Conformal anomaly, entanglement entropy and boundaries

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Plan of the talk:

1. Brief review: local Weyl anomaly, entanglement entropy

2. Integral Weyl anomaly in presence of boundaries

   a) d=4   b) d=6

3. Integral Weyl anomaly in *odd* dimensions

4. Entanglement entropy and boundaries

5. Some open questions

Based on

1. S.S., "Boundary terms of conformal anomaly," Phys. Lett. B **752**, 131 (2016) [arXiv:1510.04566 [hep-th]].

2. Dima Fursaev and S.S. "Anomalies, entropy and boundaries," arXiv:1601.06418 [hep-th].

3. work in progress with Amin Astaneh, Clement Berthiere

Other recent relevant works:

1. D. Fursaev, "Conformal anomalies of CFT's with boundaries," arXiv:1510.01427 [hep-th]

2. C. P. Herzog, K. W. Huang and K. Jensen, "Universal Entanglement and Boundary Geometry in Conformal Field Theory," arXiv:1510.00021 [hep-th].

Earlier relevant works:

1. J. S. Dowker and J. P. Schofield, "Conformal Transformations and the Effective Action in the Presence of Boundaries," J. Math. Phys. **31**, 808 (1990).

2. J. Melmed, "Conformal Invariance and the Regularized One Loop Effective Action," J. Phys. A **21**, L1131 (1988).

3. I. G. Moss, "Boundary Terms in the Heat Kernel Expansion," Class. Quant. Grav. **6**, 759 (1989).

4. T. P. Branson, P. B. Gilkey and D. V. Vassilevich, "The Asymptotics of the Laplacian on a manifold with boundary. 2," Boll. Union. Mat. Ital. **11B**, 39 (1997)

Let me first remind you

briefly the standard story

Local Weyl anomaly

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{c}{24\pi}R\,, \quad d=2$$

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = -\frac{a}{5760\pi^2}E_4 + \frac{b}{1920\pi^2}\mathrm{Tr}\,W^2\,, \;\; d=4$$

$$\mathrm{Tr}\,W^2 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$E_4 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,.$$

(For scalar field $a=b=1$)

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = 0\,, \quad d=2n+1$$

Entanglement entropy and Weyl anomaly

Σ is compact 2d entangling surface

$$S_{d=4} = \frac{A(\Sigma)}{4\pi\epsilon^2} + s_0 \ln \epsilon$$

$$s_0 = \frac{a}{180}\chi[\Sigma] - \frac{b}{240\pi}\int_\Sigma [W_{abab} - \mathrm{Tr}\, \hat{k}^2]$$

$\chi[\Sigma]$ is Euler number of Σ

W_{abab} is projection of Weyl tensor on subspace orthogonal to Σ, n^a, $a = 1, 2$ is a pair of normal vectors

$\hat{k}^a_{\mu\nu} = k^a_{\mu\nu} - \frac{1}{d-2}\gamma_{\mu\nu}k^a$, $a = 1, 2$ is trace-free extrinsic curvature of Σ

EE in d dimensions

In d dimensions compact entangling surface Σ is $(d-2)$-dimensional

Logarithmic term s_0 in entanglement entropy is given by integral over Σ of a polynomial invariant constructed from Weyl tensor $W_{\mu\alpha\nu\beta}$, even number of covariant derivatives of Weyl tensor, extrinsic curvature $\hat{k}^a_{\mu\nu}$ and projections on normal vectors n^a_μ.

If d is odd no such invariant exists so that

$$s_0 = 0 \quad if \;\; d = 2n+1$$

In this talk:

What changes if manifold has boundaries?

