{\rho_{3-\alpha}}{r}\right)^s Y_{ms}(\theta_{3-\alpha}, \phi), \tag{2.1}$$

where $Y_{mn}(\theta, \phi) = P^m_n(\cos\theta)\exp(im\phi)$ and r is the distance between the centres of the spheres. The factors of -1 that are removed from (2.1) appear, to a lesser extent, elsewhere, because it is not possible for both coordinate systems to be right-handed; we choose

$$\hat{\boldsymbol{\rho}}_\alpha \times \hat{\boldsymbol{\theta}}_\alpha = (-1)^{3-\alpha}\hat{\boldsymbol{\phi}}. \tag{2.2}$$

We use Lamb's general solution (Happel & Brenner 1965, §3.2) to write the pressure and velocity fields outside the spheres as

$$p = p^{(1)} + p^{(2)}, \tag{2.3a}$$

where

$$p^{(\alpha)} = \mu \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{1}{a_\alpha} p^{(\alpha)}_{mn} \left(\frac{a_\alpha}{\rho_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha, \phi). \tag{2.3b}$$

$$\boldsymbol{u} = \boldsymbol{u}^{(1)} + \boldsymbol{u}^{(2)}, \tag{2.4a}$$

where

$$\begin{aligned}\boldsymbol{u}^{(\alpha)} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \Bigg\{ &\boldsymbol{\nabla} \times \left[\boldsymbol{\rho}_\alpha q_{mn}^{(\alpha)} \left(\frac{a_\alpha}{\rho_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha, \phi) \right] + a_\alpha \boldsymbol{\nabla} \left[v_{mn}^{(\alpha)} \left(\frac{a_\alpha}{\rho_\alpha}\right)^{n+1} Y_{mn} \right] \\ &- \frac{n-2}{2n(2n-1)\,a_\alpha} \rho_\alpha^2 \boldsymbol{\nabla} \left[p_{mn}^{(\alpha)} \left(\frac{a_\alpha}{\rho_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha, \phi) \right] \\ &+ \frac{n+1}{n(2n-1)\,a_\alpha} \boldsymbol{\rho}_\alpha p_{mn}^{(\alpha)} \left(\frac{a_\alpha}{\rho_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha, \phi) \Bigg\}.\end{aligned} \tag{2.4b}$$

The coefficients $p_{mn}^{(\alpha)}$, $q_{mn}^{(\alpha)}$ and $v_{mn}^{(\alpha)}$ are functions only of r, have the dimensions of velocity and must be calculated from the boundary conditions.

We do not apply the boundary conditions directly to (2.4) but instead follow Happel & Brenner (1965, §3.2) in first constructing three scalar equations from the boundary conditions. We assume that the velocity on the surface of each sphere is specified as

$$\boldsymbol{u} = \boldsymbol{U}_\alpha(\theta_\alpha, \phi) \quad \text{on} \quad \rho_\alpha = a_\alpha. \tag{2.5}$$

We construct the three scalar quantities and assume that each can be expanded as a series of harmonics to obtain

$$\boldsymbol{u} \cdot \hat{\boldsymbol{\rho}}_\alpha = \boldsymbol{U}_\alpha \cdot \hat{\boldsymbol{\rho}}_\alpha = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \chi_{mn}^{(\alpha)} Y_{mn}(\theta_\alpha, \phi), \tag{2.6a}$$

$$\rho_\alpha \frac{\partial}{\partial \rho_\alpha} \boldsymbol{u} \cdot \hat{\boldsymbol{\rho}}_\alpha = -\rho_\alpha \boldsymbol{\nabla} \cdot \boldsymbol{U}_\alpha = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \psi_{mn}^{(\alpha)} Y_{mn}(\theta_\alpha, \phi), \tag{2.6b}$$

$$\boldsymbol{\rho}_\alpha \cdot \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\rho}_\alpha \cdot \boldsymbol{\nabla} \times \boldsymbol{U}_\alpha = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \omega_{mn}^{(\alpha)} Y_{mn}(\theta_\alpha, \phi). \tag{2.6c}$$

The coefficients in (2.4) can now be calculated in terms of χ, ψ and ω by using the following transformation rules.

To transform the solutions $p^{(\alpha)}$ and $\boldsymbol{u}^{(\alpha)}$ to the other coordinate system (which has the label $3-\alpha$), we use (2.1) together with

$$\boldsymbol{\rho}_\alpha = (\rho_{3-\alpha} - r\cos\theta_{3-\alpha})\,\hat{\boldsymbol{\rho}}_{3-\alpha} + r\sin\theta_{3-\alpha}\,\hat{\boldsymbol{\theta}}_{3-\alpha}, \tag{2.7a}$$

$$\rho_\alpha^2 = r^2 + \rho_{3-\alpha}^2 - 2r\rho_{3-\alpha}\cos\theta_{3-\alpha}. \tag{2.7b}$$

Doing this and equating coefficients of $Y_{mn}(\theta_\alpha, \phi)$, we obtain, using the notation

$$t_\alpha = a_\alpha / r, \tag{2.8}$$

the following equations linking the coefficients p_{mn}, v_{mn} and q_{mn} with ψ_{mn}, χ_{mn} and ω_{mn}:

$$\begin{aligned}(n+1)(2n+1)\,v_{mn}^{(\alpha)} - \tfrac{1}{2}(n+1)\,p_{mn}^{(\alpha)} + \frac{n}{2n+3} \sum_{s=m}^{\infty} \binom{n+s}{n+m} p_{ms}^{(3-\alpha)} t_\alpha^{n+1} t_{3-\alpha}^{s} \\ = \psi_{mn}^{(\alpha)} - (n-1)\,\chi_{mn}^{(\alpha)},\end{aligned} \tag{2.9a}$$

$$\begin{aligned}\frac{n+1}{2n-1} p_{mn}^{(\alpha)} + \sum_{s=m}^{\infty} \binom{n+s}{n+m} t_\alpha^{n-1} t_{3-\alpha}^{s} \Bigg[&(-1)^\alpha m(2n+1)\,\mathrm{i} q_{ms}^{(3-\alpha)} t_{3-\alpha} + n(2n+1)\,v_{ms}^{(3-1)} t_{3-\alpha}^2 \\ &+ \frac{2n+1}{2n-1} \frac{ns(n+s-2ns-2) - m^2(2ns-4s-4n+2)}{2s(2s-1)(n+s)} p_{ms}^{(3-\alpha)} + \tfrac{1}{2} n p_{ms}^{(3-\alpha)} t_\alpha^2 \Bigg] \\ = \psi_{mn}^{(\alpha)} + (n+2)\,\chi_{mn}^{(\alpha)},&\end{aligned} \tag{2.9b}$$

$$n(n+1)\,q_{mn}^{(\alpha)} + \sum_{s=m}^{\infty} \binom{n+s}{n+m} t_\alpha^{n} t_{3-\alpha}^{s} \left[-ns\,q_{ms}^{(3-\alpha)} t_{3-\alpha} + (-1)^\alpha \frac{m}{s} \mathrm{i} p_{ms}^{(3-\alpha)} \right] = \omega_{mn}^{(\alpha)}. \tag{2.9c}$$

These equations will be solved for each special case in later sections. The final general results we need are for the forces and couples exerted by the spheres.

We express the force and couple in a cartesian system $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{e}$, where $\boldsymbol{e} = \boldsymbol{r}/r$ was defined earlier and the $\boldsymbol{i}$-axis is chosen in the plane $\phi = 0$. Then from Happel & Brenner (1965; §3.2) we have

$$\begin{aligned} \boldsymbol{F}_\alpha &= 4\pi\mu\nabla[p_{m1}^{(\alpha)}\rho_\alpha^2 P_1^m(\cos\theta)] \\ &= 4\pi a_\alpha\mu[p_{01}^{(\alpha)}(-1)^{3-\alpha}\boldsymbol{e} - p_{11}^{(\alpha)}(\boldsymbol{i}+i\boldsymbol{j})], \end{aligned} \tag{2.10}$$

$$\begin{aligned} \boldsymbol{L}_\alpha &= 8\pi\mu\nabla[q_{m1}^{(\alpha)}\rho_\alpha^2 P_1^m(\cos\theta)] \\ &= 8\pi\mu a_\alpha^2[q_{01}^{(\alpha)}(-1)^{3-\alpha}\boldsymbol{e} - q_{11}^{(\alpha)}(\boldsymbol{i}+i\boldsymbol{j})]. \end{aligned} \tag{2.11}$$

The sign change from Happel & Brenner is the result of calculating the force on the fluid rather than the force on the sphere.

3. The resistance functions $X_{\alpha\beta}^A(s, \lambda)$

In proceeding to individual cases, we find that it is convenient to consider two particular problems. In the first problem, the velocities $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ of the spheres are along the line of centres and equal and opposite, i.e.

$$\boldsymbol{U}_1 = -\boldsymbol{U}_2 = U\boldsymbol{e}. \tag{3.1}$$

For the second problem, the spheres follow each other with equal velocities, i.e.

$$\boldsymbol{U}_1 = \boldsymbol{U}_2 = U\boldsymbol{e}. \tag{3.2}$$

In either case

$$\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = 0.$$

For the first problem, the quantities defined in (2.6) become

$$\chi_{mn}^{(\alpha)} = U\delta_{m0}\delta_{n1}, \quad \psi_{mn}^{(\alpha)} = \omega_{mn}^{(\alpha)} = 0. \tag{3.3a, b, c}$$

It is obvious that, in (2.9), only the coefficients for $m = 0$ will be non-zero. In addition, all the coefficients q_{mn} are zero. We expand the coefficients p_{0n} and v_{0n} as power series in r^{-1}, or, more correctly, double series in $t_\alpha = a_\alpha/r$:

$$p_{0n}^{(\alpha)} = \tfrac{3}{2}U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq}^{(\alpha)} t_\alpha^p t_{3-\alpha}^q, \tag{3.4}$$

$$v_{0n}^{(\alpha)} = \tfrac{3}{4}U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{2n+1} V_{npq}^{(\alpha)} t_\alpha^p t_{3-\alpha}^q. \tag{3.5}$$

It is clear from the initial conditions (3.3) and the form of the general recurrence relations that the coefficients will be the same for each sphere, allowing us to drop the label α. The recurrence relations for the pure numbers P_{npq} and V_{npq} are

$$P_{n00} = V_{n00} = \delta_{1n}, \tag{3.6), (3.7}$$

and

$$V_{npq} = P_{npq} - \frac{2n}{(n+1)(2n+3)} \sum_{s=1}^{q} \binom{n+s}{n} P_{s(q-s)(p-n-1)}, \tag{3.8}$$

$$\begin{aligned} P_{npq} = \sum_{s=1}^{q} \binom{n+s}{n} \bigg[& \frac{n(2n+1)(2ns-n-s+2)}{2(n+1)(2s-1)(n+s)} P_{s(q-s)(p-n+1)} \\ & - \frac{n(2n-1)}{2(n+1)} P_{s(q-s)(p-n-1)} - \frac{n(4n^2-1)}{2(n+1)(2s+1)} V_{s(q-s-2)(p-n+1)} \bigg]. \end{aligned} \tag{3.9}$$

Once we have calculated these coefficients, the force equation (2.10) gives us

$$X_{11}^A - \tfrac{1}{2}(1+\lambda)\, X_{12}^A = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{1pq}\, t_1^p\, t_2^q, \tag{3.10}$$

where the factor $\frac{1}{2}(1+\lambda)$ appears because of the non-dimensionalization. We shall change to the variables s and λ later.

