Numerical holography and colliding shocks

Laurence Yaffe
University of Washington

based on work with Paul Chesler:
arXiv: 1011.3562, 1309.1439

Oxford University, May 2014

motivation

- Use gauge/gravity duality to study, far-from-equilibrium strongly interacting dynamics
- Go beyond near-equilibrium dynamics (linear response, probe approximation)
- Honestly solve dynamics of interesting initial states

gauge/gravity duality

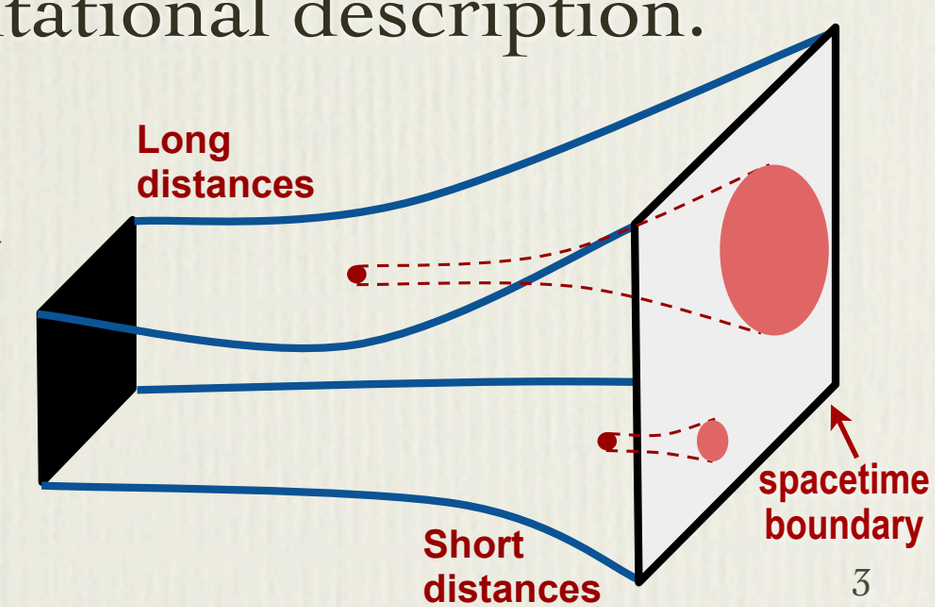
- a.k.a. "AdS/CFT duality," "gauge/string duality," "holography"

- Some non-Abelian gauge theories have exact reformulation as higher dimensional gravitational (or string) theories.

  Simplest case: maximally supersymmetric SU(N_c) Yang-Mills ($\mathcal{N}$=4 SYM) = string theory on $AdS_5 \times S^5$. More complicated generalizations for less supersymmetric, non-conformal theories.

- Strong coupling (and large N_c) limit of quantum field theory given by classical dynamics in dual gravitational description.

- Holographic description gives geometric representation of renormalization flow:



holography: features

- strongly coupled, large N QFT = classical (super)gravity in higher dimension
  - valid description on all scales
  - gravitational fluctuations: $1/N^2$ suppressed
  - QFT state ↔ asymptotically AdS geometry
  - $O(N^2)$ entropy ↔ gravitational (black brane) horizon
  - thermalization ↔ gravitational infall, horizon formation & equilibration
  - non-equilibrium QFT dynamics ↔ classical gravitational initial value problem

applications

- Heavy-ion collisions:
  - homogeneous isotropization — 0812.2053, 1309.1439 (C&Y)
  - boost invariant flow — 0906.4426 (C&Y)
  - colliding planar shocks — 1011.3562, 1309.1439 (C&Y)
  - colliding "nuclei" — 1305.4919 (Casalderrey-Solana, Heller, Mateos, van der Schee)
- Turbulence:
  - normal fluids — 1307.7267 (Adams,Chesler, Liu) 1309.1439 (C&Y)
  - superfluids — 1212.0281 (Adams,Chesler, Liu)
- Other stuff:
  - dynamical quenches — 1212.4498 (Figueras, Wiseman) 1207.4194 (Bhaseen, Gauntlett, Simons, Sonner, Wiseman)
  - black hole formation/ring-down — 1201.2132 (Bantilan, Pretorius, Gubser)

this talk

- Methods
  - characteristic formulation
  - residual diffeomorphism invariance
  - integration strategy
- Colliding planar shocks
  - dependence on shock width:
    - surviving remnants of initial shocks?
    - approximate boost invariance?
- Colliding “nuclei”

this talk

- Methods
  - characteristic formulation
  - residual diffeomorphism invariance
  - integration strategy
- Colliding planar shocks
  - dependence on shock width:
    - surviving remnants of initial shocks?
    - approximate boost invariance?
- Colliding "nuclei"

work in progress

metric ansatz

- Fefferman-Graham: good for equilibrium, bad for dynamics
- Eddington-Finklestein: good, regular across future horizon

$$ds^2 = \frac{r^2}{L^2}\, g_{\mu\nu}(x,r)\, dx^\mu\, dx^\nu - 2\, w_\mu(x)\, dx^\mu\, dr$$

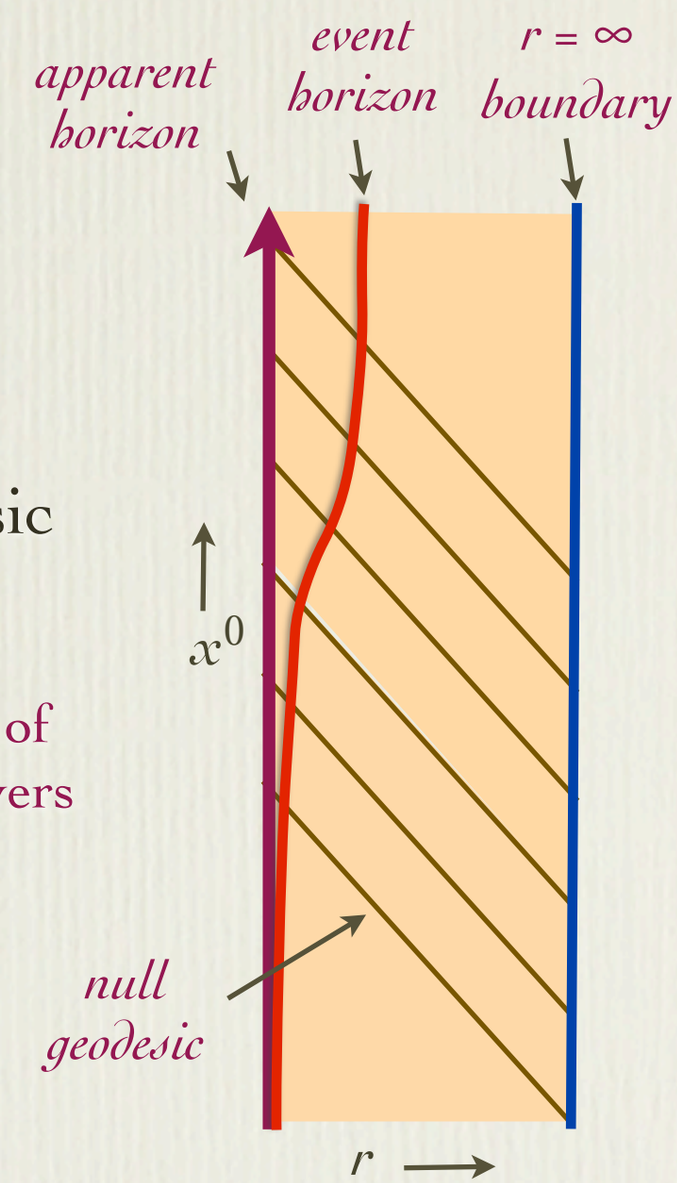
- x^μ = D-dimensional boundary coordinates
- $x^0 \equiv t$ = const. slices = null surface
- r = (non-inverted) bulk radius, ∂_r = infalling null geodesic
- $w = w_\mu\, dx^\mu$ = timelike boundary one-form
- boundary asymptotics:

preferred family of boundary observers

$$g_{\mu\nu}(x,r) \sim h_{\mu\nu}(x) + \sum_{n=1}^{\infty} g^{(n)}_{\mu\nu}(x)\, r^{-n}$$

boundary metric

$$\langle T_{\mu\nu}\rangle \equiv \frac{L^{D-1}}{16\pi G_N}\left(D\, g^{(D)}_{\mu\nu} + w^\alpha g^{(D)}_{\alpha\beta} w^\beta\, h_{\mu\nu}\right)$$



residual invariance/gauge fixing

$$ds^2 = \frac{r^2}{L^2}\, g_{\mu\nu}(x,r)\, dx^\mu\, dx^\nu - 2\, w_\mu(x)\, dx^\mu\, dr$$

