Causality in Lovelock theories of gravity

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HSR, N. Tanahashi and B. Way arXiv:1406.3379, 1409.3874
G. Papallo and HSR arXiv:1508.05303

Lovelock's theorem (1971)

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

LHS is most general symmetric tensor that is

- a function of g, ∂g, $\partial^2 g$
- divergence-free

This assumes $d = 4$ dimensions. For $d > 4$, extra terms can appear on LHS. These were classified by Lovelock.

Lovelock theories

Assume $T_{ab} = 0$

Simplest Lovelock theory is Einstein-Gauss-Bonnet:

$$G^a{}_b + \Lambda g_{ab} + \alpha\, \delta^{ac_1c_2c_3c_4}_{bd_1d_2d_3d_4} R_{c_1c_2}{}^{d_1d_2} R_{c_3c_4}{}^{d_3d_4} = 0$$

- α has dimensions of length^2: sets a scale for the theory.
- Nonlinear in $\partial^2 g$: rather exotic as PDEs.

Effective field theory perspective: α much larger than couplings for other higher derivative terms. Lovelock terms are the only terms for which this makes sense classically.

Motivation

- There has been interest in classical GR in $d > 4$ dimensions. Classically, Lovelock theories are as well-motivated as GR. They can be viewed as a deformation of GR.
- How do properties of such theories differ from GR? Is GR special? Are Lovelock theories pathological in some way?
- A Gauss-Bonnet term is predicted by some string theories. Is this inconsistent unless one includes the rest of string theory e.g. infinitely many higher derivative terms with couplings of order α? Camanho, Edelstein, Maldacena & Zhiboedov 2014

Characteristic surfaces

Causal properties of a PDE are determined by its *characteristic surfaces*.

e.g. scalar fields u^I, $I = 1, \ldots, N$, second order PDE

$$P^I{}_J{}^{\mu\nu} \partial_\mu \partial_\nu u^J = \mathcal{F}^I(u, \partial u)$$

Hypersurface Σ is characteristic iff this eq does *not* determine $\partial^2 u$ uniquely in terms of $u, \partial u$ on Σ.

1-form ξ normal to Σ: define *characteristic polynomial*

$$Q(x, \xi) = \det P^I{}_J{}^{\mu\nu} \xi_\mu \xi_\nu$$

Σ is characteristic iff $Q = 0$ everywhere on Σ.

Klein-Gordon: $P^I{}_J{}^{\mu\nu} = \delta^I_J g^{\mu\nu}$ so Σ characteristic iff $g^{\mu\nu} \xi_\mu \xi_\nu = 0$: null hypersurface.

Characteristic surfaces and causality

1. Consider a solution with continuous $u, \partial u$ but $\partial^2 u$ discontinuous across a surface Σ. Then Σ must be characteristic. Similarly discontinuities in $\partial^{100} u$ also propagate along characteristic surfaces.

2. High-frequency wave Ansatz: $\omega \gg 1$

$$u(x) = \bar{u}(x) + \frac{1}{\omega^2} v(x, \omega\phi(x)) + \dots$$

Surfaces of constant phase ϕ are characteristic w.r.t. background solution $\bar{u}$

3. Initial data prescribed on S. Region of spacetime in which solution is determined by data on $\Omega \subset S$ is bounded by ingoing characteristic surface from $\partial\Omega$.

Causality in Lovelock theories

In Klein-Gordon, Yang-Mills, GR, a hypersurface is characteristic iff it is null so causality is determined by the lightcone.

Characteristic hypersurfaces of Lovelock theories are generically non-null (Aragone 1987, Choquet-Bruhat 1988) so gravity can propagate faster or slower than light.

In AdS can have propagation that is superluminal w.r.t. boundary metric (Brigante et al 2008): problem for an AdS/CFT interpretation but is there anything wrong with the classical bulk theory?

We'll focus on asymptotically flat boundary conditions.

Superluminal propagation vs causality violation

It is widely believed that superluminal propagation in a Lorentz covariant theory implies that one can violate causality, i.e., build a "time machine".

For example, consider a scalar field with action (Adams et al 2006)

$$S = -\frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \frac{c}{\Lambda^4}\left(\eta^{\mu\nu}\partial_\mu\pi\partial_\nu\pi\right)^2\right]$$

where c is dimensionless and Λ has dimensions of mass.