We now turn to the second problem. The quantities χ, ψ and ω now become

$$\chi_{mn}^{(\alpha)} = (-1)^{3-\alpha}\,\delta_{m0}\,\delta_{n1}, \quad \psi_{mn}^{(\alpha)} = \omega_{mn}^{(\alpha)} = 0. \tag{3.11 a, b, c}$$

By substituting these conditions into (2.9), we can show that the solutions of the recurrence relations thus obtained can be related to the solutions P_{npq} and V_{npq} just defined in (3.6)–(3.9). In fact the coefficients solving the second problem can be written simply as $(-1)^{n+p+q+\alpha}\, P_{npq}$ and $(-1)^{n+p+q+\alpha}\, V_{npq}$. Thus we conclude that

$$X_{11}^A + \tfrac{1}{2}(1+\lambda)\, X_{12}^A = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^{p+q}\, P_{1pq}\, t_1^p\, t_2^q. \tag{3.12}$$

From (3.10) and (3.12) it is obvious that X_{11}^A is a series only of terms in which $p+q$ is even and X_{12}^A is a series only of terms in which $p+q$ is odd. This remarkable result is implicit in Faxen's law for a sphere, but it has not previously been pointed out. We are now in a position to give explicit results.

3.1. *Widely separated spheres*

We change to the preferred variables s and λ and write our functions as

$$X_{11}^A(s,\lambda) = \sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k} s^{-2k}, \tag{3.13}$$

$$X_{12}^A(s,\lambda) = \frac{-2}{1+\lambda}\sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1}\, s^{-2k-1}, \tag{3.14}$$

where
$$f_k(\lambda) = 2^k \sum_{q=0}^{k} P_{1(k-q)\,q}\,\lambda^q. \tag{3.15}$$

Explicitly,

$$f_0 = 1, \quad f_1 = 3\lambda,$$
$$f_2 = 9\lambda, \quad f_3 = -4\lambda + 27\lambda^2 - 4\lambda^3,$$
$$f_4 = -24\lambda + 81\lambda^2 + 36\lambda^3, \quad f_5 = 72\lambda^2 + 243\lambda^3 + 72\lambda^4,$$
$$f_6 = 16\lambda + 108\lambda^2 + 281\lambda^3 + 648\lambda^4 + 144\lambda^5,$$
$$f_7 = 288\lambda^2 + 1620\lambda^3 + 1515\lambda^4 + 1620\lambda^5 + 288\lambda^6,$$
$$f_8 = 576\lambda^2 + 4848\lambda^3 + 5409\lambda^4 + 4524\lambda^5 + 3888\lambda^6 + 576\lambda^7,$$
$$f_9 = 1152\lambda^2 + 9072\lambda^3 + 14752\lambda^4 + 26163\lambda^5 + 14752\lambda^6 + 9072\lambda^7 + 1152\lambda^8,$$
$$f_{10} = 2304\lambda^2 + 20736\lambda^3 + 42804\lambda^4 + 115849\lambda^5 + 76176\lambda^6 + 39264\lambda^7 + 20736\lambda^8 + 2304\lambda^9,$$
$$f_{11} = 4608\lambda^2 + 46656\lambda^3 + 108912\lambda^4 + 269100\lambda^5 + 319899\lambda^6 + 269100\lambda^7 + 108912\lambda^8 + 46656\lambda^9 + 4608\lambda^{10}.$$

We note that $f_{2k+1}(\lambda) = \lambda^{2k+2} f_{2k+1}(\lambda^{-1})$ in accordance with (1.9). These expressions agree with Happel & Brenner (1965, equations 6–3.51) and (6–3.52)).

3.2. *Nearly touching spheres*

We define a non-dimensional gap width by

$$\xi = \frac{r-a_1-a_2}{\frac{1}{2}(a_1+a_2)} = s-2. \tag{3.16}$$

Jeffrey (1982) has shown that for $\xi \ll 1$ and $\xi \ll \lambda$

$$X^A_{11} = g_1(\lambda)\,\xi^{-1}+g_2(\lambda)\ln\xi^{-1}+A^X_{11}(\lambda)+g_3(\lambda)\,\xi\ln\xi^{-1}+L^X_{11}(\lambda)\,\xi+O(\xi^2\ln\xi), \tag{3.17}$$

$$X^A_{12} = \frac{-2}{1+\lambda}[g_1(\lambda)\,\xi^{-1}+g_2(\lambda)\ln\xi^{-1}-\tfrac{1}{2}(1+\lambda)\,A^X_{12}(\lambda) \\ +g_3(\lambda)\,\xi\ln\xi^{-1}-\tfrac{1}{2}(1+\lambda)\,L^X_{12}(\lambda)\,\xi+O(\xi^2\ln\xi)], \tag{3.18}$$

where

$$g_1 = 2\lambda^2(1+\lambda)^{-3}, \quad g_2 = \tfrac{1}{5}\lambda(1+7\lambda+\lambda^2)\,(1+\lambda)^{-3}, \tag{3.19a,b}$$

$$g_3 = \tfrac{1}{42}(1+18\lambda-29\lambda^2+18\lambda^3+\lambda^4)\,(1+\lambda)^{-3}. \tag{3.19c}$$

Our aim now is to combine (3.17) and (3.18) with (3.13) and (3.14) to obtain alternative general expressions for the $X^A_{\alpha\beta}$ and explicit expressions for the $A^X_{\alpha\beta}$.

The process is a simple one: we write the singular terms as functions of s in such a way that their behaviour near $s=2$ is unchanged, but they tend to zero as $s\to\infty$ and they are odd or even functions of s as required. Obviously, several equivalent choices are possible; we chose

$$X^A_{11} = g_1(1-4s^{-2})^{-1}-g_2\ln(1-4s^{-2})-g_3(1-4s^{-2})\ln(1-4s^{-2})+f_0(\lambda)-g_1 \\ +\sum_{\substack{m=2\\ m\text{ even}}}^{\infty}\{2^{-m}(1+\lambda)^{-m}f_m(\lambda)-g_1-2m^{-1}g_2+4m^{-1}m_1^{-1}g_3\}\left(\frac{2}{s}\right)^m, \tag{3.20}$$

where $$m_1 = -2\delta_{m2}+(m-2)\,(1-\delta_{m2});$$

$$-\tfrac{1}{2}(1+\lambda)\,X^A_{12} = 2s^{-1}g_1(1-4s^{-2})^{-1}+g_2\ln\frac{s+2}{s-2}+g_3(1-4s^{-2})\ln\frac{s+2}{s-2}+4g_3\,s^{-1} \\ +\sum_{\substack{m=1\\ m\text{ odd}}}^{\infty}\{2^{-m}(1+\lambda)^{-m}f_m(\lambda)-g_1-2m^{-1}g_2+4m^{-1}m_1^{-1}g_3\}\left(\frac{2}{s}\right)^m. \tag{3.21}$$

To obtain expressions for the functions $A^X_{\alpha\beta}$, we expand the singular terms as series in ξ to recover the expressions (3.17) and (3.18). Thus

$$A^X_{11} = 1-\tfrac{1}{4}g_1+\sum_{\substack{m=2\\ m\text{ even}}}^{\infty}[2^{-m}(1+\lambda)^{-m}f_m-g_1-2m^{-1}g_2+4m^{-1}m_1^{-1}g_3], \tag{3.22}$$

$$-\tfrac{1}{2}(1+\lambda)\,A^X_{12} = \tfrac{1}{4}g_1+2g_2\ln 2+2g_3 \\ +\sum_{\substack{m=1\\ m\text{ odd}}}^{\infty}[2^{-m}(1+\lambda)^{-m}f_m-g_1-2m^{-1}g_2+4m^{-1}m_1^{-1}g_3]. \tag{3.23}$$

The tabulation of these functions is described in §3.4.

3.3. *Arbitrary separations*

The equations (3.13), (3.14) and (3.20), (3.21) give us two expressions for each function and both are valid for all separations. However, for numerical evaluation of the functions near $s=2$, it is obvious that (3.13) and (3.14) will be of little use because they represent the singular terms in an inefficient way, namely as an infinite series. This is true even if we consider a problem in which the singular terms appear to cancel out, such as the classical problem of the drag on two spheres following each other

Maximum power of s^{-1}	$A_{11}^X(1)$		$A_{12}^X(1)$	
	First form	Second form	First form	Second form
50	0.9963	0.99524	-0.3515	-0.35039
100	0.9960	0.99542	-0.3507	-0.35015
150	0.9957	0.99538	-0.3506	-0.35020
Cooley & O'Neill	—	0.99536	—	-0.35022

TABLE 1. Rate of convergence of the series $A_{\alpha\beta}^X(\lambda)$ for $\lambda = 1$. The series were written in two forms. The first form did not include the corrections for the known $\xi \ln \xi$ singularity and the second form did.

along their line of centres, first tackled by Smoluchowski (1911). This is because the convergence of the series is limited by a 'non-physical' pole at $s = -2$. We are able to extract this pole because of the fact that the series for X_{11}^A and X_{12}^A contain even and odd powers only. Thus the 'physical' pole at $s = 2$, which has been calculated using lubrication approximations, automatically implies the existence of the non-physical pole. Happel & Brenner (1965, p. 259) attempted to remove these same singularities from their power series, but, not having the lubrication results found only later, they did not know of the singular term in $\ln(s+2)$ and estimated the strength of the pole at $s = -2$ incorrectly.

3.4. *Numerical results*

We must first establish the rate of convergence of the series expressions derived above. Tabulations are available of $X_{11}^A + \frac{1}{2}(1+\lambda)\, X_{12}^A$ and $X_{11}^A - \frac{1}{2}(1+\lambda)\, X_{12}^A$ in Cooley & O'Neill (1969*a*, *b*) and of $A_{11}^X - \frac{1}{2}(1+\lambda)\, A_{12}^X + 2g_2 \ln \frac{1}{2}(1+\lambda)$ in Jeffrey (1982). It is clear, however, that, because the series expressions for $X_{\alpha\beta}^A$ are least accurate in the neighbourhood of $s = 2$, it is unnecessary to compare them with other published data for values of s away from 2. Further, the singular terms mask the contribution of the infinite series to $X_{11}^A - \frac{1}{2}(1+\lambda)\, X_{12}^A$ near $s = 2$. Therefore, we have compared in table 1 the present calculations of A_{11}^X and A_{12}^X with their values as deduced from Jeffrey (1982) and Cooley & O'Neill (1969*a*). The series in (3.22) and (3.23) were summed to $m = 50$, 100 and 150, both before and after the terms which rely on the function $g_3(\lambda)$ were included. This was done because the terms in g_3 are not essential to the definition of $A_{\alpha\beta}^X$, but they do increase the rate of convergence, as table 1 shows. Similar tables were constructed for other values of λ. In the limit $\lambda \to \infty$, we can deduce from Cooley & O'Neill (1969*a*) that $A_{11}^X \to -\frac{1}{5} \ln \frac{1}{2}(1+\lambda) + 0.97128$, with a similar result for A_{22}^X as $\lambda \to 0$. This singular behaviour causes the representations (3.22) and (3.23) to lose accuracy for extreme values of λ; in practical terms A_{11}^X is accurate to 4 significant figures for $\lambda < 100$ (and A_{22}^X for $\lambda > 0.01$).

In table 2 we have tabulated $A_{\alpha\beta}^X$ for values of λ between 0.1 and 1. In table 3 we have tabulated the function

$$W_2(\lambda) = L_{11}(\lambda) + (1+\lambda)\, L_{12} + \lambda L_{22}.$$

The individual functions $L_{\alpha\beta}$ are not given because only this combination is needed when calculating the asymptotic behaviour of the mobility functions in later sections.