- boundary diffeomorphisms: $x^\mu \to x'^\mu \equiv f^\mu(x)$

  ➡ choose co-moving boundary frame: $w_\mu(x) = -\delta_\mu^0$

  Eulerian → Lagrangian description

- radial shifts: $r \to \bar{r} = r + \delta\lambda(x)$

  ➡ fix radial position of apparent horizon: $r_{\rm h}(t, \mathbf{x}) = r_{\rm h}$

  ➡ rectangular computational domain

  Requires planar horizon topology

time/space split

- rename metric functions:

$$\frac{r^2}{L^2}\, g_{00}(x,r) \equiv -2A(t,\mathbf{x},r)$$
$$\frac{r^2}{L^2}\, g_{0i}(x,r) \equiv -F_i(t,\mathbf{x},r)$$
$$\frac{r^2}{L^2}\, g_{ij}(x,r) \equiv G_{ij}(t,\mathbf{x},r) \equiv \Sigma(t,\mathbf{x},r)^2\, \hat{g}_{ij}(t,\mathbf{x},r) \qquad \det(\hat{g}) \equiv 1$$

spatial scale factor

- radial shifts:

$$A(x,r) \to \bar{A}(x,\bar{r}) \equiv A(x,\bar{r}-\delta\lambda) + \partial_t\, \delta\lambda(x)$$
$$F_i(x,r) \to \bar{F}_i(x,\bar{r}) \equiv F_i(x,\bar{r}-\delta\lambda) + \partial_i\, \delta\lambda(x)$$

- modified (spatial+radial shift) covariant derivatives:

$$d_+ \equiv \partial_t + A(x,r)\,\partial_r \qquad d_i \equiv \partial_i + F_i(x,r)\,\partial_r$$

metric compatible, torsion free

$$v^i{}_{|k} \equiv (\widetilde{\nabla} v)^i{}_k = d_k(v^i) + \widetilde{\Gamma}^i{}_{jk}\, v^j = v^i{}_{,k} + v'^{\,i}\, F_k + \widetilde{\Gamma}^i{}_{jk}\, v^j$$

$$\widetilde{\Gamma}^i_{jk} \equiv \tfrac{1}{2}\, G^{il}\, (d_k\, G_{lj} + d_j\, G_{lk} - d_l\, G_{jk})$$
$$= \tfrac{1}{2}\, G^{il}\, \left(G_{lj,k} + G_{lk,j} - G_{jk,l} + G'_{lj}\, F_k + G'_{lk}\, F_j - G'_{jk}\, F_l\right)$$

$' \equiv \partial_r$

Einstein equations

$' \equiv \partial_r$

$$0 = \mathrm{tr}\left(G'' - \tfrac{1}{2}G'^{\,2}\right),$$

$$0 = A'' + \tfrac{1}{2}\widetilde{\nabla}\cdot F' + \tfrac{1}{2}F'\cdot F' + \tfrac{1}{2}(\mathrm{tr}\ d_+G)' + \tfrac{1}{4}\,\mathrm{tr}\,(G'\,d_+G) + 2\Lambda/\nu\,,$$

spatial dimensionality, D-1

$$0 = \mathrm{tr}\,[d_+(d_+G) - A'\,(d_+G) - \tfrac{1}{2}(d_+G)^2] + 2\,\widetilde{\nabla}\cdot E + \tfrac{1}{2}\mathrm{tr}\,(\Omega^2)\,.$$

radial shift "field strength": $\Omega_{ij} \equiv F_{j|i} - F_{i|j} = F_{j,i} - F_{i,j} + F_iF'_j - F_jF'_i$

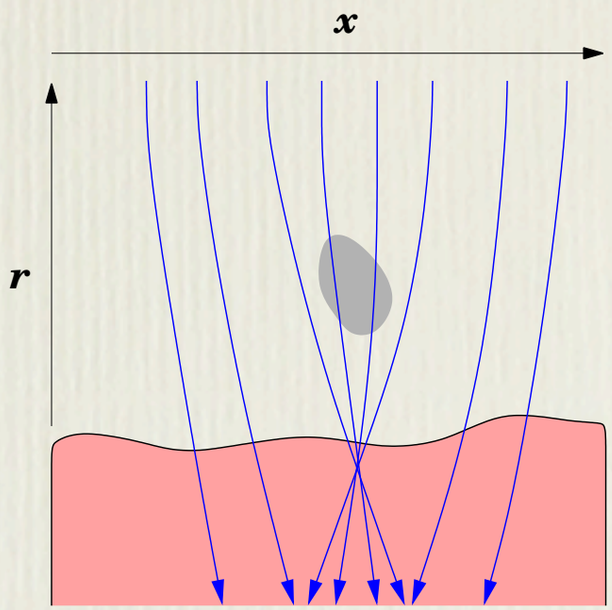
$$E_i \equiv d_+F_i - d_i\,A = F_{i,t} - A_{,i} + A\,F'_i - F_i\,A'$$

$$0 = G_{ik}\left[G^{1/2}\,F'^{\,k}\right]'\,G^{-1/2} - G'^{\,k}{}_{i|k} + (\mathrm{tr}\,G')_{|i}\,,$$

$$0 = d_+F'_i + (d_+G)^k{}_{i|k} - (\mathrm{tr}\,d_+G)_{|i} + \tfrac{1}{2}(\mathrm{tr}\ d_+G)F'_i - 2A'_{|i} - G'^{\,k}_i E_k + \Omega^k{}_{i|k} + F'_k\,\Omega^k{}_i\,,$$

$$0 = \left\{G_{ik}\left[G^{1/4}(d_+G)^k_j\right]'\,G^{-1/4} + \tfrac{1}{4}G'_{ij}\,\mathrm{tr}\,(d_+G) - \widetilde{R}_{ij} + \tfrac{2}{\nu}\,\Lambda\,G_{ij} + F'_{i|j} + \tfrac{1}{2}F'_iF'_j\right\} + (i \leftrightarrow j)\,,$$

apparent horizon



- require planar topology horizon = IR cutoff

  ➡ any caustics hidden behind horizon

- fixed-*r* horizon condition

  ➡ $d_+\Sigma|_{\bar{r}_\mathrm{h}} = S_{\dot{\Sigma}_\mathrm{h}}[\hat{g},\Sigma,F] \equiv -\frac{1}{2}\,\Sigma'\,F^2 - \frac{1}{\nu}\,\Sigma\,\nabla\cdot F$

- horizon stationarity ➡ $\partial_t\, d_+\Sigma|_{\bar{r}_\mathrm{h}} = \partial_t\, S_{\dot{\Sigma}_\mathrm{h}}[\hat{g},\Sigma,F]$

  ➡ $$\begin{aligned} 0 = {} & \nabla^2 A + \nabla A\cdot(F'-G'F) + \tfrac{1}{2}A\Big[-R+2\Lambda+\tfrac{1}{2}(F'-G'F)\cdot(F'-G'F)+\nabla\cdot(F'-G'F)\Big] \\ & + F\cdot F\Big[-\tfrac{1}{2}\mathrm{tr}\,[(d_+G)'] + (\nabla\cdot F)' + F_{i;j}G'^{ji} - \tfrac{1}{4}(F\cdot F)'\mathrm{tr}\,G'\Big] \\ & - \tfrac{1}{2}\mathrm{tr}\,[(d_+G)^2] + 2(d_+G)^{ji}F_{i;j} + 2F\cdot\nabla^2 F + (F'-G'F)\cdot\nabla(F\cdot F) \\ & - \tfrac{1}{2}(F_{i;j}-F_{j;i})(F^{j;i}-F^{i;j})\Big|_{r=r_h}, \end{aligned}$$

equation structure

$$\left(\partial_r^2 + Q_\Sigma[\hat{g}]\right)\Sigma = 0\,, \tag{1}$$

$$\left(\delta_i^j\,\partial_r^2 + P_F[\hat{g},\Sigma]_i^j\,\partial_r + Q_F[\hat{g},\Sigma]_i^j\right)F_j = S_F[\hat{g},\Sigma]_i\,, \tag{2}$$

$$\left(\partial_r + Q_{\dot{\Sigma}}[\Sigma]\right)d_+\Sigma = S_{\dot{\Sigma}}[\hat{g},\Sigma,F]\,, \tag{3}$$

$$\left(\delta_{(i}^k\delta_{j)}^l\,\partial_r + Q_{\dot{\hat{g}}}[\hat{g},\Sigma]_{ij}^{kl}\right)d_+\hat{g}_{kl} = S_{\dot{\hat{g}}}[\hat{g},\Sigma,F,\dot{\Sigma}]_{ij}\,, \tag{4}$$

$$\partial_r^2\,A = S_A[\hat{g},\Sigma,F,\dot{\Sigma},\dot{\hat{g}}]\,. \tag{5}$$

nested linear radial ODEs!