Conformal boundary conditions

General (mixed) boundary condition is a combination of Robin and Dirichlet b.c.

$$(\nabla_n + S)\Pi_+\varphi|_{\partial\mathcal{M}} = 0\,, \quad \Pi_-\varphi|_{\partial\mathcal{M}} = 0\,, \quad \Pi_+ + \Pi_- = 1$$

Conformal scalar field in d dimensions

Dirichlet b.c. ($\Pi_+ = 0$)

$$\phi|_{\partial\mathcal{M}} = 0$$

Conformal Robin b.c. ($\Pi_- = 0$)

$$(\nabla_n + \frac{(d-2)}{2(d-1)}K)\phi|_{\partial\mathcal{M}} = 0$$

Remark: in $d = 4$ exists one more (complex) Robin b.c.

$$S = \frac{1}{3}K \pm \frac{i}{10}\sqrt{10\mathrm{Tr}\,\widehat{K}^2}, \quad \widehat{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}K$$

for which (classical and quantum) theory is conformal

Dirac field in $d = 4$ dimensions

$$\Pi_- \psi|_{\partial\mathcal{M}} = 0 \,, \quad (\nabla_n + K/2)\Pi_+ \psi|_{\partial\mathcal{M}} = 0$$

$\Pi_\pm = \frac{1}{2}(1 \pm i\gamma_* n^\mu \gamma_\mu)$, γ_* is chirality gamma function

Integral Weyl anomaly

Variation of effective action under constant rescaling of metric

$$\mathcal{A} \equiv \partial_\sigma W[e^{2\sigma} g_{\mu\nu}] = \int_{\mathcal{M}_d} \left\langle T^\mu_\mu \right\rangle$$

For free fields integral Weyl anomaly reduces to computation of heat kernel coefficient A_d.

General structure

$$\int_{\mathcal{M}_d} \sqrt{g}\, \langle T_{\mu\nu} \rangle\, g^{\mu\nu} = a\ \chi(\mathcal{M}_d) + b_k \int_{\mathcal{M}_d} \sqrt{\gamma} I_k(W)$$

$$+ b'_k \int_{\partial\mathcal{M}_d} \sqrt{\gamma} J_k(W, \widehat{K}) + c_n \int_{\partial\mathcal{M}_d} \sqrt{\gamma} \mathcal{K}_n(\widehat{K})\,,$$

$\chi[\mathcal{M}_d]$ is Euler number of manifold $\mathcal{M}_d$, $I_k(W)$ are conformal invariants constructed from the Weyl tensor, $\mathcal{K}_n(\widehat{K})$ are polynomial of degree $(d-1)$ of the trace-free extrinsic curvature, $K_{\mu\nu} = K_{\mu\nu} - \frac{1}{d-2}\gamma K$ is trace free extrinsic curvature of boundary; $\widehat{K}_{\mu\nu} \to e^{\sigma}\widehat{K}_{\mu\nu}$ if $g_{\mu\nu} \to e^{\sigma} g_{\mu\nu}$.

Q: Does it mean that there are new conformal charges b'_n, c_n?

A: we suggest that in appropriate normalization $b'_n = b_n$ and the corresponding boundary term $J_k(W, \widehat{K})$ is in fact the Hawking-Gibbons type term for the bulk action $I_k(W)$

c_n are indeed new *boundary* conformal charges

Gibbons-Hawking type terms

Re-writing functional of curvature in a form linear in Riemann tensor

$$I_{bulk} = \int_{\mathcal{M}_d} \Big(U^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - U^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu} + F(V) \Big)$$

In order to cancel normal derivatives of the metric variation on the boundary one should add a boundary term,

$$I_{boundary} = -\int_{\partial\mathcal{M}_d} U^{\alpha\beta\mu\nu} P^{(0)}_{\alpha\beta\mu\nu}$$

$$P^{(0)}_{\alpha\beta\mu\nu} = n_\alpha n_\nu K_{\beta\mu} - n_\beta n_\nu K_{\alpha\mu} - n_\alpha n_\mu K_{\beta\nu} + n_\beta n_\mu K_{\alpha\nu}$$

n^μ is normal vector and $K_{\mu\nu}$ is extrinsic curvature of $\partial\mathcal{M}_d$

Barvinsky-SS (95)