λ	A^X_{11}	A^X_{12}	A^X_{22}
1.0	0.9954	−0.3502	0.9954
0.5	1.0881	−0.2957	0.8083
0.25	1.0836	−0.1880	0.6253
0.2	1.0730	−0.1556	0.5789
0.125	1.0496	−0.0993	0.5001
0.1	1.0398	−0.0787	0.4692

TABLE 2. The functions $A^X_{\alpha\beta}(\lambda)$ for $0.1 < \lambda < 1$

λ	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$
$W_2(\lambda)$	0.1163	0.0575	0.0146	0.0084	0.0022	0.0011

TABLE 3. Values of $W_2 = L^X_{11} + (1+\lambda)\,L^X_{12} + \lambda L^X_{22}$

4. The resistance functions $Y^A_{\alpha\beta}(s,\lambda)$

We now consider motion perpendicular to the line of centres, and again define 2 problems. First,

$$\boldsymbol{U}_1 = -\boldsymbol{U}_2 = U\boldsymbol{i}, \tag{4.1}$$

and secondly

$$\boldsymbol{U}_1 = \boldsymbol{U}_2 = U\boldsymbol{i}, \tag{4.2}$$

with in each case $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = 0$.

Thus we have, for the first problem,

$$\chi^{(\alpha)}_{mn} = (-1)^\alpha \delta_{m1}\delta_{n1}, \quad \psi^{(\alpha)}_{mn} = \omega^{(\alpha)}_{mn} = 0.$$

The expansions we use this time are

$$p^{(\alpha)}_{1n} = (-1)^\alpha \tfrac{3}{2} U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq}\, t^p_\alpha\, t^q_{3-\alpha}, \tag{4.3}$$

$$v^{(\alpha)}_{1n} = (-1)^\alpha \tfrac{3}{4} U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{2n+1} V_{npq}\, t^p_\alpha\, t^q_{3-\alpha}, \tag{4.4}$$

$$q^{(\alpha)}_{1n} = \mathrm{i}U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{npq}\, t^p_\alpha\, t^q_{3-\alpha}, \tag{4.5}$$

where again $t_\alpha = a_\alpha/r$. The recurrence relations for this set of coefficients are

$$P_{n00} = V_{n00} = \delta_{1n}, \quad Q_{n00} = 0, \qquad (4.6), (4.7), (4.8)$$

$$V_{npq} = P_{npq} + \frac{2n}{(n+1)(2n+3)} \sum_{s=1}^{q} \binom{n+s}{n+1} P_{s(q-s)(p-n+1)}, \tag{4.9}$$

$$\begin{aligned} P_{npq} = \sum_{s=1}^{q} \binom{n+s}{n+1} \bigg[& \frac{2n+1}{2(n+1)} \frac{3(n+s)-(ns+1)(2ns-s-n+2)}{s(n+s)(2s-1)} P_{s(q-s)(p-n+1)} \\ & + \frac{n(2n-1)}{2(n+1)} P_{s(q-s)(p-n-1)} + \frac{n(4n^2-1)}{2(n+1)(2s+1)} V_{s(q-s-2)(p-n+1)} \\ & - \frac{2(4n^2-1)}{3(n+1)} Q_{s(q-s-1)(p-n+1)} \bigg], \end{aligned} \tag{4.10}$$

$$Q_{npq} = \sum_{s=1}^{q} \binom{n+s}{n+1} \left[\frac{s}{n+1} Q_{s(q-s-1)(p-n)} - \frac{3}{2ns(n+1)} P_{s(q-s)(p-n)} \right]. \tag{4.11}$$

The equation for the force now gives us

$$Y_{11}^A - \tfrac{1}{2}(1+\lambda)\,Y_{12}^A = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{1pq}\, t_1^p\, t_2^q. \tag{4.12}$$

By considering the second problem, we obtain the same result as that obtained in §3, namely that Y_{11}^A consists of the even powers of s and Y_{12}^A of the odd powers of s. We can thus jump straight to the explicit results.

4.1. *Widely separated spheres*

Switching to the variables s and λ, we again write

$$Y_{11}^A = \sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k}\, s^{-2k}, \tag{4.13}$$

and

$$Y_{12}^A = \frac{-2}{1+\lambda}\sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1}\, s^{-2k-1}. \tag{4.14}$$

Further,

$$\begin{aligned}
f_0 &= 1, \quad f_1 = \tfrac{3}{2}\lambda,\\
f_2 &= \tfrac{9}{4}\lambda, \quad f_3 = 2\lambda + \tfrac{27}{8}\lambda^2 + 2\lambda^3,\\
f_4 &= 6\lambda + \tfrac{81}{16}\lambda^2 + 18\lambda^3, \quad f_5 = \tfrac{63}{2}\lambda^2 + \tfrac{243}{32}\lambda^3 + \tfrac{63}{2}\lambda^4,\\
f_6 &= 4\lambda + 54\lambda^2 + \tfrac{1241}{64}\lambda^3 + 81\lambda^4 + 72\lambda^5,\\
f_7 &= 144\lambda^2 + \tfrac{1053}{8}\lambda^3 + \tfrac{19083}{128}\lambda^4 + \tfrac{1053}{8}\lambda^5 + 144\lambda^6,\\
f_8 &= 279\lambda^2 + \tfrac{4261}{8}\lambda^3 + \tfrac{126369}{256}\lambda^4 - \tfrac{117}{8}\lambda^5 + 648\lambda^6 + 288\lambda^7,\\
f_9 &= 576\lambda^2 + 1134\lambda^3 + \tfrac{60443}{32}\lambda^4 + \tfrac{766179}{512}\lambda^5 + \tfrac{60443}{32}\lambda^6 + 1134\lambda^7 + 576\lambda^8,\\
f_{10} &= 1152\lambda^2 + \tfrac{7857}{4}\lambda^3 + \tfrac{98487}{16}\lambda^4 + \tfrac{10548393}{1024}\lambda^5 + \tfrac{67617}{8}\lambda^6 - \tfrac{351}{2}\lambda^7 + 3888\lambda^8 + 1152\lambda^9,\\
f_{11} &= 2304\lambda^2 + 7128\lambda^3 + \tfrac{22071}{2}\lambda^4 + \tfrac{2744505}{128}\lambda^5 + \tfrac{95203835}{2048}\lambda^6 + \tfrac{2744505}{128}\lambda^7 + \tfrac{22071}{2}\lambda^8\\
&\quad + 7128\lambda^9 + 2304\lambda^{10}.
\end{aligned}$$

The minus signs in f_8 and f_{10} are not printing errors.

4.2. *Nearly touching spheres*

Jeffrey & Onishi (1984) have shown that, for $\xi \ll 1$ and $\xi \ll \lambda$,

$$Y_{11}^A = g_2(\lambda)\ln\xi^{-1} + A_{11}^Y(\lambda) + g_3(\lambda)\,\xi\ln\xi^{-1}, \tag{4.15}$$

and

$$-\tfrac{1}{2}(1+\lambda)\,Y_{12}^A = g_2(\lambda)\ln\xi^{-1} - \tfrac{1}{2}(1+\lambda)\,A_{12}^Y(\lambda) + g_3(\lambda)\,\xi\ln\xi^{-1}, \tag{4.16}$$

where now

$$g_2(\lambda) = \tfrac{4}{15}\lambda(2+\lambda+2\lambda^2)\,(1+\lambda)^{-3},$$
$$g_3(\lambda) = \tfrac{2}{375}(16 - 45\lambda + 58\lambda^2 - 45\lambda^3 + 16\lambda^4)\,(1+\lambda)^{-3}.$$

The functions $A_{\alpha\beta}^Y$ can be obtained by repeating the steps of §3. The expressions are

$$A_{11}^Y = 1 + \sum_{\substack{m=2\\ m\ \text{even}}}^{\infty} [2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3], \tag{4.17}$$

$$-\tfrac{1}{2}(1+\lambda)\,A_{12}^Y = 2g_2\ln 2 + 2g_3 + \sum_{\substack{m=1\\ m\ \text{odd}}}^{\infty} [2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3]. \tag{4.18}$$

Here m_1 has the same definition as in (3.20).

λ	A_{11}^Y	A_{12}^Y	A_{22}^Y
1.0	0.9983	−0.2737	0.9983
0.5	1.0193	−0.2246	0.9009
0.25	1.0073	−0.1181	0.6871
0.2	1.0015	−0.0844	0.5930
0.125	0.9907	−0.0268	0.3637
0.1	0.9869	−0.0072	0.2438

TABLE 4. The functions $A_{\alpha\beta}^Y(\lambda)$ for $0.1 < \lambda < 1$

4.3. *Arbitrary separations*

Combining the results of the previous sections in the same manner as in §3, we obtain expressions accurate for any separation:

$$Y_{11}^A = -g_2 \ln(1-4s^{-2}) - g_3(1-4s^{-2})\ln(1-4s^{-2}) + f_0(\lambda) + \sum_{\substack{m=2\\ m\,\text{even}}}^{\infty} \{2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3\}\left(\frac{2}{s}\right)^m, \tag{4.19}$$

$$-\tfrac{1}{2}(1+\lambda)\, Y_{12}^A = g_2 \ln\frac{s+2}{s-2} + g_3(1-4s^{-2})\ln\frac{s+2}{s-2} + 4g_3 s^{-1} + \sum_{\substack{m=1\\ m\,\text{odd}}}^{\infty} \{2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3\}\left(\frac{2}{s}\right)^m. \tag{4.20}$$

4.4. *Numerical results*

O'Neill & Majumdar (1970*a*) have tabulated $Y_{\alpha\beta}^A$ and O'Neill (1969) has calculated $A_{11}^Y(1)+A_{12}^Y(1)$. For the last quantity, O'Neill gives 0.72426 whereas we calculate 0.72462, suggesting a printing error in O'Neill's paper. It is interesting to note the $g_3(1) = 0$, so that for $\lambda = 1$, the $\xi \ln \xi^{-1}$ correction makes no impact on the accuracy obtained. In table 4, $A_{\alpha\beta}^Y$ has been tabulated for $0.1 < \lambda < 1$ using terms up to $m = 120$, and as with the functions $A_{\alpha\beta}^A$, there is singular behaviour as $\lambda \to 0$, because $A_{22}^Y \to -\frac{8}{15}\ln\frac{1}{2}(1+\lambda)$.

5. The resistance functions $Y_{\alpha\beta}^B(s,\lambda)$

The recurrence relations (4.6)–(4.11) given in §4 can be used to find the coefficients in the series for $Y_{\alpha\beta}^B$. This is because the problem specified in (4.1) and (4.2) requires a couple to act on the spheres to prevent them rotating. Thus, recalling how Y^B is non-dimensionalized,

$$L_1 = -4\pi\mu a_1^2 (Y_{11}^B - \tfrac{1}{4}(1+\lambda)^2 Y_{12}^B)\, Uj. \tag{5.1}$$

Therefore, recalling (2.10),

$$Y_{11}^B - \tfrac{1}{4}(1+\lambda)^2 Y_{12}^B = 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{1pq} t_1^p t_2^q. \tag{5.2}$$

There is a difference between this section and §§3 and 4, however, which is that, when the secondary problem is considered, Y_{11}^B now consists of the odd powers of s and Y_{12}^B the even powers.