horizon stationarity condition:

$$\left(\nabla^2 + P_{A_h}[\hat{g},\Sigma,F]\cdot\nabla + Q_{A_h}[\hat{g},\Sigma,F]\right)A_h = S_{A_h}[\dot{\hat{g}},\hat{g},\dot{\Sigma},\Sigma,F] \tag{$\bigstar$}$$

linear elliptic PDE

radial integration constants

- near-boundary behavior:

radial shift

$$\Sigma \sim 1\, r + \lambda + O(r^{1-2D})$$

boundary momentum density

$$F_i \sim 0\, r^2 + \partial_i \lambda + f_i^{(D)}\, r^{2-D} + \cdots$$

boundary energy density

$$d_+\Sigma \sim \tfrac{1}{2}(r+\lambda)^2 + \tfrac{1}{2} a^{(D)}\, r^{2-D} + \cdots$$

$$d_+\hat{g} \sim 0\, r^{2-D} + O(r^{1-D})$$

$$A \sim r^2 + 2\lambda\, r + (\lambda^2 - 2\,\partial_t \lambda) + O(r^{2-D})$$

- near-horizon behavior:

$$A \to A_{\mathrm{h}}$$

integration strategy

- $t = t_0$: given $\hat{g}_{ij}(t_0, \mathbf{x}, r)$, $\lambda(t_0, \mathbf{x})$, $a^{(D)}(t_0, \mathbf{x})$, $f_i^{(D)}(t_0, \mathbf{x})$
  - integrate radial ODE (1) (for each $\mathbf{x}$) ➡ $\Sigma(t_0, \mathbf{x}, r)$
  - integrate (2) ➡ F
  - integrate (3) ➡ $d_+(\Sigma)$
  - integrate (4) ➡ $d_+(\hat{g})$
  - solve horizon eqn. (★) ➡ A_{horizon}
  - integrate (5) ➡ A

  $\Big\}$ ➡ $\partial_t \hat{g}$

  - extract $\partial_t \lambda$ from A
  - extract $\partial_t a^{(D)}$, $\partial_t f^{(D)}$ from stress-energy conservation:

$$\partial_t a^{(D)} = -\frac{D}{D-1} \partial_i f_i^{(D)}, \qquad \partial_t f_i^{(D)} = -\frac{1}{D} \partial_i a^{(D)} - \partial_j \hat{g}_{ji}^{(D)}$$

- do time step, $t \to t_0 + \epsilon$

details, details

- field redefinitions: remove leading pieces, rescale
- spatial compactification = addl. IR cutoff
- discretization = UV cutoff, pseudo-spectral derivatives
- choice of time integrator
- low-pass filtering to alleviate aliasing, spectral blocking
- domain decomposition

field redefinitions

- map radial direction to compact domain: $u = 1/r$
- remove singular $u \to 0$ terms
- rescale to make boundary asymptotics $O(u^0)$ or $O(u^1)$:

$$\sigma(x,u) \equiv \Sigma(x,1/u) - 1/u\,, \qquad \gamma_{ij}(x,u) \equiv u^{1-D}\left[\hat{g}_{ij}(x,1/u) - \delta_{ij}\right],$$
$$a(x,u) \equiv A(x,1/u) - \Sigma(x,1/u)^2\,, \quad \dot{\gamma}_{ij}(x,u) \equiv u^{2-D}\left[d_+\hat{g}_{ij}(x,1/u)\right],$$
$$f_i(x,u) \equiv F_i(x,1/u)\,, \qquad \dot{\sigma}(x,u) \equiv u^{3-D}\left[d_+\Sigma(x,1/u) - \tfrac{1}{2}\,\Sigma(x,1/u)^2\right]$$

UV cutoff: discretization

- use (pseudo)spectral methods: represent functions using finite Fourier (x) & Chebyshev (r) basis expansion

$$f(x) = \sum_{n=-M}^{M} \alpha_n \, e^{ik_n x} \qquad g(u) = \sum_{n=0}^{M} \alpha_n \, T_n(2u-1)$$

  - trade series coefficients for function values on collocation grid:

$$x_m = L\left(\frac{m}{2M+1}\right) \qquad u_m = \tfrac{1}{2}\left(1 - \cos\frac{m\pi}{M}\right)$$

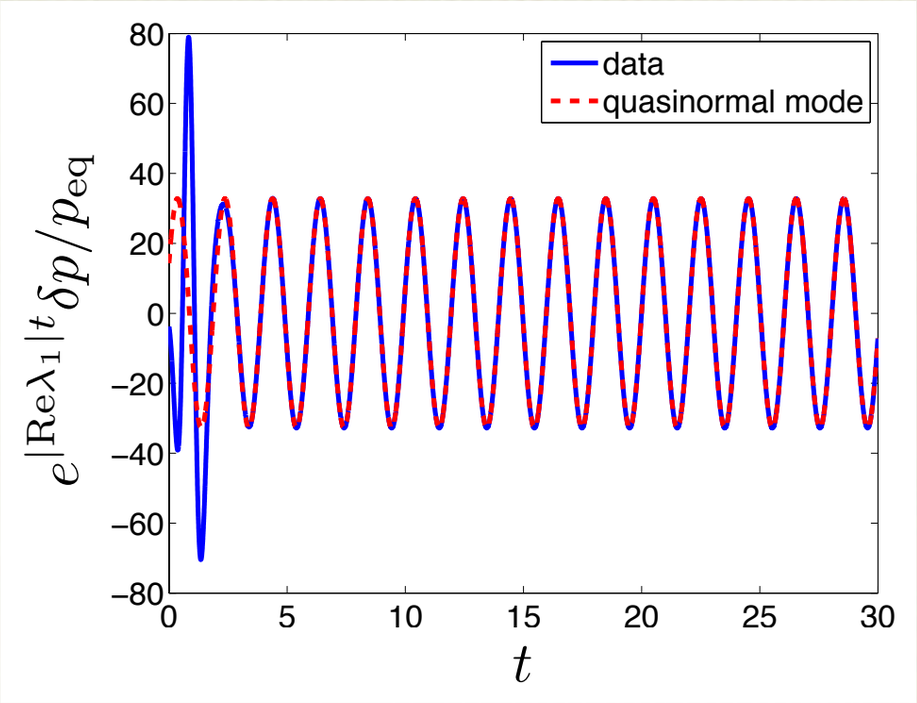
  - linear radial ODEs ➡ linear matrix equation with dense coefficient matrix

  - better accuracy, much faster convergence than traditional finite range finite difference approximations of derivatives

  - no need to excise $u = 0$ endpoint

performance

- ✓ stable evolution
- ✓ accurate results
- ✓ fast computation



(rescaled) pressure anisotropy
vs. lowest quasinormal mode

colliding planar null shocks

- energy density localized on infinite planar sheets
- caricature of large, Lorentz-contracted nuclei
- questions:
  - domain of validity of hydrodynamic approximation?
  - dependence on longitudinal profile?
  - surviving remnants?
  - approximate boost invariance?

colliding shocks

- 2D translation invariance

➡ $g_{\mu\nu} = g_{\mu\nu}(t, z, r)$ $\qquad \|\hat{g}_{ij}\| = \mathrm{diag}(e^B, e^B, e^{-2B})$

➡ 2+1D PDEs

- Initial conditions: superposition of counter-propagating planar shocks
  - Single shock, arbitrary longitudinal profile: known solution:

$$ds^2 = r^2[-dx_+ dx_- + d\boldsymbol{x}_\perp^2] + \frac{1}{r^2}\,[dr^2 + h(x_\pm)\,dx_\pm^2]$$

Janik & Peschanski

  - Choose Gaussian profile with width w, surface energy density μ^3:

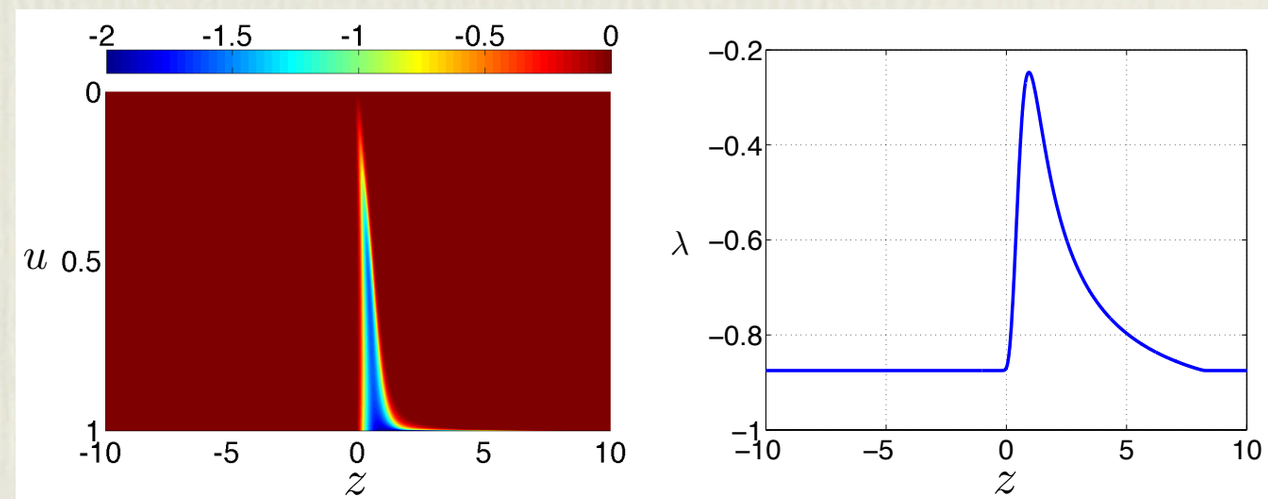
$$h(x_\pm) \equiv \mu^3\,(2\pi w^2)^{-1/2}\,e^{-\frac{1}{2}x_\pm^2/w^2}$$

    - Results depend on dimensionless width parameter $w\mu$

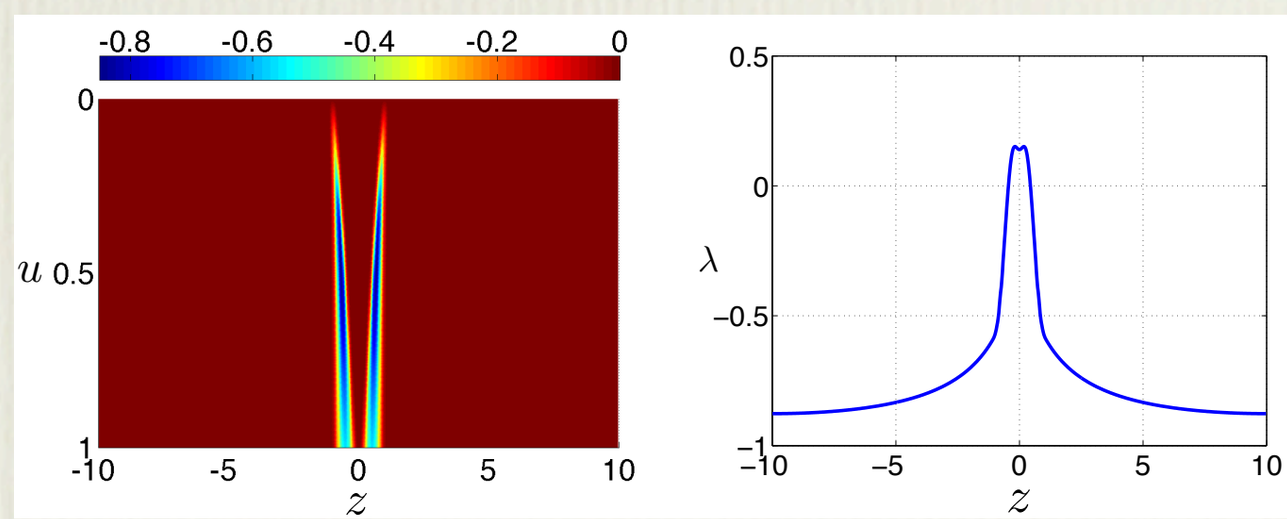
initial data

- transformation to infalling coordinates:
  - must solve coupled 1+1D PDEs
  - shocks extend "forward" deep in bulk
  - apparent horizon exists regardless of separation