Equation of motion is $G^{\mu\nu}\partial_\mu\partial_\nu\pi = 0$ where

$$G^{\mu\nu} = \left[1 - \frac{2c}{\Lambda^4}(\partial\pi\cdot\partial\pi)\right]\eta^{\mu\nu} - \frac{4c}{\Lambda^4}\partial^\mu\pi\partial^\nu\pi.$$

A surface is characteristic iff it is null w.r.t. $G^{\mu\nu}$.

The "effective metric" $G^{\mu\nu}$ determines causality, not $\eta^{\mu\nu}$.

If $c > 0$ then causal cones of $G^{\mu\nu}$ lie inside those of $\eta^{\mu\nu}$: subluminal propagation. if $c < 0$ then it is the other way round: possible superluminal propagation (e.g. of small fluctuations around a background solution).

Adams *et al* argued that the $c < 0$ theory must be rejected because one can build a solution with closed causal curves w.r.t. $G^{\mu\nu}$ (i.e. a "time machine") by considering two blobs of non-trivial π-field that are highly boosted w.r.t. to each other.

Consider initial data $(\pi, \partial_0 \pi)$ on the surface $\Sigma = \{x^0 = 0\}$ describing such a configuration. Σ is everywhere spacelike w.r.t. $\eta^{\mu\nu}$ but *not* w.r.t. $G^{\mu\nu}$.

The initial value problem is not well-posed. One expects that either no solution of the equation of motion exists or the solution does not depend continuously on the initial data, i.e., it is infinitely fine-tuned. So there is no reason to believe that one can build a "time machine" when $c < 0$.

Conclusion: the argument that superluminal propagation in a Lorentz covariant field theory implies causality violation is not convincing. (cf Geroch 2010)

For *small* initial data (i.e. $\pi, \partial\pi, \ldots \partial^N\pi$ all small) it is well known that solutions simply disperse, for either sign of c (Christodoulou 1986, Klainerman 1986). Superluminal propagation is not a problem.

Characteristic surfaces

What do Lovelock characteristic surfaces look like?

Example 1: Ricci flat type N spacetime

Type N: $\exists$ null ℓ^a such that $\ell^a C_{abcd} = 0$ (e.g. pp-wave).
Solves Lovelock eq. of motion with $\Lambda = 0$.

A hypersurface is characteristic iff it is null w.r.t. one of $N = d(d-3)/2$ "effective metrics" of form

$$G_{(I)ab} = g_{ab} - \alpha \omega_{(I)} \ell_a \ell_b \qquad I = 1, \ldots, N$$

where ω_I is homogeneous (degree 1) function of curvature.

- Different graviton polarizations propagate with different speeds: multirefringence.
- Null cones of $G_{(I)ab}$ form a nested set, tangent along ℓ^a, causality determined by outermost cone

Example 2: Killing horizon

Gravitational signals can travel faster than light. Can they escape from inside a black hole?

Izumi (2014): a *Killing horizon* is characteristic for all graviton polarizations in Einstein-Gauss-Bonnet theory. We generalized this to any Lovelock theory.

If we deform the metric inside a Killing horizon, the deformation cannot escape the horizon.

Event horizon of a static BH must be a Killing horizon. True also for stationary BHs in GR - what about Lovelock?

Non-stationary BHs?

Example 3: static black hole spacetime

Consider black hole solution (Boulware & Deser 1985)

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-2}^2$$

Can determine characteristic surfaces from equations for linearized perturbations: decompose into scalar, vector and tensor types.

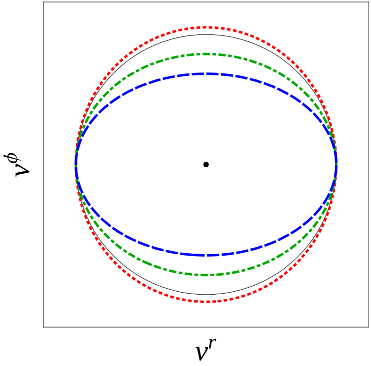
For each type, there is an "effective metric" G^I_{ab} ($I = S, V, T$). A surface is characteristic iff it is null w.r.t. one of the G^I_{ab}.

$$G_I = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{c_I(r)} d\Omega_{d-2}^2$$

$c_I \to 1$ as $r \to \infty$.
(Reduction to effective metrics is a consequence of symmetry.)