5.1. *Widely separated spheres*

Following the pattern above, we write

$$Y_{11}^B = \sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1}\,s^{-2k-1}, \tag{5.3}$$

$$Y_{12}^B = \frac{-4}{(1+\lambda)^2}\sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k}\,s^{-2k}. \tag{5.4}$$

The explicit expressions for the $f_n(\lambda)$ are this time

$$f_0 = f_1 = 0, \quad f_2 = -6\lambda, \quad f_3 = -9\lambda, \quad f_4 = -\tfrac{27}{2}\lambda^2,$$

$$f_5 = -12\lambda - \tfrac{81}{4}\lambda^2 - 36\lambda^3, \quad f_6 = -108\lambda^2 - \tfrac{243}{8}\lambda^3 - 72\lambda^4,$$

$$f_7 = -189\lambda^2 - \tfrac{8409}{16}\lambda^3 - 243\lambda^4 - 144\lambda^5,$$

$$f_8 = -432\lambda^2 - 486\lambda^3 - \tfrac{77451}{32}\lambda^4 - 405\lambda^5 - 288\lambda^6,$$

$$f_9 = -864\lambda^2 - \tfrac{3159}{4}\lambda^3 - \tfrac{283041}{64}\lambda^4 - \tfrac{30525}{4}\lambda^5 - 1620\lambda^6 - 576\lambda^7,$$

$$f_{10} = -1728\lambda^2 - 3888\lambda^3 - \tfrac{59553}{4}\lambda^4 - \tfrac{1125603}{128}\lambda^5 - 22002\lambda^6 - 2916\lambda^7 - 1152\lambda^8,$$

$$f_{11} = -3456\lambda^2 - 6804\lambda^3 - \tfrac{614481}{16}\lambda^4 - \tfrac{4579497}{256}\lambda^5 - \tfrac{536679}{16}\lambda^6 - 73989\lambda^7 - 9072\lambda^8 - 2304\lambda^9.$$

5.2. *Nearly touching spheres*

Jeffrey & Onishi (1984) have shown that

$$Y_{11}^B = g_2(\lambda)\ln\xi^{-1} + B_{11}^Y(\lambda) + g_3(\lambda)\,\xi\ln\xi^{-1}, \tag{5.5}$$

$$-\tfrac{1}{4}(1+\lambda)^2 Y_{12}^B = g_2(\lambda)\ln\xi^{-1} - \tfrac{1}{4}(1+\lambda)^2 B_{12}^Y(\lambda) + g_3(\lambda)\,\xi\ln\xi^{-1}, \tag{5.6}$$

where

$$g_2(\lambda) = -\tfrac{1}{5}\lambda(4+\lambda)(1+\lambda)^{-2},$$

$$g_3(\lambda) = -\tfrac{1}{250}(32 - 33\lambda + 83\lambda^2 + 43\lambda^3)(1+\lambda)^{-2}.$$

As with previous cases, the functions $B_{\alpha\beta}^Y$ are given by summations, although the even and odd terms are interchanged from previous sections:

$$B_{11}^Y = \sum_{\substack{m=1\\ m\ \text{odd}}}^{\infty} [2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3], \tag{5.7}$$

$$-\tfrac{1}{4}(1+\lambda)^2 B_{12}^Y = 2g_2\ln 2 + 2g_3 + \sum_{\substack{m=2\\ m\ \text{even}}}^{\infty} [2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3]. \tag{5.8}$$

Again m_1 has the definition in (3.20).

5.3. *Arbitrary separations*

The series for use at arbitrary separations become

$$Y_{11}^B = g_2\ln\frac{s+2}{s-2} + g_3(1-4s^{-2})\ln\frac{s+2}{s-2} + 4g_3 s^{-1} + \sum_{\substack{m=1\\ m\ \text{odd}}}^{\infty} \{2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3\}\left(\frac{2}{s}\right)^m, \tag{5.9}$$

$$-\tfrac{1}{4}(1+\lambda)^2 Y_{12}^B = -g_2\ln(1-4s^{-2}) - g_3(1-4s^{-2})\ln(1-4s^{-2}) + \sum_{\substack{m=2\\ m\ \text{even}}}^{\infty} \{2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1}g_2 + 4m^{-1}m_1^{-1}g_3\}\left(\frac{2}{s}\right)^m. \tag{5.10}$$

λ	B_{11}^Y	B_{12}^Y	B_{21}^Y	B_{22}^Y
1.0	0.2390	−0.0017	0.0017	−0.2390
0.5	0.1201	0.0817	0.0686	−0.4016
0.25	0.0620	0.0592	0.0594	−0.5664
0.2	0.0532	0.0320	0.0489	−0.6154
0.125	0.0433	−0.0306	0.0288	−0.7124
0.1	0.0408	−0.0560	0.0214	−0.7559

TABLE 5. The functions $B_{\alpha\beta}^Y(\lambda)$ for $0.1 < \lambda < 1$

5.4. *Numerical results*

The functions $Y_{\alpha\beta}^B$ have been tabulated by O'Neill & Majumdar (1970*a*). The function $\frac{1}{2}[B_{11}^Y(1)+B_{12}^Y(1)]$ has been calculated by O'Neill (1969). Table 5 gives the results for $B_{\alpha\beta}^Y(\lambda)$ obtained from (5.7) and (5.8).

6. The resistance functions $X_{\alpha\beta}^C(s,\lambda)$

To obtain these functions, we must solve problems in which the spheres rotate about their line of centres. It turns out to be most convenient to express the rotation in terms of a surface speed U:

$$a_1\boldsymbol{\Omega}_1 = \mp a_2\boldsymbol{\Omega}_2 = U\boldsymbol{e}. \tag{6.1}$$

Now the velocities are zero:

$$\boldsymbol{U}_1 = \boldsymbol{U}_2 = 0. \tag{6.2), (6.3}$$

The only non-zero coefficients in (2.9) are $q_{0n}^{(\alpha)}$, which we expand as

$$q_{0n}^{(\alpha)} = U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{npq}\, t_\alpha^p\, t_{3-\alpha}^q, \tag{6.4}$$

assuming that the minus sign in (5.1) is taken, thus making $Q_{npq}^{(1)} = Q_{npq}^{(2)}$. The recurrence relations are

$$Q_{n00} = \delta_{1n}, \tag{6.5a}$$

$$Q_{npq} = \sum_{s=0}^{q}\binom{n+s}{n}\frac{s}{n+1}Q_{s(q-s-1)(p-n)}. \tag{6.5b}$$

From the couple on a sphere, we find

$$X_{11}^C - \frac{(1+\lambda)^3}{8\lambda}X_{12}^C = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{1pq}\, t_1^p\, t_2^q. \tag{6.6}$$

6.1. *Widely separated spheres*

In the standard notation

$$X_{11}^C = \sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k}\, s^{-2k}, \tag{6.7}$$

$$X_{12}^C = \frac{-8}{(1+\lambda)^3}\sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1}\, s^{-2k-1}, \tag{6.8}$$

where

$$f_0 = 1,\quad f_1 = f_2 = 0,\quad f_3 = 8\lambda^3,\quad f_4 = f_5 = 0,$$

$$f_6 = 64\lambda^3,\quad f_7 = 0,\quad f_8 = 768\lambda^5,\quad f_9 = 512\lambda^6,$$

$$f_{10} = 6144\lambda^7,\quad f_{11} = 6144\lambda^6 + 6144\lambda^8.$$

There appear to be no published results with which these can be compared.

6.2. *Nearly touching spheres*

It has been shown by Majumdar (1967) and Jeffrey & Onishi (1984) that

$$X_{11}^C = \frac{\lambda^3}{(1+\lambda)^3}\zeta\left(3, \frac{\lambda}{1+\lambda}\right) - \frac{\lambda^2}{4(1+\lambda)}\xi \ln \xi^{-1}, \tag{6.9}$$

$$X_{12}^C = \frac{-8\lambda^3}{(1+\lambda)^6}\zeta(3) + \frac{2\lambda^2}{(1+\lambda)^4}\xi \ln \xi^{-1} + O(\xi), \tag{6.10}$$

where

$$\zeta(z, a) = \sum_{k=0}^{\infty} (k+a)^{-z}. \tag{6.11}$$

In this case, therefore, we have two expressions for each of the functions $C_{\alpha\beta}^X$: the expressions above, derived using tangent-sphere coordinates, and the usual series expressions.

6.3. *Arbitrary separations*

Since the function $X_{\alpha\beta}^C$ are not infinite at $s = 2$, we could use the forms (6.7) and (6.8) unchanged. Convergence is hastened, however, by removing the $\xi \ln \xi^{-1}$ term:

$$X_{11}^C = \frac{\lambda^2}{2(1+\lambda)}\ln(1-4s^{-2}) + \frac{\lambda^2}{1+\lambda}s^{-1}\ln\frac{s+2}{s-2} + 1 + \sum_{k=1}^{\infty}\left\{(1+\lambda)^{-2k}f_{2j} - 2^{2k+1}k^{-1}(2k-1)^{-1}\frac{\lambda^2}{4(1+\lambda)}\right\}s^{-2k}, \tag{6.12}$$

$$X_{12}^C = \frac{4\lambda^2}{(1+\lambda)^4}\ln\frac{s+2}{s-2} + \frac{8\lambda^2}{(1+\lambda)^4}s^{-1}\ln(1-4s^{-2}) - \frac{8}{(1+\lambda)^3}\sum_{k=1}^{\infty}\left\{(1+\lambda)^{-2k-1}f_{2k+1} - 2^{2k+2}k^{-1}(2k+1)^{-1}\frac{\lambda^2}{1+\lambda}\right\}s^{-2k-1}. \tag{6.13}$$

6.4. *Numerical results*

Tables of $X_{\alpha\beta}^C$ were given by Jeffery (1915) and Majumdar (1967) has tabulated $C_{\alpha\beta}^X$.

7. The resistance functions $Y_{\alpha\beta}^C(s, \lambda)$

The final resistance functions are obtained from problems which are also defined using surface velocities to specify a rate of rotation:

$$a_1 \boldsymbol{\Omega}_1 = +a_2 \boldsymbol{\Omega}_2 = U\boldsymbol{i}. \tag{7.1}$$

Again

$$\boldsymbol{U}_1 = \boldsymbol{U}_2 = 0. \tag{7.2}$$

The expansions (4.3)–(4.5) can be used again, as can the recurrence relations (4.9)–(4.11). The only change is in the initial conditions in that (4.6)–(4.8) are replaced by

$$P_{n00} = V_{n00} = 0, \quad Q_{n00} = \delta_{1n}. \qquad (7.3), (7.4), (7.5)$$

Having solved the recurrence relations, we can find the force exerted by a sphere on the fluid and reproduce the results of §4, as predicted by the reciprocal theorem. From the couple exerted on the fluid we find

$$Y_{11}^C + \frac{(1+\lambda)^3}{8\lambda}Y_{12}^C = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{1pq}t_1^p t_2^q. \tag{7.6}$$

7.1. *Widely separated spheres*

We have

$$Y^C_{11} = \sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k} s^{-2k} \tag{7.7}$$

and

$$Y^C_{12} = \frac{8}{(1+\lambda)^3} \sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1} s^{-2k-1}. \tag{7.8}$$

We have

$$f_0 = 1,\quad f_1 = f_2 = 0,\quad f_3 = 4\lambda^3,\quad f_4 = 12\lambda,$$

$$f_5 = 18\lambda^4,\quad f_6 = 27\lambda^2 + 256\lambda^3,\quad f_7 = 72\lambda^4 + \tfrac{81}{2}\lambda^5 + 72\lambda^6,$$

$$f_8 = 216\lambda^2 + \tfrac{243}{4}\lambda^3 + 216\lambda^4 + 2496\lambda^5,$$

$$f_9 = 288\lambda^4 + 486\lambda^5 - \tfrac{6439}{8}\lambda^6 + 486\lambda^7 + 288\lambda^8,$$

$$f_{10} = 864\lambda^2 + 972\lambda^3 + \tfrac{151179}{16}\lambda^4 + 972\lambda^5 + 1296\lambda^6 + 18432\lambda^7,$$

$$f_{11} = 1152\lambda^4 + 3240\lambda^5 - \tfrac{10947}{2}\lambda^6 + \tfrac{518049}{32}\lambda^7 - \tfrac{10947}{2}\lambda^8 + 3240\lambda^9 + 1152\lambda^{10}.$$