anisotropy function $B(z,r)$ | radial shift $\lambda(z)$



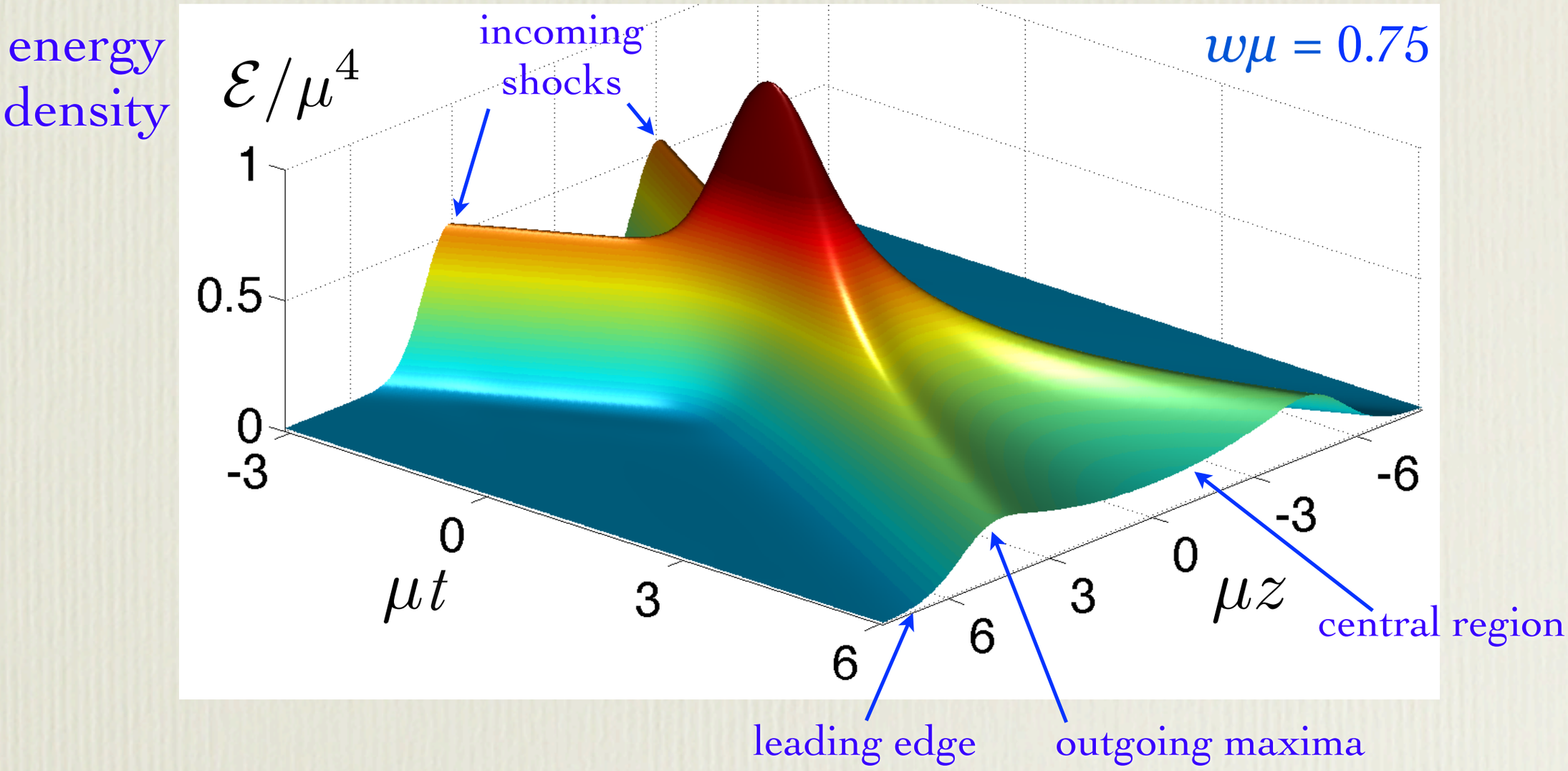
single right-moving shock



superposed shocks

old results

C&Y: 1011.3562



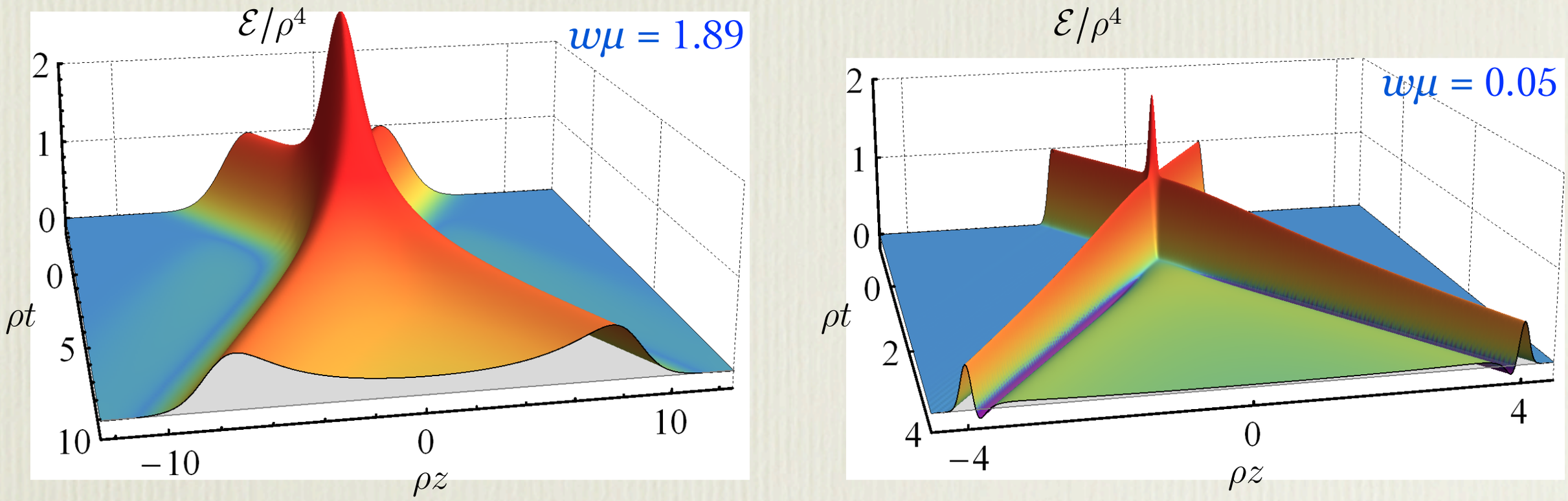
background energy density = 2% of single shock peak energy density

more recent results

From full stopping to transparency in a holographic model of heavy ion collisions

Jorge Casalderrey-Solana,[1] Michal P. Heller,[2,*] David Mateos,[3,4] and Wilke van der Schee[5]

1305.4919



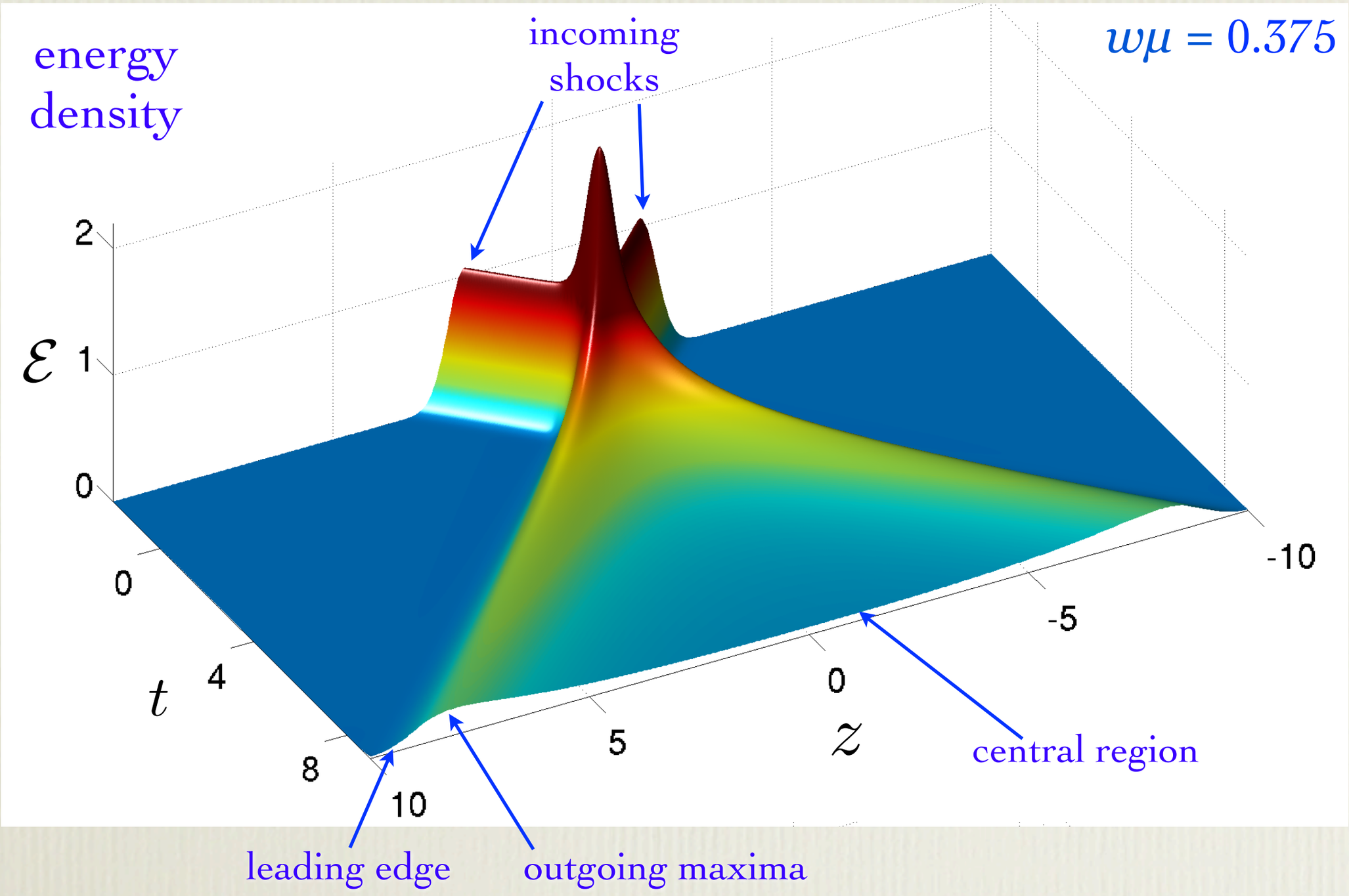
background energy density = 1.5 - 7.5% of single shock peak energy density

"We uncover a cross-over between two different dynamical regimes...
At high energies, receding fragments move outward at the speed of light."

most recent results (I)