For a bulk invariant expressed in terms of Weyl tensor only,

$$I[W] = \int_{\mathcal{M}_d} \Big(U^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu} - U^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu} + F(V) \Big)$$

$$- \int_{\partial\mathcal{M}_d} U^{\alpha\beta\mu\nu} P_{\alpha\beta\mu\nu}$$

$$P_{\alpha\beta\mu\nu} = P^{(0)}_{\alpha\beta\mu\nu} - \frac{1}{d-2} (g_{\alpha\mu} P^{(0)}_{\beta\nu} - g_{\alpha\nu} P^{(0)}_{\beta\mu} - g_{\beta\mu} P^{(0)}_{\alpha\nu}$$

$$+ g_{\beta\nu} P^{(0)}_{\alpha\mu}) + \frac{P^{(0)}}{(d-1)(d-2)} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\nu})$$

$$P^{(0)}_{\mu\nu} = n_\mu n^\alpha K_{\alpha\beta} + n_\mu n^\alpha K_{\alpha\nu} - K_{\mu\nu} - n_\mu n_\nu K$$

$$P^{(0)} = -2K$$

$P_{\alpha\beta\mu\nu}$ has same symmetries as the Weyl tensor. In particular, $P^{\alpha}{}_{\mu\alpha\nu} = 0$.

$P_{\alpha\beta\mu\nu}$ can be expressed in terms of $\widehat{K}_{\mu\nu}$

Examples

1. $$\int_{\mathcal{M}_d} \mathsf{Tr}\,(W^n) - \int_{\partial\mathcal{M}_d} n\,\mathsf{Tr}\,(PW^{n-1})$$

2. $$\int_{\mathcal{M}_d} \mathsf{Tr}\,(W\nabla^2 W) - 2\int_{\partial\mathcal{M}_d} \mathsf{Tr}\,(P\nabla^2 W)$$

Integral Weyl anomaly in $d = 4$: anomaly of type A

First of all, bulk integral of E_4 is supplemented by some boundary terms to form a topological invariant, the Euler number,

$$\chi[\mathcal{M}_4] = \frac{1}{32\pi^2}\int_{\mathcal{M}_4} E_4$$

$$-\frac{1}{4\pi^2}\int_{\partial\mathcal{M}_4}(K^{\mu\nu}R_{n\mu n\nu} - K^{\mu\nu}R_{\mu\nu} - KR_{nn} + \frac{1}{2}KR$$

$$-\frac{1}{3}K^3 + K\,\mathrm{Tr}\,K^2 + \frac{2}{3}\mathrm{Tr}\,K^3)$$

$R_{\mu n\nu n} = R_{\mu\alpha\nu\beta}n^\alpha n^\beta$ and $R_{nn} = R_{\mu\nu}n^\mu n^\nu$

Dowker-Schofield (90)

Herzog-Huang-Jensen (2015)

Integral Weyl anomaly in $d = 4$: anomaly of type B

Gibbons-Hawking type boundary term:

$$\int_{\mathcal{M}_4} \mathrm{Tr}\, W^2 - 2 \int_{\partial\mathcal{M}_4} \mathrm{Tr}\,(WP)$$

Due to properties of Weyl tensor:

$$\mathrm{Tr}\,(WP) = \mathrm{Tr}\,(WP^{(0)}) = 4 W^{\mu\nu\alpha\beta} n_\mu n_\beta \hat{K}_{\nu\alpha}$$

Integral Weyl anomaly in $d = 4$

$$\int_{\mathcal{M}_4} \langle T \rangle = -\frac{a}{180}\chi[\mathcal{M}_4]$$

$$+\frac{b}{1920\pi^2}\left(\int_{\mathcal{M}_4} \mathrm{Tr}\, W^2 - 8\int_{\partial\mathcal{M}_4} W^{\mu\nu\alpha\beta} n_\mu n_\beta \hat{K}_{\nu\alpha}\right)$$

$$+\frac{c}{280\pi^2}\int_{\partial\mathcal{M}_4} \mathrm{Tr}\, \hat{K}^3$$

For B-anomaly balance between bulk and boundary terms agrees with calculation for free fields of spin s=0,1/2, 1

Fursaev (2015)

also Herzog-Huang-Jensen (2015)

Values of boundary charge c:

(Malmed (88), Dowker-Schofield (95), Fursaev (2015))

$c = 1$ for $s = 0$ (Dirichlet b.c.)

$c = 7/9$ for $s = 0$ (Robin b.c.)

$c = 5$ for $s = 1/2$ (mixed b.c.)