7.2. *Nearly touching spheres*

O'Neill & Majumdar (1970*b*) and Jeffrey & Onishi (1984) have shown that

$$Y^C_{11} = g_2 \ln \xi^{-1} + C^Y_{11}(\lambda) + g_3\, \xi \ln \xi^{-1}, \tag{7.9}$$

$$Y^C_{12} = g_4 \ln \xi^{-1} + C^Y_{12}(\lambda) + g_5\, \xi \ln \xi^{-1}, \tag{7.10}$$

where

$$g_2 = \tfrac{2}{5}\lambda(1+\lambda)^{-1},\quad g_3 = \tfrac{1}{125}(8 + 6\lambda + 33\lambda^2)(1+\lambda)^{-1},$$

$$g_4 = \tfrac{4}{5}\lambda^2(1+\lambda)^{-4},\quad g_5 = \tfrac{4}{125}\lambda(43 - 24\lambda + 43\lambda^2)(1+\lambda)^{-4}.$$

The functions $C^Y_{\alpha\beta}$ are expressed as

$$C^Y_{11} = 1 + \sum_{\substack{m=2\\ m\,\text{even}}}^{\infty} [2^{-m}(1+\lambda)^{-m} f_m - 2m^{-1} g_2 + 4m^{-1} m_1^{-1} g_3], \tag{7.11}$$

$$C^Y_{12} = 2g_4 \ln 2 + 2g_5 + \sum_{\substack{m=1\\ m\,\text{odd}}}^{\infty} [2^{3-m}(1+\lambda)^{3-m} f_m - 2m^{-1} g_4 + 4m^{-1} m_1^{-1} g_5]. \tag{7.12}$$

7.3. *Arbitrary separations*

The interesting special feature of the function $Y^C_{\alpha\beta}$ is that the simple rolling motion $\Omega_1 = -\lambda \Omega_2$ does not lead to a finite couple as $\xi \to 0$. The force and couple remain finite, however, for the rigid-body motion $\Omega_1 = \Omega_2$, $U_2 = U_1 + \Omega_1 r$. From this we conclude the easily verified results

$$3(1+\lambda)\, g_2^{(4)}(\lambda) + 2g_2^{(5)}(\lambda) + 2\lambda^2 g_2^{(5)}(\lambda^{-1}) = 0, \tag{7.13a}$$

$$8 g_2^{(7)}(\lambda) + (1+\lambda)^3 g_4^{(7)} + 4(1+\lambda)\, g_2^{(5)} = 0, \tag{7.13b}$$

where the function $g_4^{(4)}$ was defined in (4.15), the function $g_2^{(5)}$ in (5.5) and the functions $g^{(7)}$ in (7.9) and (7.10). For arbitrary separations,

$$Y^C_{11} = -g_2 \ln(1 - 4s^{-2}) - g_3(1 - 4s^{-2}) \ln(1 - 4s^{-2}) + f_0(\lambda) + \sum_{\substack{m=2\\ m\,\text{even}}}^{\infty} \{2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1} g_2 + 4m^{-1} m_1^{-1} g_3\} \left(\frac{2}{s}\right)^m, \tag{7.14}$$

$$\tfrac{1}{8}(1+\lambda)^3 Y^C_{12} = g_4 \ln \frac{s+2}{s-2} + g_5(1 - 4s^{-2}) \ln \frac{s+2}{s-2} + 4g_5 s^{-1} + \sum_{\substack{m=1\\ m\,\text{odd}}}^{\infty} \{2^{-m}(1+\lambda)^{-m} f_m(\lambda) - 2m^{-1} g_4 + 4m^{-1} m_1^{-1} g_5\} \left(\frac{2}{s}\right)^m. \tag{7.15}$$

λ	C^Y_{11}	C^Y_{12}	C^Y_{22}
1.0	0.7028	−0.0274	0.7028
0.5	0.8489	−0.0349	0.4839
0.25	0.9280	−0.0349	0.2097
0.2	0.9427	−0.0312	0.1144
0.125	0.9625	−0.0210	−0.0918
0.1	0.9683	−0.0165	−0.1909

TABLE 6. The functions $C^Y_{\alpha\beta}(\lambda)$ for $0.1 < \lambda < 1$

7.4. *Numerical results*

A tabulation of the functions $Y^C_{\alpha\beta}$ has been given by O'Neill & Majumdar (1970*a*) and $C^Y_{11}-C^Y_{12}-\frac{3}{20}\ln 2$ has been calculated by O'Neill (1969). The results differ only in the fourth decimal place. The functions $C^Y_{\alpha\beta}$ are tabulated in table 6.

8. The mobility functions $x^a_{\alpha\beta}(s,\lambda)$

We now turn to the mobility functions and consider external forces acting on the spheres given by

$$(6\pi\mu a_1)^{-1}\boldsymbol{F}_1 = -(6\pi\mu a_2)^{-1}\boldsymbol{F}_2 = U\boldsymbol{e}, \tag{8.1}$$

where U is the velocity that either sphere would have in the absence of the other. In addition, we specify

$$\boldsymbol{L}_1 = \boldsymbol{L}_2 = 0. \qquad (8.2), (8.3)$$

We wish to calculate the motion of each sphere, which by symmetry is given by

$$\boldsymbol{U}_1 = U^{(1)}\boldsymbol{e}, \quad \boldsymbol{U}_2 = -U^{(2)}\boldsymbol{e}, \qquad (8.4), (8.5)$$

and

$$\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = 0. \tag{8.6}$$

From (2.10) we obtain

$$p^{(\alpha)}_{01} = \tfrac{3}{2}U, \quad q^{(\alpha)}_{01} = 0. \qquad (8.7), (8.8)$$

From the fact that the spheres move in rigid-body motion according to (8.4)–(8.6), we obtain

$$\chi^{(\alpha)}_{mn} = U^{(\alpha)}\delta_{0m}\,\delta_{1n}, \quad \psi^{(\alpha)}_{mn} = \omega^{(\alpha)}_{mn} = 0. \qquad (8.9), (8.10)$$

In view of these results and the symmetry in the equations, we expand $p^{(\alpha)}$, $q^{(\alpha)}$ and $U^{(\alpha)}$ as

$$p^{(\alpha)}_{0n} = \tfrac{3}{2}U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq}\, t^p_\alpha\, t^q_{3-\alpha}, \tag{8.11}$$

$$v^{(\alpha)}_{0n} = \tfrac{3}{4}U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{2n+1} V_{npq}\, t^p_\alpha\, t^q_{3-\alpha}, \tag{8.12}$$

$$U^{(\alpha)} = U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} U_{pq}\, t^p_\alpha\, t^q_{3-\alpha}, \tag{8.13}$$

where $t_\alpha = a_\alpha/r$ as in previous parts. From (8.7) we known

$$P_{1pq} = \delta_{0p}\,\delta_{0q}. \tag{8.14}$$

The relation (2.9*a*) remains valid for all $n \geqslant 1$, giving us

$$V_{npq} = P_{npq} - \frac{2n}{(n+1)(2n+3)} \sum_{s=1}^{q} \binom{n+s}{n} P_{s(q-s)(p-n-1)}. \tag{8.15}$$

The equation (2.9*b*) takes on different roles for $n = 1$ and $n > 1$. For $n = 1$ it is an equation for U_{pq}:

$$U_{pq} = -\sum_{s=1}^{q}(s+1)\left[\frac{3}{4(2s-1)}P_{s(q-s)\,p} - \tfrac{1}{4}P_{s(q-s)\,(p-2)} - \frac{3}{4(2s+1)}V_{s(q-s-2)\,p}\right]. \quad (8.16)$$

For $n > 1$ the equation is a recurrence relation for P_{npq}:

$$P_{npq} = \sum_{s=1}^{q}\binom{n+s}{n}\left[\frac{n(2n+1)(2ns-n-s+2)}{2(n+1)(2s-1)(n+s)}P_{s(q-s)\,(p-n+1)} - \frac{n(2n-1)}{2(n+1)}P_{s(q-s)\,(p-n-1)} - \frac{n(4n^2-1)}{2(n+1)(2s+1)}V_{s(q-s-2)\,(p-n+1)}\right]. \quad (8.17)$$

These equations form a complete set from which we can solve for U_{pq}. These coefficients are then related to the mobility functions by

$$x_{11}^a - \frac{2\lambda}{1+\lambda}x_{12}^a = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} U_{pq}t_1^p t_2^q.$$

We now consider the problem in which the forces are given by

$$(6\pi\mu a_1)^{-1}\boldsymbol{F}_1 = (6\pi\mu a_2)^{-1}\boldsymbol{F}_2 = U\boldsymbol{e}. \quad (8.18)$$

We find that if the velocities $U^{(\alpha)}$ in the first problem are described by coefficients U_{pq}, then the velocities in the second problem are described by coefficients $(-1)^{p+q}U_{pq}$. Thus, as before, the mobility functions are given by a series either of even combinations of p and q (even powers of s^{-1}), or odd combinations.

8.1. *Widely separated spheres*

The mobility functions x_{11}^a and x_{12}^a are given by

$$x_{11}^a = \sum_{k=0}^{\infty} f_{2k}(\lambda)(1+\lambda)^{-2k}s^{-2k}, \quad (8.19)$$

$$x_{12}^a = -\tfrac{1}{2}(1+\lambda)\sum_{k=0}^{\infty} f_{2k+1}(\lambda)(1+\lambda)^{-2k-1}s^{2k-1}, \quad (8.20)$$

where

$$f_k(\lambda) = 2^k\sum_{q=0}^{\infty} U_{(k-q)\,q}\lambda^{q-i}, \quad i = \begin{cases}0 & (k \text{ even}),\\ 1 & (k \text{ odd}).\end{cases}$$

Explicitly

$$f_0 = 1,\quad f_1 = -3,\quad f_2 = 0,\quad f_3 = 4+4\lambda^2,$$

$$f_4 = -60\lambda^3,\quad f_5 = 0,\quad f_6 = 480\lambda^3 - 128\lambda^5,$$

$$f_7 = -2400\lambda^3,\quad f_8 = -960\lambda^3 + 4224\lambda^5 - 576\lambda^7,$$

$$f_9 = 1920\lambda^3 + 1920\lambda^5,$$

$$f_{10} = -17920\lambda^5 - 96000\lambda^6 + 30720\lambda^7 - 2304\lambda^9,$$

$$f_{11} = -15360\lambda^3 + 231936\lambda^5 - 15360\lambda^7.$$

8.2. *Nearly touching spheres*

When the spheres touch, the mobility functions do not diverge in the way the resistance functions do, but the apparently smooth behaviour hides singularities in their derivatives. This is reflected in the fact that the series (8.19) and (8.20) lose accuracy near $s = 2$. To obtain alternative expressions for $x_{\alpha\beta}^a$ near $s = 2$, we turn to

(2.2) relating the $x^a_{\alpha\beta}$ to the $X^A_{\alpha\beta}$. We remark in passing that the widely separated forms given above and in §3.1 have been checked against each other using this equation. We have given, in §3.2, asymptotic forms for $X^A_{\alpha\beta}$ with $\xi = s-2 \ll 1$, and now, substituting those forms into (1.15) we obtain the following expressions for the $x^a_{\alpha\beta}$. We introduce the notation

$$W_1 = A^X_{11}+(1+\lambda)\,A^X_{12}+\lambda A^X_{22}, \tag{8.21}$$

$$W_2 = L^X_{11}+(1+\lambda)\,L^X_{12}+\lambda L^X_{22}. \tag{8.22}$$

The physical interpretation of these quantities is that the total non-dimensional force exerted by two nearly touching spheres following each other with the same velocity along their line of centres is $W_1+\xi W_2+O(\xi^2)$. Then

$$x^a_{\alpha\beta} = d^{(1)}_{\alpha\beta}(\lambda)+d^{(2)}_{\alpha\beta}\,\xi+d^{(3)}_{\alpha\beta}\,\xi^2\ln\xi+d^{(4)}_{\alpha\beta}\,\xi^2+O(\xi^3\,(\ln\xi)^2). \tag{8.23}$$

For $\alpha = 1$, $\beta = 1$,

$$d^{(1)}_{11} = W_1^{-1},$$

$$d^{(2)}_{11} = \frac{(1+\lambda)^3}{2\lambda^2}\,W_1^{-2}[\lambda A^X_{22}(\lambda A^X_{22}+(1+\lambda)\,A^X_{12})+\tfrac{1}{4}(1+\lambda)^2\,(A^X_{12})^2]-W_1^{-2}\,W_2,$$

$$d^{(3)}_{11} = \tfrac{1}{20}\lambda^{-3}(1+7\lambda+\lambda^2)\,(1+\lambda)^3\,W_1^{-2}[\lambda A^X_{22}(\lambda A^X_{22}+(1+\lambda)\,A^X_{12})+\tfrac{1}{4}(1+\lambda)^2\,(A^X_{12})^2].$$

Rather than calculating $d^{(4)}_{11}$ in terms of resistance functions, it was found simply by matching (8.23) to (8.19) in the region where both forms were accurate (see §8.4 below).