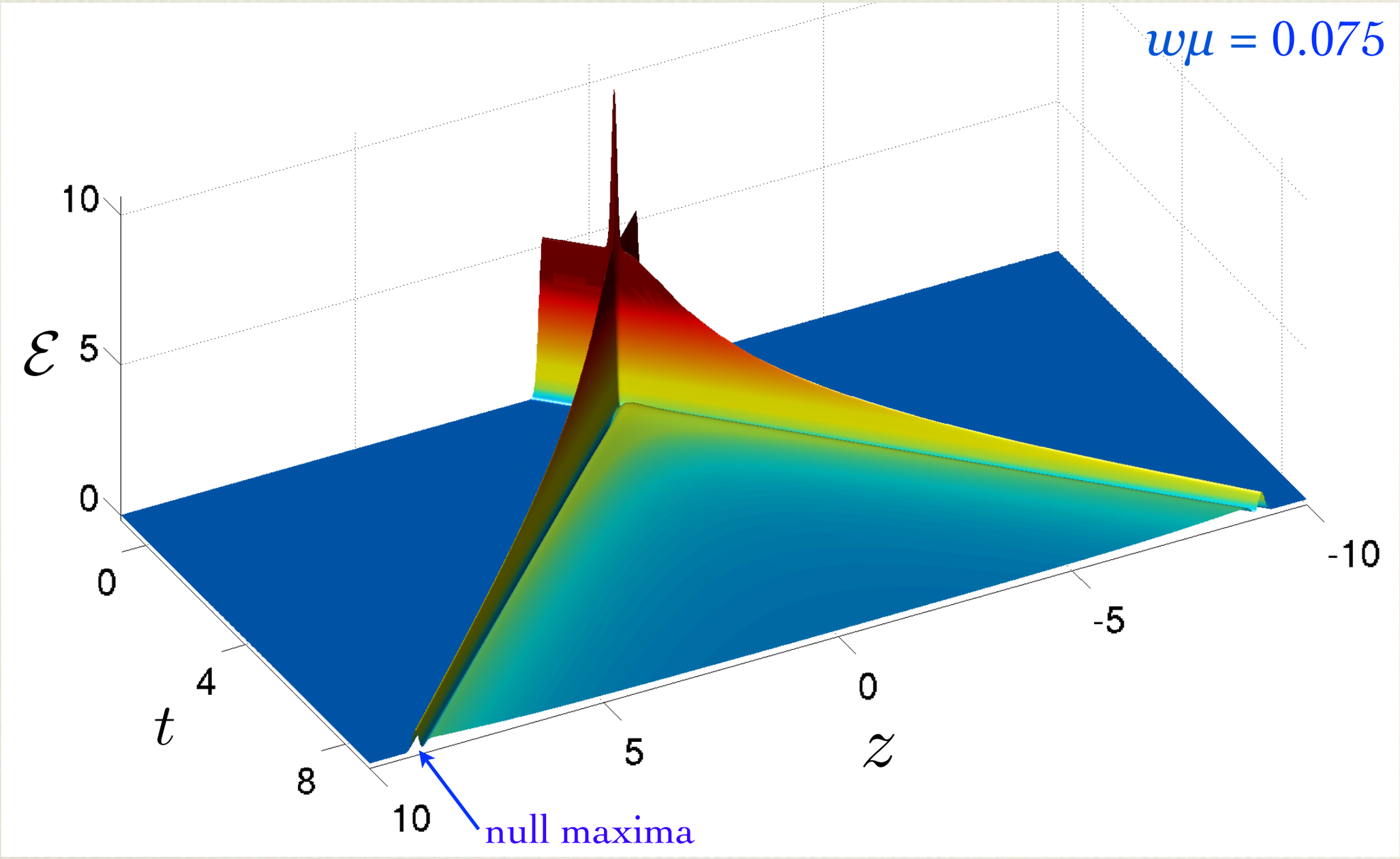
C&Y: 1309.1439

New: no background energy density

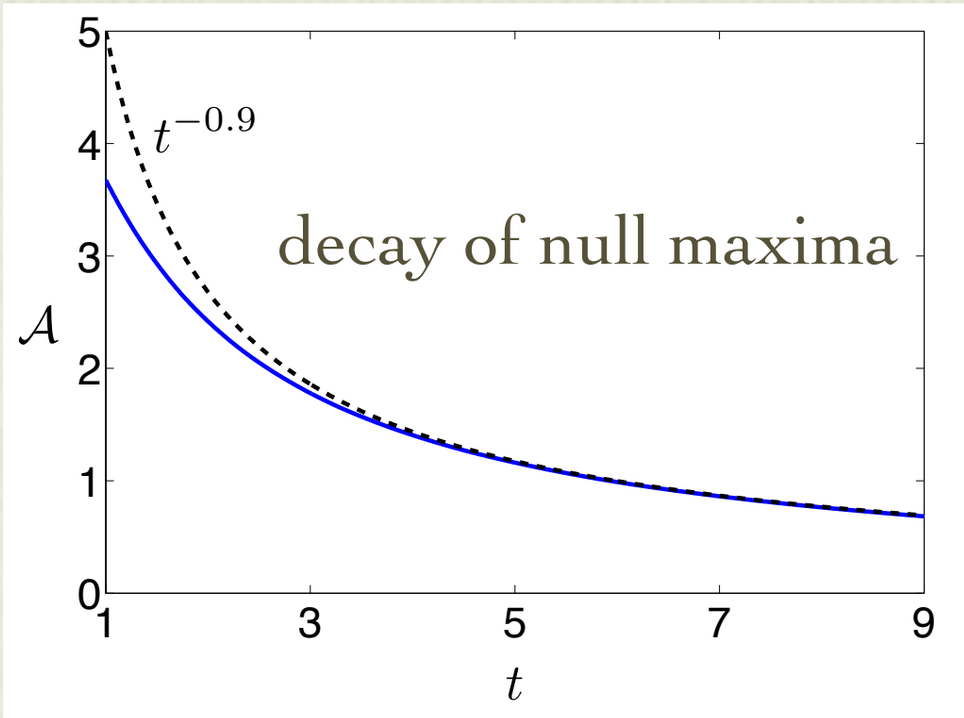


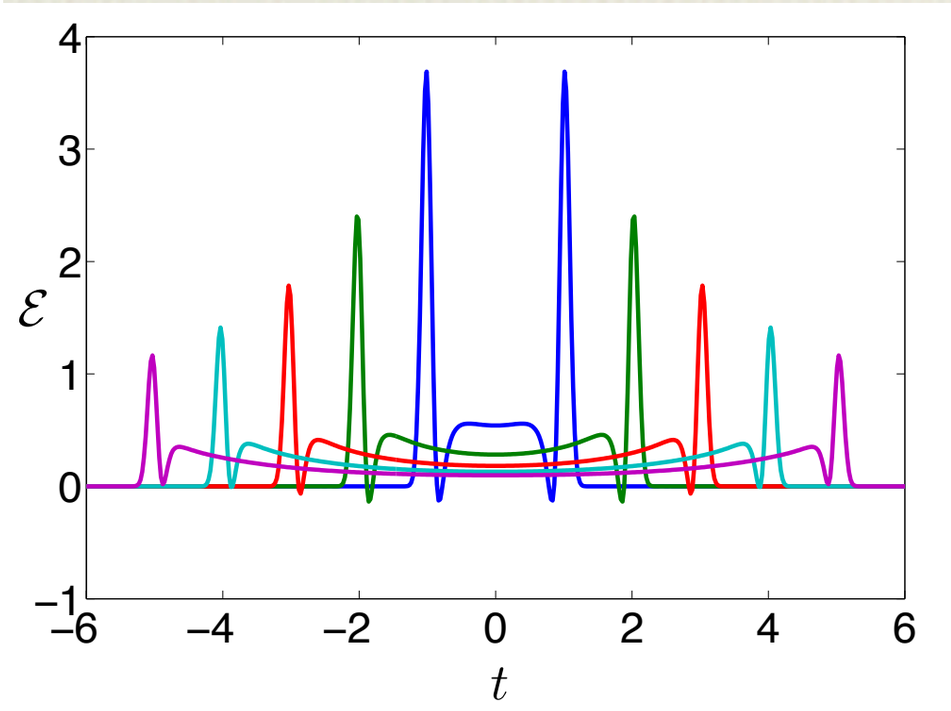
most recent results (II)

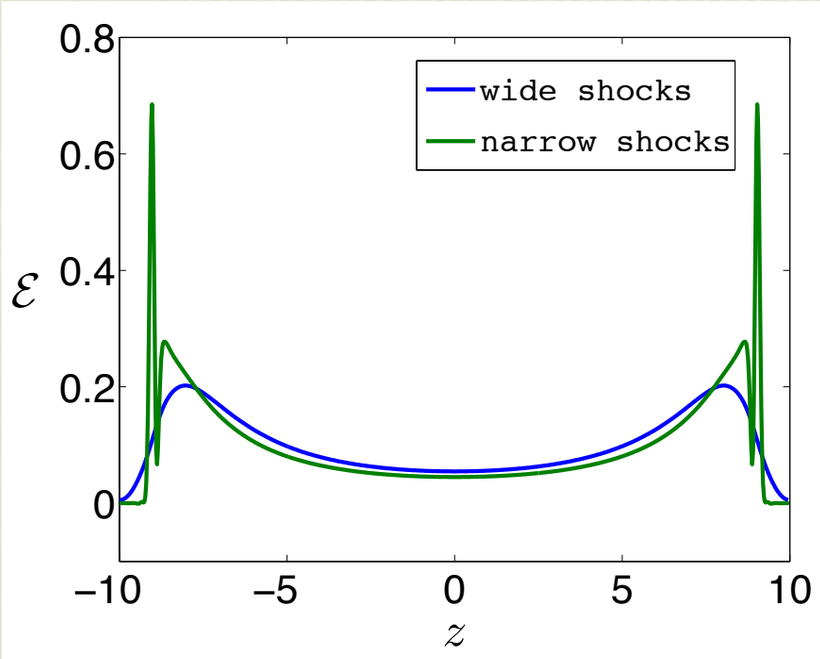
no background energy density,
longer time evolution