$c = 8$ for $s = 1$ (absolute or relative b.c)

Local Weyl anomaly in $d = 6$

$$\langle T \rangle = \mathcal{A} = aE_6 + b_1 I_1 + b_2 I_2 + b_3 I_3 + TD$$

where E_6 is the Euler density in $d = 6$ and we defined

$$I_1 = \mathrm{Tr}_1(W^3) = W_{\alpha\mu\nu\beta} W^{\mu\sigma\rho\nu} W_{\sigma}{}^{\alpha\beta}{}_{\rho}$$

$$I_2 = \mathrm{Tr}_2(W^3) = W_{\alpha\beta}{}^{\mu\nu} W_{\mu\nu}{}^{\sigma\rho} W_{\sigma\rho}{}^{\alpha\beta}$$

$$I_3 = \mathrm{Tr}\,(W \nabla^2 W) + \mathrm{Tr}_2(WXW)$$

$$X_{\alpha\beta}{}^{\mu\nu} = X^{[\mu}_{[\alpha}\delta^{\nu]}_{\beta]}, \;\; X^\mu_\nu = 4R^\mu_\nu - \frac{6}{5} R \delta^\mu_\nu$$

Integral Weyl anomaly in $d = 6$

$$\int_{\mathcal{M}_6} \langle T \rangle = a' \chi[\mathcal{M}_6]$$

$$+ b_1 \left(\int_{\mathcal{M}_6} \mathrm{Tr}\,_1 W^3 - 3 \int_{\partial\mathcal{M}_6} \mathrm{Tr}\,_1 (PW^2) \right)$$

$$+ b_2 \left(\int_{\mathcal{M}_6} \mathrm{Tr}\,_2 W^3 - 3 \int_{\partial\mathcal{M}_6} \mathrm{Tr}\,_2 (PW^2) \right)$$

$$+ b_3 [\int_{\mathcal{M}_6} \mathrm{Tr}\,(W \nabla^2 W) - 2 \int_{\partial\mathcal{M}_6} \mathrm{Tr}\,(P \nabla^2 W)$$

$$+ \int_{\mathcal{M}_6} \mathrm{Tr}\,_2 (WXW) - \int_{\partial\mathcal{M}_6} \mathrm{Tr}\,_2 (WQW)]$$

$$+ \int_{\partial\mathcal{M}_6} \left(c_1 \mathrm{Tr}\, \hat{K}^2 \mathrm{Tr}\, \hat{K}^3 + c_2 \mathrm{Tr}\, \hat{K}^5 \right)$$

two new *boundary* charges c_1 and c_2

there may exist additional invariant with derivatives of extrinsic curvature

Integral Weyl anomaly in *odd* dimensions

Euler number of $\mathcal{M}_d$ vanishes if d is odd

Euler number of boundary $\partial\mathcal{M}_d$ may appear in integral anomaly

$\mathbf{d = 3}$:

$$\int_{\mathcal{M}_3} \langle T \rangle = \frac{c_1}{96}\chi[\partial\mathcal{M}_3] + \frac{c_2}{256\pi}\int_{\partial\mathcal{M}_3} \mathrm{Tr}\,\hat{K}^2$$

(c_1, c_2):

$(-1, 1)$ for scalar filed (Dirichlet b.c.)

$(1, 1)$ for scalar field (conformal Robin b.c)

$(0, 2)$ for Dirac field (mixed b.c.)

Remark: similar anomaly for defects Jensen-O'Bannon (2015)

Integral Weyl anomaly in $\mathbf{d=5}$

$$\int_{\mathcal{M}_5}\langle T\rangle = c_1\chi[\partial\mathcal{M}_5]$$

$$+\int_{\partial\mathcal{M}_5}[c_2\mathrm{Tr}\,W^2+c_3W_{\alpha n\beta n}W^{\alpha\ \beta}_{\ n\ n}+c_4W_{n\alpha\beta\mu}W_n^{\ \alpha\beta\mu}$$

$$+c_5W^{\alpha\mu\beta\nu}\hat{K}_{\alpha\beta}\hat{K}_{\mu\nu}+c_6W^{\alpha\ \beta}_{\ n\ n}\hat{K}_{\alpha\sigma}\hat{K}^{\sigma}_{\ \beta}$$

$$+c_7(\mathrm{Tr}\,\hat{K}^2)^2+c_8\mathrm{Tr}\,\hat{K}^4+c_9\mathrm{Tr}\,(\hat{K}\mathcal{D}\hat{K})]$$