For $\alpha = 1$, $\beta = 2$,

$$d^{(1)}_{12} = \tfrac{1}{2}(1+\lambda)\,W_1^{-1},$$

$$d^{(2)}_{12} = \frac{-(1+\lambda)^4}{4\lambda^2}\,W_1^{-2}[\tfrac{1}{2}(1+\lambda)\,A^X_{12}(A^X_{11}+\lambda A^X_{22}+\tfrac{1}{2}(1+\lambda)\,A^X_{12})+\lambda A^X_{11}\,A^X_{22}]-\tfrac{1}{2}(1+\lambda)\frac{W_2}{W_1^2},$$

$$d^{(3)}_{12} = -\tfrac{1}{40}\lambda^{-3}(1+7\lambda+\lambda^2)\,(1+\lambda)^4\,W_1^{-2}[\tfrac{1}{2}(1+\lambda)\,A^X_{12}(A^X_{11}+\lambda A^X_{22}$$
$$+\tfrac{1}{2}(1+\lambda)\,A^X_{12})+\lambda A^X_{11}\,A^X_{22}].$$

As with $d^{(4)}_{11}$, $d^{(4)}_{12}$ was found by matching to (8.20).

From the above results, it is easy to deduce the properties

$$x^a_{11}(2,\lambda) = \frac{2}{1+\lambda}x^a_{12}(2,\lambda) = \frac{1}{\lambda}x^a_{22}(2,\lambda),$$

and
$$x^a_{11}-\frac{4}{1+\lambda}x^a_{12}+\frac{1}{\lambda}x^a_{22} = \frac{(1+\lambda)^3}{2\lambda^2}\xi+O(\xi^2\ln\xi),$$

quoted by Batchelor (1982). Tables 7 and 8 tabulate the values of the constants.

8.3. *Arbitrary separations*

It might be expected that we should now follow the procedure used for resistance functions and combine the widely separated and nearly touching forms to obtain more convergent series. This step, however, has little effect on the convergence of the series. The singularities which limit the convergence of the series are thus not ones that can be found from asymptotic analysis of physically significant limits. The only way forward in this direction would thus be to attempt to locate the singularities using one of the battery of numerical techniques available (Van Dyke 1975); however, the good overlap between the two expressions derived above made this unnecessary.

λ	$d_{11}^{(1)}$	$d_{11}^{(2)}$	$d_{11}^{(3)}$	$d_{11}^{(4)}$
1.125	0.9997	−0.002	0.003	0
0.25	0.9951	0.009	0.026	0
0.5	0.9537	0.152	0.194	−0.3
1	0.7750	0.930	0.900	−2.0
2	0.4768	2.277	2.188	−4.5
4	0.2488	3.610	4.061	−6.4
8	0.1250	5.620	8.500	−9.2

TABLE 7. The functions $d_{11}(\lambda)$ appearing in the asymptotic expression for $x_{11}^{a}(s,\lambda)$

λ	$d_{12}^{(1)}$	$d_{12}^{(2)}$	$d_{12}^{(3)}$	$d_{12}^{(4)}$
1.0	0.7750	−1.070	−0.900	2.0
0.5	0.7152	−0.766	−0.691	1.0
0.25	0.6219	−0.372	−0.408	0.2
0.125	0.5623	−0.170	−0.256	−0.2

TABLE 8. The functions $d_{12}(\lambda)$ appearing in the asymptotic form for $x_{12}^{a}(s,\lambda)$

$s\backslash\lambda$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
2.00	0.7750	0.9537	0.9951	0.9997
2.02	0.7914	0.9563	0.9952	0.9997
2.1	0.8356	0.9634	0.9956	0.9996
2.5	0.9234	0.9791	0.9964	0.9995
3.0	0.9613	0.9880	0.9976	0.9996
Asymptote	(0.9614)	(0.9880)	(0.9976)	(0.9996)

TABLE 9. The function $x_{11}^{a}(s,\lambda)$. The asymptotic value for $s = 3.0$ was obtained using the terms up to s^{-11} quoted in §8.

$s\backslash\lambda$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
2.00	0.7750	0.7152	0.6219	0.5623
2.02	0.7558	0.7014	0.6152	0.5592
2.1	0.7007	0.6606	0.5940	0.5487
2.5	0.5627	0.5479	0.5204	0.4992
3.0	0.4708	0.4644	0.4518	0.4411
Asymptote	(0.4706)	(0.4643)	(0.4518)	(0.4411)

TABLE 10. The function $x_{12}^{a}(s,\lambda)$. The asymptotic value for $s = 3.0$ was obtained using terms up to s^{-11} quoted in §8.

8.4. *Numerical results*

The numerical results were obtained by summing the series (8.19) and (8.20) to terms $O(s^{-150})$. Comparison of the results obtained from these summations with results obtained from other sources, such as Batchelor (1976) and Adler (1981), showed that the series were accurate to 4 decimal places for $s \geqslant 2.02$. For $2 \leqslant s \leqslant 2.02$, the nearly touching forms (8.23) were used. The coefficients $d_{\alpha\beta}^{(1)}$, $d_{\alpha\beta}^{(2)}$ and $d_{\alpha\beta}^{(3)}$ were calculated from the formulae given above and the coefficient $d_{\alpha\beta}^{(4)}$ was calculated from the requirement that the two forms agree at $s = 2.02$. The choice of $s = 2.02$ was not critical, however, and the results changed little for other choices between 2.015 and 2.02. Tables 9, 10 and 11 tabulate $x_{\alpha\beta}^{a}$ for $\lambda \leqslant 1$.

$s \backslash \lambda$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
2.00	0.7750	0.4768	0.2488	0.1250
2.02	0.7915	0.5171	0.3121	0.2204
2.1	0.8356	0.6266	0.4810	0.4527
2.5	0.9234	0.9413	0.7980	0.8178
3.0	0.9613	0.9261	0.9121	0.9263
Asymptote	(0.9614)	(0.9265)	(0.9124)	(0.9269)

TABLE 11. The function x^a_{22}. The asymptotic value for $s = 3.0$ was obtained using the terms up to s^{-11} quoted in §8.

9. The mobility functions $y^a_{\alpha\beta}(s, \lambda)$

If $\boldsymbol{i}$ is a unit vector in a direction perpendicular to the line of centres, we consider the problem of motion under the conditions

$$(6\pi\mu a_1)^{-1}\boldsymbol{F}_1 = -(6\pi\mu a_2)^{-1}\boldsymbol{F}_2 = U\boldsymbol{i}, \tag{9.1}$$

$$\boldsymbol{L}_1 = \boldsymbol{L}_2 = 0. \qquad (9.2), (9.3)$$

By symmetry, we expect to find velocities and rotations

$$\boldsymbol{U}_1 = U^{(1)}\boldsymbol{i}, \quad \boldsymbol{U}_2 = -U^{(2)}\boldsymbol{i}, \qquad (9.4), (9.5)$$

$$\boldsymbol{\Omega}_1 = -\Omega^{(1)}\boldsymbol{j}, \quad \boldsymbol{\Omega}_2 = -\Omega^{(2)}\boldsymbol{j}. \qquad (9.6), (9.7)$$

From (2.6) we obtain

$$p^{(\alpha)}_{11} = (-1)^{3-\alpha}\tfrac{3}{2}U, \quad q^{(\alpha)}_{11} = 0. \qquad (9.8), (9.9)$$

From the rigid-body motion of the spheres we obtain

$$\chi^{(\alpha)}_{mn} = (-1)^{3-\alpha}U^{(\alpha)}\delta_{m1}\delta_{n1}, \quad \psi^{(\alpha)}_{mn} = 0, \qquad (9.10), (9.11)$$

$$\omega^{(\alpha)}_{mn} = 2\Omega^{(\alpha)}a_\alpha\delta_{m1}\delta_{n1}. \tag{9.12}$$

We now expand the quantities $p^{(\alpha)}_{1n}$, $q^{(\alpha)}_{1n}$, $v^{(\alpha)}_{1n}$, $U^{(\alpha)}$ and $\Omega^{(\alpha)}$ in ways similar to those of §8:

$$p^{(\alpha)}_{1n} = (-1)^{3-\alpha}\tfrac{3}{2}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}P_{npq}t^p_\alpha t^q_{3-\alpha},$$

$$q^{(\alpha)}_{1n} = U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}Q_{npq}t^p_\alpha t^q_{3-\alpha},$$

$$v^{(\alpha)}_{1n} = (-1)^{3-\alpha}\tfrac{3}{4}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2n+1}V_{npq}t^p_\alpha t^q_{3-\alpha},$$

$$U^{(\alpha)} = (-1)^{3-\alpha}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}U_{pq}t^p_\alpha t^q_{3-\alpha},$$

$$a_\alpha\Omega^{(\alpha)} = U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{pq}t^p_\alpha t^q_{3-\alpha}.$$

We have the initial conditions that

$$P_{1pq} = \delta_{0p}\delta_{0q}, \quad Q_{1pq} = 0.$$

All V_{npq} are calculated from the relation

$$V_{npq} = P_{npq} + \frac{2n}{(n+1)(2n+3)}\sum_{s=1}^{q}\binom{n+s}{n+1}P_{s(q-s)(p-n-1)}. \tag{9.13}$$

For $n \geqslant 2$, the recurrence relations for P_{npq} and Q_{npq} are

$$P_{npq} = \sum_{s=1}^{q} \binom{n+s}{n+1} \left[\frac{2n+1}{2(n+1)} \frac{(n+s)(4+ns)-2(ns+1)^2}{s(n+s)(2s-1)} P_{s(q-s)(p-n+1)} + \frac{n(2n-1)}{2(n+1)} P_{s(q-s)(p-n-1)} + \frac{n(4n^2-1)}{2(n+1)(2s+1)} V_{s(q-s-2)(p-n+1)} - \frac{2(4n^2-1)}{3(n+1)} Q_{s(q-s-1)(p-n+1)} \right], \tag{9.14}$$

$$Q_{npq} = \sum_{s=1}^{q} \binom{n+s}{n+1} \left[\frac{s}{n+1} Q_{s(q-s-1)(p-n)} - \frac{3}{2ns(n+1)} P_{s(q-s)(p-n)} \right]. \tag{9.15}$$

Substituting $n = 1$ into (2.9*b, c*) gives expressions for U_{pq} and Ω_{pq}:

$$U_{pq} = P_{1pq} - \sum_{s=1}^{q} \binom{s+1}{2} \left[\frac{3(2-s)}{4s(2s-1)} P_{s(q-s)p} + \tfrac{1}{4} P_{s(q-s)(p-2)} + \frac{3}{4(2s+1)} V_{s(q-s-2)p} - Q_{s(q-s-1)p} \right], \tag{9.16}$$

$$\Omega_{pq} = Q_{1pq} - \sum_{s=1}^{q} \binom{s+1}{2} \left[\tfrac{1}{2} s Q_{s(q-s-1)(p-1)} - \frac{3}{4s} P_{s(q-s)(p-1)} \right]. \tag{9.17}$$

Taking the values for U_{pq} from (9.16) and observing that, as in previous cases, the even and odd powers in the series go to the functions y^a_{11} and y^a_{12} respectively, we arrive at our series expressions for $y^a_{\alpha\beta}$.