Effective metrics

The null cones of G^I_{ab} form a nested set, with causality determined by the outermost null cone.



Cones coincide in radial direction (cf Brigante et al 2008).

Effective metrics

$$G_I = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{c_I(r)}d\Omega_{d-2}^2$$

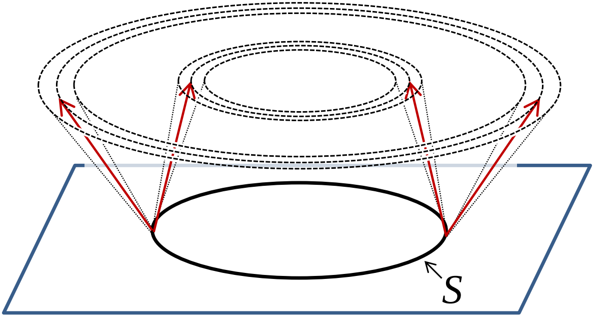
For some *small* black holes, $c_I(r)$ changes sign at $r = r_*$ outside black hole.

In Einstein-Gauss-Bonnet, this happens for $d = 5, 6$

This means that the equation of motion is *not hyperbolic* for $r \leq r_*$

Hyperbolicity

Pick some "initial" hypersurface Σ (non-characteristic) and a $(d-2)$-dimensional surface $S \subset \Sigma$.



$N = d(d-3)/2$ independent graviton polarizations. Theory is *hyperbolic* if there are N "ingoing" and N "outgoing" characteristic hypersurfaces through S (allow for degeneracy).

Hyperbolicity

Lovelock equations of motion are not always hyperbolic. Initial value problem not well-posed if not hyperbolic.

Expect hyperbolic equations when curvature is small.

Can hyperbolicity be violated dynamically? Yes - consider *large* black hole: hyperbolicity violated in region near singularity. But seems to be unstable: linear perturbations blow up there. Maybe nonlinear theory prevents itself from becoming non-hyperbolic. Reminiscent of strong cosmic censorship. (Work in progress.)

Initial value problem

Initial data in Lovelock theories, as in GR, consists of a hypersurface Σ together with the induced metric and extrinsic curvature of Σ. The following are necessary conditions for a well-posed initial value problem:

- The constraint equations are satisfied.
- The equation of motion is hyperbolic on Σ
- Σ is spacelike w.r.t. the *causal structure defined by the equation of motion*

A hyperbolic PDE defines a causal structure on spacetime (e.g. division of vectors into timelike, spacelike, null).

In Lovelock theories, this is not the same as the causal structure defined by the metric (it is defined by the *effective* metrics in our type N and static black hole examples but in general it is more complicated).

Shapiro time delay for gravitons

(Camanho, Edelstein, Maldacena & Zhiboedov 2014)

GR time delay: gravitons/photons travel between two points in curved spacetime slower than between "same" two points in flat spacetime.

Camanho *et al* argued that gravitons can experience a negative time delay, i.e, time *advance* in theories with exotic graviton 3-point coupling e.g. Einstein-Gauss-Bonnet theory.

They argued that the time advance can be eliminated by including contributions from infinite tower of massive higher spin particles, as in string theory.

In an Appendix, they argued that time advance is a *pathology* because it could be exploited to build a "time machine". We will explain why this argument is incorrect.

Aichelburg-Sexl solutions

The "time machine" construction involves superposing two Aichelburg-Sexl "shock-wave" solutions. These are solutions of any Lovelock theory ($\Lambda = 0$)

They are flat except for delta-function curvature localised on a null hypersurface, with amplitude of delta function diverging along null line within this surface (worldline of high energy particle)

Are these singular solutions physical? Can construct as a limit: boost a black hole solution, take boost to infinity, scale mass to zero, keeping energy fixed. So can "regulate" an AS solution by replacing it with a small, highly boosted, black hole.

This is fine in GR (no scale). But in Einstein-Gauss-Bonnet we will argue that it is not possible to boost a *small* (compared to $\sqrt{\alpha}$) black hole arbitrarily close to the speed of light.

Speed limit for small black holes in EGB theory

Isotropic coordinates:

$$ds^2 = -f(R)dt^2 + H(R)dx_i dx_i \qquad R = \sqrt{x_i x_i}$$

Perform boost $t = \gamma(t' - vx_1')$, $x_1 = \gamma(x_1' - vt')$ and consider the initial data induced on the surface $t' = 0$. This describes a black hole with speed v.