$\mathcal{D}$ is conformal operator acting on trace free symmetric tensor in 4 dimensions

values of c_k for conformal scalar field: work in progress with Clement Berthiere

Entanglement entropy: $d = 3$

(recent work with Fursaev)

Renyi entropy

$$S^{(n)} \simeq c(n)L/\epsilon - \ln(\epsilon)s^{(n)}$$

$$s^{(n)} = \eta\frac{nA_3(1) - A_3(n)}{n-1}$$

$A_3(n)$ is heat kernel coefficient on replica manifold $\mathcal{M}_n$

Consider $\mathcal{M} = R^2 \times L$, L is an interval with 2 end points P_1 and P_2

Entangling surface $\Sigma = L$, replica space $\mathcal{M}_n = \mathcal{C}_n \times L$

$$A_3(n) = A_2(\mathcal{C}_n) \times A_1(L)\,,$$

$$A_2(\mathcal{C}_n) = \frac{1}{12n}(1 - n^2)$$

is the heat kernel coefficient on two-dimensional cone, and

$$A_1(L) = \frac{1}{4}\sum_{P_k} \text{tr}\ \chi\,, \quad \chi = \Pi_+ - \Pi_-$$

for scalar field

$$s^{(n)} = \frac{c_1}{48}\frac{n+1}{n}\sum_P \ , \quad s^{(n=1)} = \frac{c_1}{24}\sum_P$$

for Dirac field

$$s^{(n)} = 0 \ , \quad s^{(n=1)} = 0$$

INTERESTING PREDICTION:

dependence on angle between entangling surface Σ and boundary ∂M

$\cos\alpha = (n,t)$,
n^μ normal vector to $\partial\mathcal{M}_3$,
t^μ tangent vector to Σ.

Assume that the bulk $\mathcal{M}_n$ contains a conical singularity then:

scalar curvature of the boundary

$$\int_{\partial\mathcal{M}_n} \hat{R} \simeq 4\pi\cos\alpha\ (1-n)\,, \quad n\to 1$$

and extrinsic curvature of the boundary

$$\int_{\partial\mathcal{M}_n} K^2 \simeq \int_{\partial\mathcal{M}_n} \mathrm{Tr}\ K^2 \simeq 8\pi(1-n)f(\alpha) \quad ,$$

$$f(\alpha) = -\frac{1}{32}\frac{\sin^2\alpha}{\cos\alpha}(1+2\cos^2\alpha+5\cos^4\alpha)$$

OTHER DIMENSIONS

$P = \Sigma \cap \partial\mathcal{M}_d \quad dim(P) = d - 3$

$p_a^\mu\,, a = 1, 2$ normal vectors to P in $\partial\mathcal{M}_d$
$\hat{k}^a_{\mu\nu}$ is respective extrinsic curvature of P

$\hat{K}_{ab} = p_a^\alpha p_b^\beta \hat{K}_{\alpha\beta}$

$\mathbf{d = 3}: dim(P) = 0 \qquad s_0(P) \sim \sum_P$

$\mathbf{d = 4}: dim(P) = 1 \qquad s_0(P) \sim \int_P \hat{K}_{aa}$

$\mathbf{d = 5}: dim(P) = 2$

possible terms in $s_0(P)$: $\chi(P)\,,\ W_{nana}\,,\ W_{abab}\,,$ $(\hat{K}_{aa})^2\,,\quad \hat{K}_{ab}\hat{K}_{ab}\,,\quad \mathrm{tr}\,\hat{k}^2$ and terms with two derivatives of extrinsic curvature

RESUME

in presence of boundaries integral Weyl anomaly is modified by boundary terms

boundary terms for B-anomaly are of Gibbons-Hawking type

additional new *boundary* charges

in odd dimensions integral Weyl anomaly is non-vanishing (!) and is entirely due to boundary terms

if intersection of entangling surface and boundary is P then there appear new contributions to EE (and RE) due to P

in odd dimensions log term in EE (and RE) is non-vanishing (!) and is entirely due to P

SOME OPEN QUESTIONS

1. how derive boundary charges from n-point correlation functions in CFT?

2. what is holographic description of boundary terms in anomaly and in EE?
(work in progress with Amin Astaneh)

THANK YOU!