9.1. *Widely separated spheres*

We have

$$y^a_{11} = \sum_{k=0}^{\infty} f_{2k}(\lambda)(1+\lambda)^{-2k} s^{-2k}, \tag{9.18}$$

$$y^a_{12} = \tfrac{1}{2}(1+\lambda) \sum_{k=0}^{\infty} f_{2k+1}(\lambda)(1+\lambda)^{-2k-1} s^{-2k-1}, \tag{9.19}$$

where

$$f_0 = 1, \quad f_1 = \tfrac{3}{2}, \quad f_2 = 0, \quad f_3 = 2+2\lambda^2,$$
$$f_4 = f_5 = 0, \quad f_6 = -68\lambda^5, \quad f_7 = 0,$$
$$f_8 = -320\lambda^3 + 288\lambda^5 - 288\lambda^7, \quad f_9 = 0,$$
$$f_{10} = -6720\lambda^5 + 3456\lambda^7 - 1152\lambda^9,$$
$$f_{11} = 8960\lambda^3 - 8848\lambda^5 + 8960\lambda^7.$$

9.2. *Nearly touching spheres*

The singular behaviour of $y^a_{\alpha\beta}$ when $\xi = s-2 \ll 1$ requires even more elaborate formulae than those of the previous section. We must invert the 4×4 matrix given in (1.16) when the resistance functions are given by their asymptotic representations for $\xi \ll 1$. The inversion was carried out using the CAMAL language for algebraic manipulation by computer. It was found that the general form for $y^a_{\alpha\beta}$ is

$$y^a_{\alpha\beta} = \frac{a^{(1)}_{\alpha\beta}(\ln \xi^{-1})^2 + a^{(2)}_{\alpha\beta} \ln \xi^{-1} + a^{(3)}_{\alpha\beta}}{(\ln \xi^{-1})^2 + e^{(1)} \ln \xi^{-1} + e^{(2)}} + O(\xi (\ln \xi)^3). \tag{9.20}$$

The fact that $\ln \xi^{-1}$ appears only to the second power is a result of the relation (7.13). This might not seem to be a natural form in which to write the result, but it is

λ	$a_{11}^{(1)}$	$a_{11}^{(2)}$	$a_{11}^{(3)}$	$a_{12}^{(1)}$	$a_{12}^{(2)}$	$a_{12}^{(3)}$	$e^{(1)}$	$e^{(2)}$
0.125	0,994	1.53	−1.55	0.553	0.46	−0.65	1.52	−1.54
0.25	0.973	3.84	0.34	0.571	1.53	−0.06	3.79	0.32
0.5	0.927	5.61	4.40	0.535	2.50	1.24	5.60	4.18
1	0.891	5.77	7.07	0.489	2.81	1.98	6.04	6.33
2	0.764	5.02	5.60	0.535	2.50	1.24	5.60	4.18
4	0.473	3.71	1.89	0.571	1.53	−0.06	3.79	0.32
8	0.238	2.70	−0.85	0.553	0.46	−0.65	1.52	−1.54

TABLE 12. The functions $a_{\alpha\beta}(\lambda)$ and $e(\lambda)$ appearing in the asymptotic expression for $y^a_{\alpha\beta}(s, \lambda)$ near $s = 2$

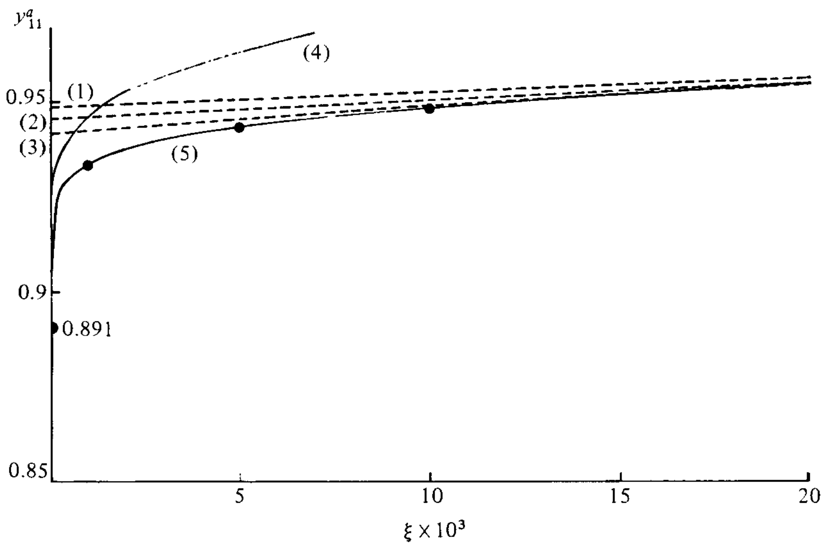

FIGURE 1. Various representations of the function $y^a_{11}(s, 1)$ in the neighbourhood of $s = 2$ plotted against $\xi = s-2$. The broken curves labelled (1)–(3) correspond to the series form summed to (1) terms to $O(s^{-80})$, (2) terms to $O(s^{-120})$, and (3) terms to $O(s^{-220})$. The solid lines correspond to asymptotic expressions: (4) simplified expression, and (5) full expression. The solid circles are numerical data taken from Batchelor (1976).

chosen because the error term is multiplied by ξ; other, simpler, forms such as (9.21) below are obviously possible, but the error associated with them is $O((\ln \xi)^{-2})$ which is much larger. Each coefficient $a_{\alpha\beta}$ and e is a function of λ and can be written out explicitly in terms of the functions A^Y, B^Y and C^Y. When, however, these explicit expressions are derived using a computer, they turn out to be too lengthy to reproduce here. They may be obtained from the authors either as algebraic expressions printed by the computer, or as Fortran function subprograms. Table 12 tabulates numerical values for the factors.

9.3. *Numerical results*

The series expressions were extended to terms $O(s^{-120})$ and in the special case $\lambda = 1$ to terms $O(s^{-220})$. Comparisons between the various representations of the functions $y^a_{\alpha\beta}(s, 1)$ near $s = 2$ are given in figures 1 and 2. Two asymptotic forms are given, the full form (9.20) and the simpler form

$$a^{(1)}_{\alpha\beta} + \frac{a^{(2)}_{\alpha\beta} - a^{(1)}_{\alpha\beta}\, e^{(1)}}{\ln \xi^{-1}}; \tag{9.21}$$

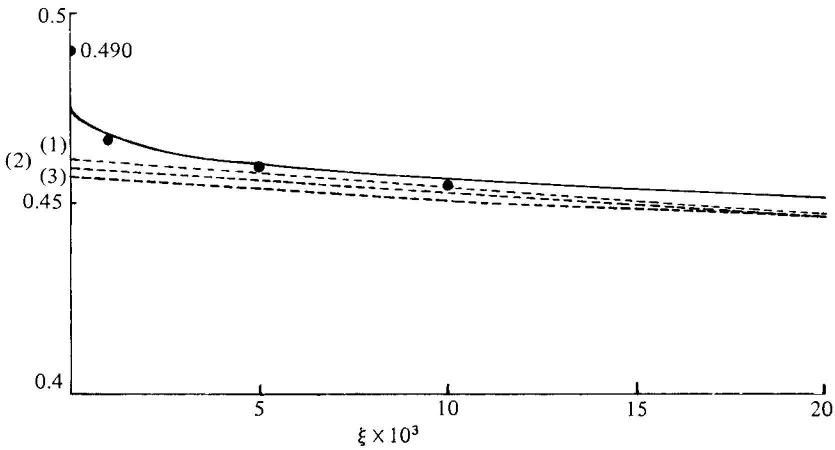

FIGURE 2. Various representations of the function $y^a_{12}(s, 1)$ in the neighbourhood of $s = 2$, plotted against $\xi = s-2$. The broken curves labelled (1)–(3) correspond to the series form summed to (1) terms to $O(s^{-80})$, (2) terms to $O(s^{-120})$, and (3) terms to $O(s^{-220})$. The solid line corresponds to both asymptotic expressions because the simplified and full expressions are indistinguishable on this plot. The solid circles are numerical data taken from Batchelor (1976).

$s\backslash\lambda$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
2.00	0.891	0.927	0.973	0.994
2.015	0.953	0.978	0.992	0.998
2.1	0.974	0.989	0.996	0.999
2.5	0.994	0.998	0.999	1.000
3.0	0.998	0.999	1.000	1.000
Asymptote	(0.998)	(0.999)	(1.000)	(1.000)

TABLE 13. The function $y^a_{11}(s, \lambda)$. For $s > 2.015$, the series form was used. At $s = 2.015$ the average of the nearly touching asymptotic form and the series form was used. At $s = 3$, the widely separated asymptotic form is given.

three series forms are given, corresponding to 80, 120 and 220 terms; finally some data points calculated for $\lambda = 1$ by Batchelor (1976) and by Nir & Acrivos (1973) are plotted. We see that there is a smooth transition between the full asymptotic form and the series forms for y^a_{11}, but for y^a_{12} the asymptotic curves and the series curves do not intersect. Since, for general values of λ, the series are known only to terms $O(s^{-120})$ the point 2.015 was chosen for the transition from the series form to the asymptotic form, this point being one where both forms are of comparable accuracy. For the tabulations in tables 13, 14 and 15, the average of the two values at $s = 2.015$ is given.

10. The mobility functions $y^b_{\alpha\beta}(s, \lambda)$

The calculations of §9 yield, from (9.17), the coefficients Ω_{pq} in a series expansion for $\Omega^{(\alpha)}$, the rate of rotation of a sphere translating under an applied force. The functions $y^b_{\alpha\beta}$ are deduced from these in the now standard way with the difference that odd powers of s^{-1} go to y^b_{11} and even powers to y^b_{12}.

$s\backslash\lambda$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
2.00	0.490	0.535	0.571	0.553
2.015	0.452	0.464	0.482	0.488
2.1	0.418	0.426	0.440	0.450
2.5	0.333	0.336	0.344	0.352
3.0	0.269	0.271	0.275	0.280
Asymptote	(0.269)	(0.271)	(0.275)	(0.280)

TABLE 14. The function $y^a_{12}(s, \lambda)$. For $s > 2.015$, the series form was used. At $s = 2.015$ the average of the series form and the nearly touching asymptotic form was used. At $s = 3$, the widely separated asymptotic form is given.

$s\backslash\lambda$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
2.00	0.891	0.764	0.473	0.238
2.015	0.953	0.889	0.768	0.663
2.1	0.974	0.934	0.868	0.820
2.5	0.994	0.983	0.970	0.965
3.0	0.998	0.995	0.992	0.991
Asymptote	(0.998)	(0.995)	(0.992)	(0.991)

TABLE 15. The function $y^a_{22}(s, \lambda)$. For $s > 2.015$, the series form was used. For $s = 2.015$, the average of the nearly touching asymptotic form and the series form was used. For $s = 3$, the widely separated asymptotic form is given.