This is the same as considering the data induced on the surface $t = -vx_1$ in the original coordinates.

This surface is spacelike w.r.t. the metric provided $|v| < 1$. But for this to be legitimate initial data, this surface needs to be spacelike w.r.t. all of the *effective* metrics.

In EGB theory, for the surface to be spacelike w.r.t. the tensor effective metric we need $|v| < v_{\max}$.

For a large black hole (compared to $\sqrt{\alpha}$) we have $v_{\max} = 1$.

But for a small black hole

$$v_{\max} = \frac{1}{3 - 2\sqrt{1 - \frac{1}{(d-4)^2}}} < 1$$

A hypersurface with a small black hole moving with speed $|v| > v_{\max}$ isn't spacelike w.r.t. the effective metrics: it does not describe an "instant of time". Such a black hole cannot arise from Cauchy evolution of any initial data.

This means that the time machine construction won't work.

Time advance/delay

To define time delay, need to identify points in a curved spacetime with points in flat spacetime. No gauge invariant way of doing this in general (Gao & Wald 2000).

Time delay *can* be defined unambiguously in static, spherically symmetric, spacetimes: consider proper time to propagate across spherical cavity (Cabrera-Palmer & Marolf 2002).

Seems worthwhile calculating time delay/advance for gravitons this way. (Camanho *et al* derivation used scattering amplitudes and AS spacetime.)

Graviton trajectories in EGB

Characteristic surfaces are ruled by *bicharacteristic curves*. For GR (or Yang-Mills), characteristic surfaces are null and bicharacteristic curves are null geodesics.

Geometric optics: high frequency gravitons follow bicharacteristic curves.

For a spherically symmetric Lovelock black hole, the bicharacteristic curves are the null geodesics of the effective metrics.

High frequency tensor-polarized gravitons follow null geodesics of the effective metric for the tensor modes etc.

Consider null geodesics of

$$G_I = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{c_I(r)} d\Omega_{d-2}^2$$

Reduces to motion in effective potential:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_I(r) = \frac{1}{2b^2}$$

where b is impact parameter of graviton trajectory and

$$V_I(r) = \frac{f(r)c_I(r)}{r^2}$$

$c_I(r)$ varies over length scale $L \equiv (\alpha\mu)^{1/(d-1)}$ where μ is mass parameter.

For $r \gg L$ we find

$$V_I(r) = \frac{1}{2r^2} - \frac{\mu}{2r^{d-1}} + \beta_I \frac{\alpha\mu}{r^{d+1}} + \ldots$$

where $\beta_I < 0$ for scalar/vector polarizations, $\beta_I > 0$ for tensor polarisation ($\beta_I = 0$ for geodesics of physical metric).

First two terms are usual GR terms. Final term is GB effect: this term is *repulsive* for tensor polarisation. For a small black hole this term dominates in range $L \ll r \ll \sqrt{\alpha}$: tensor-polarized gravitons with small b will experience repulsive gravitational interaction.

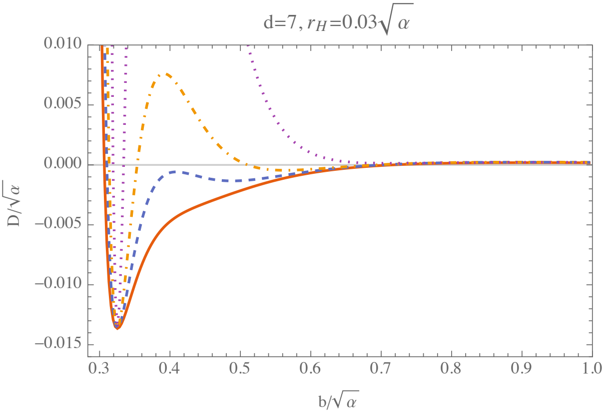
Results: perturbative

For small EGB black holes a perturbative calculation reveals that graviton trajectories with $(\alpha\mu)^{1/(d-1)} \ll b \lesssim \sqrt{\alpha}$ exhibit

- a deflection angle less than π, characteristic of repulsive central force
- a time advance scaling same way as found by Camanho *et al.*

Size of time advance increases as b decreases. How large can it get?