qualitative features







wide vs. narrow shocks,
$t = 9$

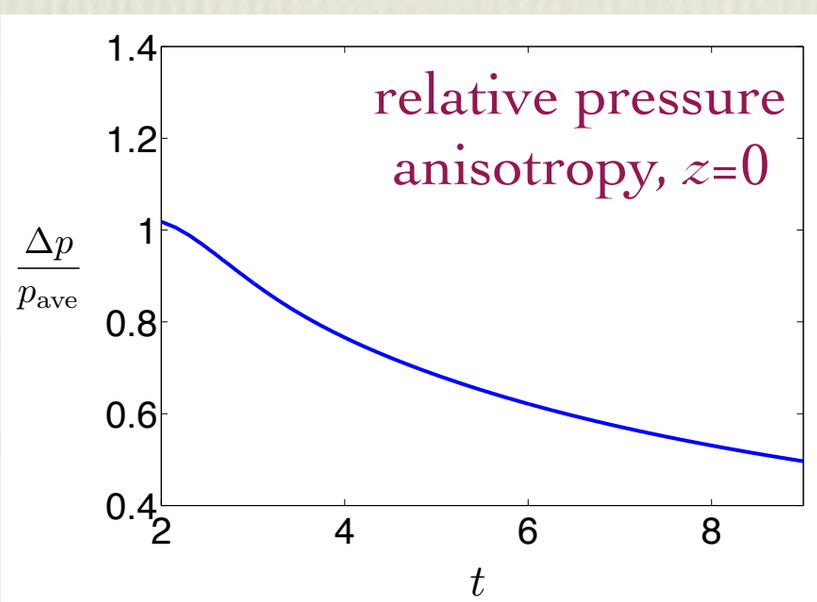
validity of hydrodynamics

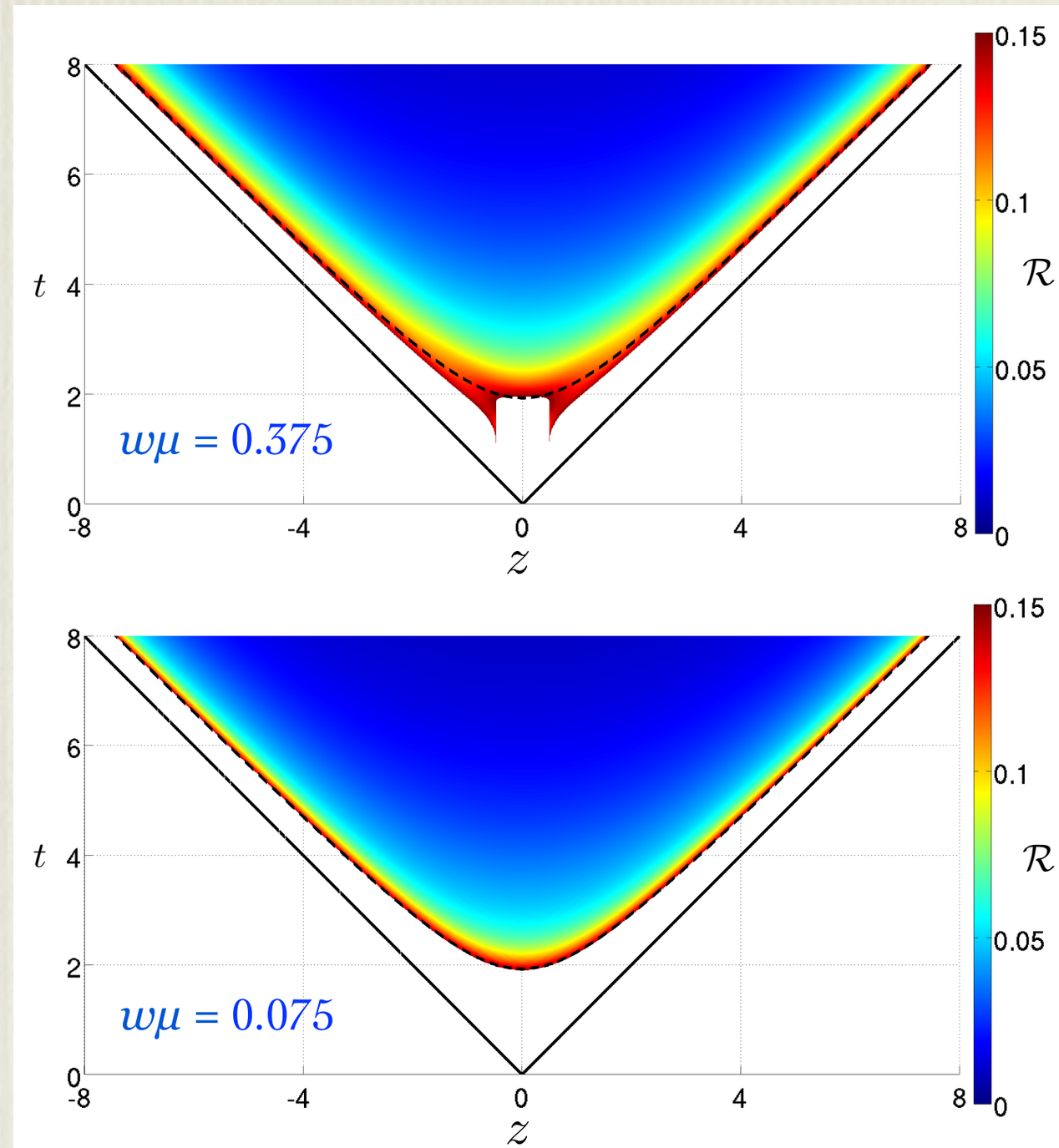
figure of merit:

$$\mathcal{R} \equiv \frac{1}{\bar{p}}\left[(\langle T^{xx}\rangle - T^{xx}_{\rm hydro})^2 + (\langle T^{zz}\rangle - T^{zz}_{\rm hydro})^2\right]^{1/2} \leq 15\%$$

first order viscous

hydro works even when viscous effects are $O(1)$:





local boost invariance

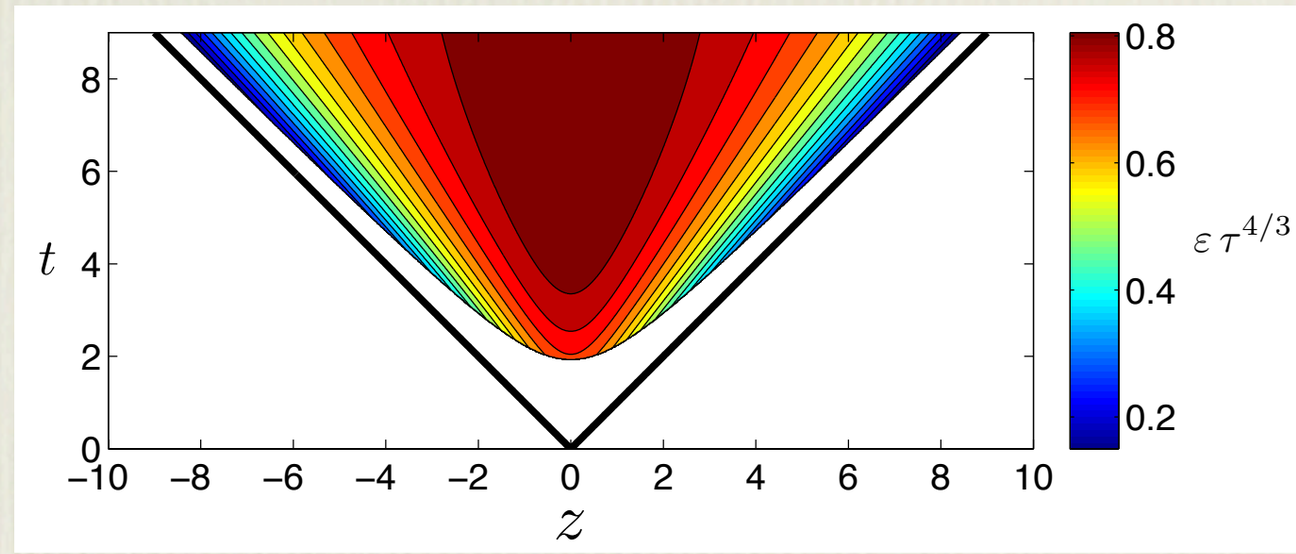
boost invariant flow:

$$u_\mu\, dx^\mu = d\tau \equiv \cosh y\, dt + \sinh y\, dz\,,$$

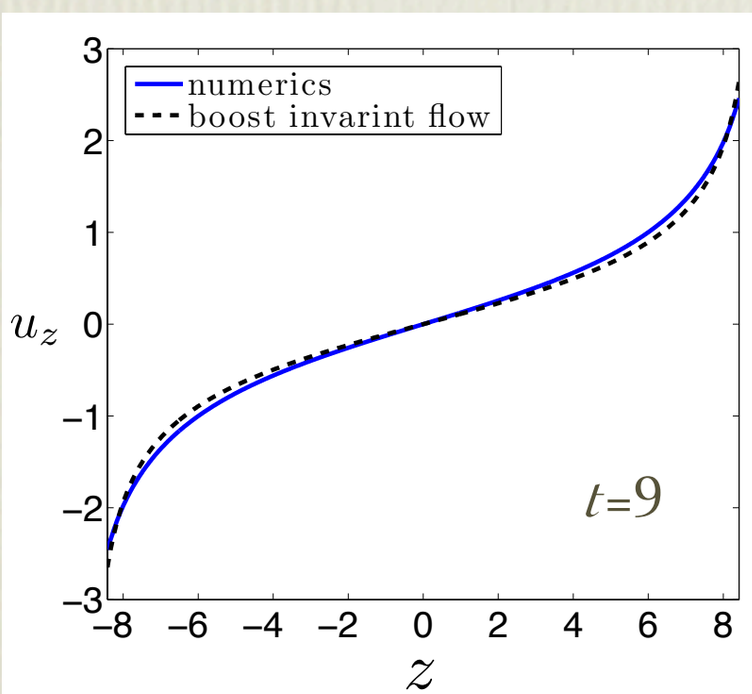
$$\varepsilon = \frac{3}{4}\frac{(\pi\Lambda)^4}{(\Lambda\tau)^{4/3}}\left[1 - \frac{C_1}{(\Lambda\tau)^{2/3}} + \frac{C_2}{(\Lambda\tau)^{4/3}} + O\Big(\frac{1}{(\Lambda\tau)^2}\Big)\right]$$

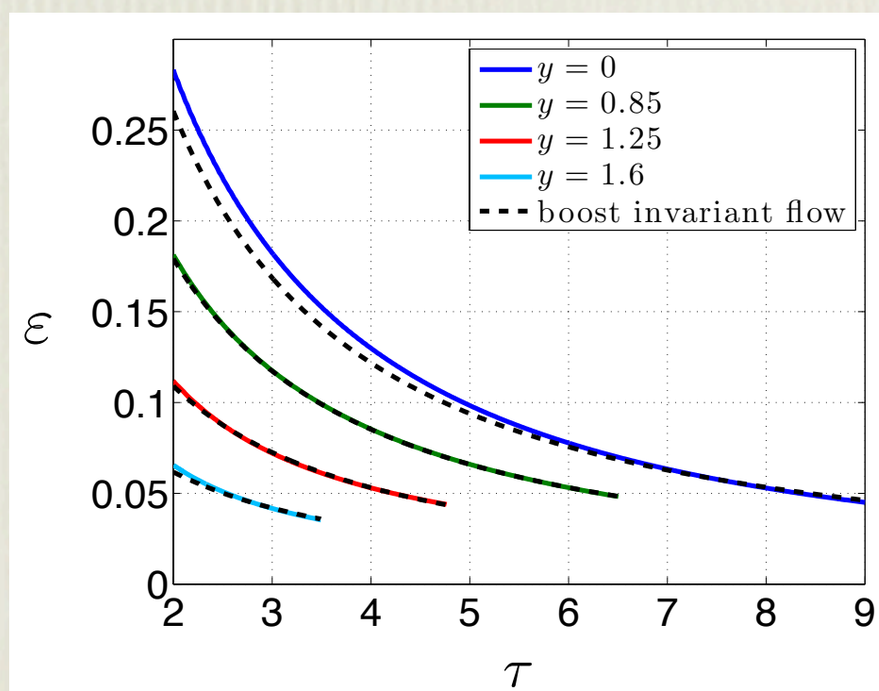
proper time $\tau \equiv \sqrt{t^2 - z^2}$

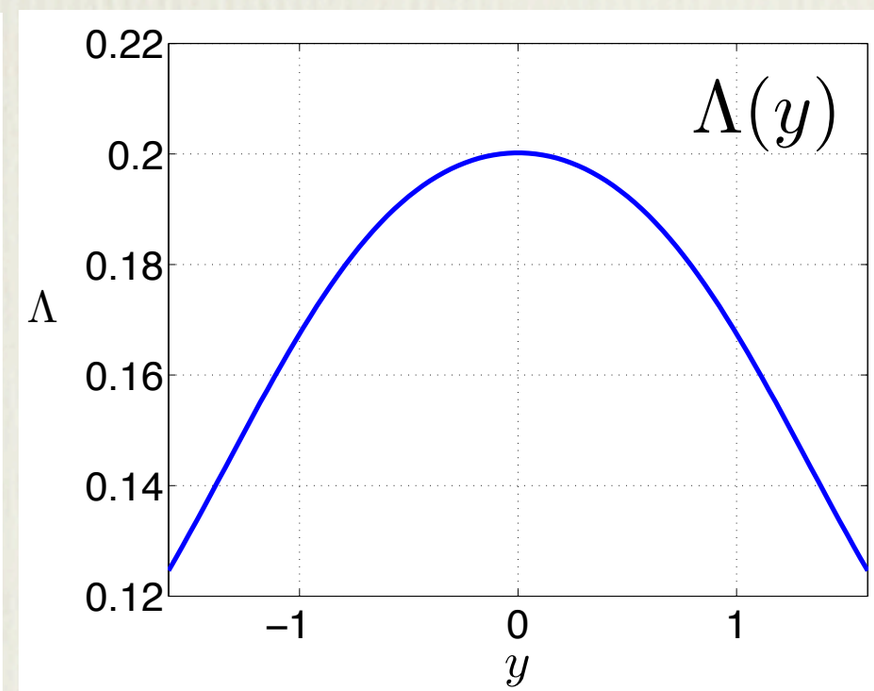
rapidity $y \equiv \tanh^{-1}(z/t)$



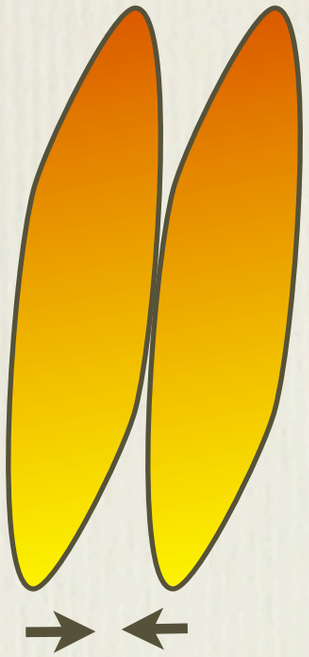
local boost invariance: $\Lambda \to \Lambda(y)$







colliding "nuclei"



- finite transverse extent, cylindrically symmetric
- single "nucleus": smooth, localized null "shock"
  - ✓ exact solution = linear superposition of infinitely boosted point sources Gubser, Pufu, Yarom
  - ✓ transformation to null infalling coordinates
- Matlab implementation for general 4+1D case: work in progress

remarks (I)

- using gauge/gravity duality to study strongly coupled far-from-equilibrium dynamics works for interesting variety of problems
  - characteristic formulation, adapted to gravitational infall ➡ remarkably simple equations allowing efficient integration
  - can achieve stable evolution
  - desktop resources suffice for 1+1D, 2+1D, and even 3+1D problems
  - no need to be professional numerical relativist!

remarks (II)

- work to date has only scratched the surface; many interesting generalizations await:
  - collisions:
    - asymmetric shocks
    - planar shocks with non-zero charge density
    - “nuclei” with finite transverse extent
  - turbulence in three spatial dimensions:
    - normal fluids
    - superfluids
  - dynamics in non-conformal theories with more complicated dual gravitational descriptions