10.1. *Widely separated spheres*

We have

$$y^b_{11} = \sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1}\,s^{-2k-1}, \tag{10.1}$$

$$y^b_{12} = \tfrac{1}{4}(1+\lambda)^2 \sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k}s^{-2k}, \tag{10.2}$$

where

$$f_0 = f_1 = 0, \quad f_2 = -2, \quad f_3 = f_4 = f_5 = f_6 = 0,$$
$$f_7 = 160\lambda^3 + 48\lambda^5, \quad f_8 = 0, \quad f_9 = 2240\lambda^5 + 192\lambda^7,$$
$$f_{10} = -4480\lambda^3 + 3200\lambda^5, \quad f_{11} = 21504\lambda^7 + 768\lambda^9.$$

The coefficients do not have the symmetry of the other cases, because $y^b_{12}(s, \lambda) \neq y^b_{12}(s, \lambda^{-1})$.

10.2. *Nearly touching spheres*

The same matrix inversion that gave the asymptotic expression for $y^a_{\alpha\beta}$ also gives an expression for $y^b_{\alpha\beta}$. We write

$$y^b_{\alpha\beta} = \frac{b^{(1)}_{\alpha\beta}(\ln\xi^{-1})^2 + b^{(2)}_{\alpha\beta}\ln\xi^{-1} + b^{(3)}_{\alpha\beta}}{(\ln\xi^{-1})^2 + e^{(1)}\ln\xi^{-1} + e^{(2)}} + O(\xi\ln\xi). \tag{10.3}$$

The fact that two touching spheres move as a rigid body gives us some conditions between the $b^{(1)}_{\alpha\beta}$ and the $a^{(1)}_{\alpha\beta}$, and the $c^{(1)}_{\alpha\beta}$ defined below in §11.2. First, the two spheres

λ	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$b_{11}^{(1)}$	0.032	0.095	0.134	0.091	0.033
$b_{11}^{(2)}$	−0.056	−0.039	0.200	0.527	0.618
$b_{11}^{(3)}$	−0.018	−0.239	−0.792	−1.451	−1.748
$b_{12}^{(1)}$	−0.204	−0.204	−0.134	−0.054	−0.012
$b_{12}^{(2)}$	−0.317	−0.681	−0.927	−0.992	−0.872
$b_{12}^{(3)}$	0.077	−0.101	−0.188	0.045	0.469
$c_{11}^{(1)}$	0.887	0.610	0.267	0.076	0.014
$c_{11}^{(2)}$	3.991	5.804	5.609	4.625	3.756
$c_{11}^{(3)}$	0.371	4.983	9.281	9.875	6.513
$c_{12}^{(1)}$	0.217	0.257	0.267	0.257	0.217
$c_{12}^{(2)}$	−1.097	−1.096	−1.058	−1.091	−1.087
$c_{12}^{(3)}$	0.785	0.450	0.300	0.428	0.750

TABLE 16. The functions $b_{\alpha\beta}$ and $c_{\alpha\beta}$ appearing in the asymptotic expressions for $y^b_{\alpha\beta}$ and $y^c_{\alpha\beta}$

$s\backslash\lambda$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
2.00	0.032	0.095	0.134	0.091	0.033
2.015	0.009	0.027	0.047	0.051	0.043
2.1	0.005	0.014	0.025	0.027	0.022
2.5	0.001	0.003	0.005	0.005	0.003

TABLE 17. The function $y^b_{11}(s,\lambda)$. The value at $s = 2.015$ is the average of the nearly touching asymptotic and series values. Widely separated asymptotic forms are not given because the values are so small.

$s\backslash\lambda$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
2.00	−0.204	−0.204	−0.134	−0.054	−0.012
2.015	−0.142	−0.144	−0.130	−0.112	−0.102
2.1	−0.121	−0.121	−0.116	−0.109	−0.105
2.5	−0.081	−0.081	−0.080	−0.079	−0.079
(Asymptote)	(−0.080)	(−0.080)	(−0.080)	(−0.080)	(−0.080)

TABLE 18. The function $y^b_{12}(s,\lambda)$. The value at $s = 2.015$ is the average of the nearly touching asymptotic and series values. The asymptotic value at $s = 2.5$ uses the widely separated form.

move with the same angular velocity $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = \Omega \boldsymbol{j}$. If we imagine, therefore, that forces $\boldsymbol{F}_1 = F_1\boldsymbol{i}$, $\boldsymbol{F}_2 = F_2\boldsymbol{i}$ and couples $\boldsymbol{L}_1 = L_1\boldsymbol{j}$, $\boldsymbol{L}_2 = L_2\boldsymbol{j}$ act on the spheres, then

$$\begin{aligned}(4\pi a_1^2)^{-1} y^b_{11} F_1 + (\pi(a_1+a_2)^2)^{-1} y^b_{12} F_2 + (8\pi a_1^3)^{-1} y^c_{11} L_1 + (\pi(a_1+a_2)^3)^{-1} y^c_{12} L_2 \\ = (\pi(a_1+a_2)^2)^{-1} y^b_{21} F_1 + (4\pi a_2^2)^{-1} y^b_{22} F_2 + (\pi(a_1+a_2)^3)^{-1} y^c_{12} L_1 + (8\pi a_2^3)^{-1} y^c_{22} L_2.\end{aligned}$$

Thus, since F_1, F_2, L_1 and L_2 are arbitrary, we obtain

$$y^b_{11} = 4(1+\lambda)^{-2} y^b_{21}, \quad 4(1+\lambda)^{-2} y^b_{12} = y^b_{22} \quad \text{at} \quad s = 2.$$

Therefore $$b_{11}^{(1)} = 4(1+\lambda)^{-2} b_{21}^{(1)}, \quad 4(1+\lambda)^{-2} b_{12}^{(1)} = b_{22}^{(1)}.$$

Similarly, from $\boldsymbol{U}_1 = \boldsymbol{U}_2 + \boldsymbol{\Omega} \wedge (a_1+a_2)\boldsymbol{e}$,

$$a_{11}^{(1)} = 2(1+\lambda)^{-1} a_{12}^{(1)} + \tfrac{3}{2}(1+\lambda) b_{11}^{(1)}.$$

Table 16 gives values of the $b_{\alpha\beta}$ for various λ.

10.3. *Numerical results*

The terms in the series were calculated to terms $O(s^{-120})$ and numerical results for summing these are given in tables 17 and 18.

11. The mobility functions $y^c_{\alpha\beta}(s, \lambda)$

The recurrence relations used to calculate y^a and y^b can be used with only the initial conditions changed to find y^b and y^c. The results for y^b reproduce those given in §10. The new initial conditions are

$$Q_{1pq} = \delta_{0p}\,\delta_{0q}, \quad P_{1pq} = 0 \quad (= V_{1pq}).$$

11.1. *Widely separated spheres*

As usual,

$$y^c_{11} = \sum_{k=0}^{\infty} f_{2k}(\lambda)\,(1+\lambda)^{-2k} s^{-2k}, \tag{11.1}$$

$$y^c_{12} = \tfrac{1}{8}(1+\lambda)^3 \sum_{k=0}^{\infty} f_{2k+1}(\lambda)\,(1+\lambda)^{-2k-1}\, s^{-2k-1}, \tag{11.2}$$

where

$$f_0 = 1, \quad f_1 = f_2 = 0, \quad f_3 = -4, \quad f_4 = f_5 = 0,$$
$$f_6 = -240\lambda^3, \quad f_7 = 0, \quad f_8 = -2496\lambda^5,$$
$$f_9 = 4800\lambda^3, \quad f_{10} = -18432\lambda^7, \quad f_{11} = 30720(\lambda^3+\lambda^5).$$

11.2. *Nearly touching spheres*

Using the same matrix inversion that gave y^a and y^b, we obtain

$$y^c_{\alpha\beta} = \frac{c^{(1)}_{\alpha\beta}\,(\ln \xi^{-1})^2 + c^{(2)}_{\alpha\beta} \ln \xi^{-1} + c^{(3)}_{\alpha\beta}}{(\ln \xi^{-1})^2 + e^{(1)} \ln \xi^{-1} + e^{(2)}}. \tag{11.3}$$

From the fact that touching spheres move as a rigid body, we obtain the relations

$$c^{(1)}_{11} = 8(1+\lambda)^{-3}\, c^{(1)}_{12} = \lambda^{-3} c^{(1)}_{22},$$

$$b^{(1)}_{11} = 4(1+\lambda)^{-2}\, b^{(1)}_{12} + \tfrac{1}{2}(1+\lambda)\, c^{(1)}_{11}, \quad \lambda^{-2}\, b^{(1)}_{22} = 4(1+\lambda)^{-2}\, b^{(1)}_{21} - 4(1+\lambda)^{-2}\, c^{(1)}_{12}.$$

Numerical values for the $c_{\alpha\beta}$ are given in table 10.

11.3. *Numerical results*

The series was summed to terms $O(s^{-120})$ and the results are tabulated in tables 19 and 20.

The change in sign of y^c_{12} with decreasing s shows that, when the two spheres are far apart, the application of a couple to one sphere results in a motion of the other sphere that is reminiscent of one sphere rolling on the other, but when the spheres are very close together, the lubricating layer of fluid between the spheres prevents this motion and forces the second sphere into a rigid-body motion as part of a doublet with the first sphere.

$s\diagdown\lambda$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
2.00	0.887	0.610	0.267	0.076	0.014
2.015	0.968	0.888	0.768	0.684	0.670
2.1	0.983	0.941	0.879	0.844	0.859
2.5	0.996	0.986	0.974	0.973	0.983
(Asymptote)	(0.996)	(0.987)	(0.976)	(0.977)	(0.988)

TABLE 19. The function $y^c_{11}(s,\lambda)$. The value at $s = 2.015$ is the average of the nearly touching asymptotic and series values. The value at $s = 2.5$ of the widely separated asymptotic form is given.

$s\diagdown\lambda$	$\frac{1}{4}$	$\frac{1}{2}$	1
2.00	0.217	0.257	0.267
2.015	−0.002	0.004	0.007
2.1	−0.032	−0.026	−0.023
2.5	−0.030	−0.029	−0.028
(Asymptote)	(−0.031)	(−0.029)	(−0.028)

TABLE 20. The function $y^c_{12}(s,\lambda)$. The value at $s = 2.015$ is the average of the nearly touching asymptotic and series values. The value at $s = 2.5$ of the widely separated asymptotic form is given.

12. The mobility functions $x^c_{\alpha\beta}(s,\lambda)$

The coefficients for this function are obtained from the relations

$$Q_{n00} = \delta_{1n}, \tag{12.1}$$

$$Q_{npq} = \sum_{s=0}^{q} \binom{n+s}{n} \frac{s}{n+1} Q_{s(q-s-1)(p-n)}, \tag{12.2}$$

for $n \geqslant 2$, and

$$\Omega_{pq} = Q_{1pq} - \sum_{s=0}^{\infty} \binom{s+1}{1} \frac{s}{2} Q_{s(q-s-1)(p-1)}. \tag{12.3}$$

12.1. *Widely separated spheres*

We have

$$x^c_{11} = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} f_n(\lambda)(1+\lambda)^{-n}\left(\frac{2}{s}\right)^n, \tag{12.4}$$

$$x^c_{12} = -\tfrac{1}{8}(1+\lambda)^3 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} f_n(\lambda)(1+\lambda)^n\left(\frac{2}{s}\right)^n, \tag{12.5}$$

where

$$f_0(\lambda) = 1, \quad f_1 = f_2 = 0,$$

$$f_3 = 1, \quad f_4 = f_5 = f_6 = f_7 = 0, \quad f_8 = -3\lambda^5,$$

$$f_9 = 0, \quad f_{10} = -6\lambda^7, \quad f_{11} = 0.$$

We shall not proceed further with this function, because there is little scope for applications.

During this work, one of us (DJJ) was supported by the Natural Environment Research Council of Great Britain, while the other (YO) was supported by the British Council.

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