Results: numerical



Largest possible time advance scales as $(\alpha\mu)^{1/(d-1)}$ for small μ.

Nonlinear stability of Minkowski spacetime

Highly non-trivial in 4d GR (Christodoulou & Klainerman 1993). Much easier in $d > 4$ dimensions because linear perturbations disperse faster.

What about Lovelock?

Lovelock eqs of motion in harmonic coordinates, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$\Box h_{\mu\nu} = \mathcal{F}_{\mu\nu}(h, \partial h, \partial^2 h)$$

where RHS is second order in $h_{\mu\nu}$.

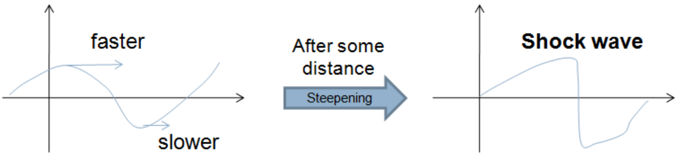
Toy model: replace $h_{\mu\nu}$ with scalar field h. The $h = 0$ solution is *stable* for $d \geq 5$ (Hörmander 1997).

This suggests that Minkowski spacetime is nonlinearly stable in Lovelock theories (J. Keir, work in progress.)

Shock formation in Lovelock theories

Can we make a wavepacket so that back of wavepacket travels faster than front?

cf compressible perfect fluid: speed of sound depends on pressure $\Rightarrow$ wave steepening $\Rightarrow$ shock!



Compressible perfect fluid in $3+1$ dimensions. Initial data: fluid at rest outside a ball. Shock formation occurs for generic *small* initial data (Sideris 1985, Christodoulou 2007) because nearby outgoing characteristic surfaces intersect.

Lovelock: ”speed of gravity” can vary in spacetime: does shock formation occur?

Shock formation won’t occur for small (almost flat) initial data if Minkowski stable.

Transport equations

Consider a solution with curvature discontinuous across hypersurface Σ. Then Σ must be characteristic.

Characteristic surfaces are ruled by *bicharacteristic curves* (e.g. null geodesics in GR).

Can derive a transport equation for amplitude of discontinuity: an ODE along a bicharacteristic curve.

GR is *exceptional* because transport equation is linear.

Shocks

For Lovelock theories (as for compressible perfect fluid), transport equation is *nonlinear*. Discontinuity can blow up in finite time. Blow up occurs because nearby outgoing characteristic surfaces intersect: shock!

Blow-up occurs whenever amplitude of initial curvature discontinuity is large enough.

Similar results for high frequency, small amplitude gravitational waves in a background spacetime (nonlinear geometrical optics).

What about smooth initial data? (Numerics?)

Weak cosmic censorship

Shocks are curvature singularities. Are these naked or hidden inside black holes?

Reduce amplitude of initial outgoing disturbance: takes longer for shock to form $\Rightarrow$ requires bigger black hole, but initial energy smaller...suggests shock won't be hidden by black hole.

Evolution of shocks

In fluid dynamics, shock formation is not the end of time evolution: can extend as a *weak* solution by allowing fields to be discontinuous. Rankine-Hugoniot junction conditions from conservation of energy-momentum and particle number. Shocks propagate along *non*characteristic hypersurfaces.

Analogous situation in Lovelock theories: once shock forms, allow ∂g to be discontinuous across hypersurface Σ. Weak solution: extremize action $\Rightarrow$ canonical momentum π^{ij} should be continuous across Σ. Possible because π^{ij} is a polynomial in ∂g. Does such a surface describes a dynamical shock?

Conclusions

- Naive argument relating superluminal propagation to causality violation is unconvincing.
- Small black holes cannot be boosted arbitrarily close to the speed of light in Einstein-Gauss-Bonnet theory.
- This theory exhibits a repulsive short-distance interaction leading to time advance of Camanho *et al*.
- Heuristic argument suggests that Minkowski spacetime is stable: if so, Lovelock theories "make sense" for small curvature initial data.
- Unlike GR, these theories probably form shocks for large curvature initial data. It may be possible to develop a theory of shock evolution.

Open questions

- Well-posedness of initial value problem (cf Willison 2014)
- Positive energy theorem (or counterexample)
- Definition of a black hole in Lovelock theories. Use "bicharacteristic cone" to define causal structure. The "gravity horizon" would be an outermost outgoing characteristic surface. Is there a second law for this surface?