2^{nd}-Order Hydrodynamics & Universality in Non-Conformal Holographic Fluids

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Why relativistic hydrodynamics?

- ubiquitous low-energy effective theory:
  applies to slowly-varying fluctuations in any interacting field theory at finite temperature
- only theory-dependent constants: transport coefficients
- successful description of early-stage QGP

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Why relativistic hydrodynamics?

- ubiquitous low-energy effective theory:
  applies to slowly-varying fluctuations in any interacting field theory at finite temperature
- only theory-dependent constants: transport coefficients
- successful description of early-stage QGP

Why non-conformal?

- most physical systems, including the QGP, are non-conformal

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Why holography?

- in order to compute transport coefficients:
  need to match effective hydro result for suitable real-time correlators with corresponding microscopic result
- for QGP however:
  - perturbative calculations impossible due to strong coupling
  - lattice calculations unsuitable for *real-time* correlators
- only currently available tool for real-time correlators at strong coupling: gauge/gravity duality or holography

Why universality?

- gravity dual of realistic theories such as QCD unknown
  - ➢ best one can hope for:
    identify and investigate universal properties that hold for a large class of holographic theories
  - ➢ being insensitive to microscopic details, such universal properties may be common to all strongly-coupled field theories, including the ones realised in nature
- example: $\eta/s = 1/4\pi$ for all strongly-coupled theories with a two-derivative gravity dual
  - ➢ experiment shows that for QGP indeed $\eta/s \approx 1/4\pi$

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Why second order?

- hydrodynamics = effective theory for slowly-varying low-energy fluctuations
  - ➢ systematic expansion in momenta/gradients
    (0th order: thermodynamic quantities, 1st order: η and ζ)
- hydro simulations of QGP require 2nd-order gradients to remove unstable superluminal modes
- 2nd-order coefficients can be measured in principle (dispersion relations, equation of state in curved space, …)
  - ➢ theoretical advancement: what can holography teach us?
- within gravity: thermodynamics/black holes --> fluid/gravity

The Haak-Yarom Identity

The Haak-Yarom Identity

One particular relation between 2nd-order transport coefficients

$$H \equiv 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0$$

has been shown to hold universally…

- …for *conformal* holographic fluids with any number of U(1) charges at finite density [Haack,Yarom '08]
- …when taking into account leading corrections to the infinite coupling limit in N=4 [Grozdanov,Starinets '14] and in the dual of Gauss-Bonnet [Shaverin,Yarom '12 & '15][Grozdanov,Starinets '15]
- …for the non-conformal Chamblin-Reall background [Bigazzi,Cotrone '10] (however: this is a compactification of AdS!)
- ...for a non-conformal compactification of D4-branes [Wu,Chen,Huang '16]

➢ Is H=0 generally satisfied by *non-conformal* holographic fluids?

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Summary of our results

we

- ...derived new Kubo formulae for five 2nd-order coefficients which are valid for any uncharged relativistic fluid in (3+1)-d
- ...applied these Kubo formulae to a large class of non-conformal holographic models: namely holographic RG flows triggered by any relevant scalar operator of dimension Δ=3
- ...found that the following combination always vanishes:

$$\tilde{H} \equiv 2\eta\tau_\pi - 2\left(\kappa - \kappa^*\right) - \lambda_2 = 0$$

- ...proved analytically that H=0 still holds when taking into account leading non-conformal corrections
- ...showed numerically that H vanishes along two specific families of RG flows beyond leading order

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3. Quick recap of hydro

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- Assumption: all relevant dynamics in the hydrodynamic regime is governed by microscopic conservation laws
  - only relevant d.o.f. are expectation values of global charge densities, averaged over small patches of local equilibrium
- uncharged relativistic fluid: only conserved charges are energy $\langle T^{00}(x)\rangle$ and momentum density $\langle T^{0i}(x)\rangle$
  - microscopic conservation equations: $\nabla_\mu \langle T^{\mu\nu}(x)\rangle = 0$
- need to be supplemented by constitutive relations that express the current densities $\langle T^{\mu\nu}(x)\rangle$ as functions of the charge densities $\langle T^{0\mu}(x)\rangle$
  - in the form of an expansion in small momenta/gradients
- in the spirit of effective field theory, at every order all terms compatible with the underlying symmetries are written down
  - each multiplied by a free parameter=transport coefficient

3. Quick recap of hydro

Uncharged relativistic fluid in (3+1) dimensions:

$$\langle T^{\mu\nu}(x)\rangle = \epsilon(x)u^\mu(x)u^\nu(x) + p\left(\epsilon(x)\right)\overbrace{\left(u^\mu(x)u^\nu(x) + g^{\mu\nu}_{(0)}(x)\right)}^{\equiv\Delta^{\mu\nu}}$$

$$-\,\eta\,\underbrace{2\nabla^{<\mu}u^{\nu>}}_{\text{projection to transverse, symmetric, traceless}} - \zeta\,\underbrace{(\nabla\cdot u)\,\Delta^{\mu\nu}}_{\text{transverse, symmetric trace-part}}$$

+ 5 traceless 2nd-order tensor structures multiplied by $\eta\tau_\pi,\ \kappa,\ \lambda_1,\ \lambda_2,\ \lambda_3$

+ 10 non-traceless 2nd-order tensor structures multiplied by κ^* and 9 other transport coefficients

+ 3rd-order gradients

4. New Kubo formulae

4. New Kubo formulae

- in order to compute transport coefficients: need to match effective hydro result for suitable real-time correlators of $T^{\mu\nu}$ with the corresponding microscopic result
  - ➢ Kubo formulae tell you which correlators exactly to look at for a specific transport coefficient
- correlators are encoded in the response of $\langle T^{\mu\nu}\rangle$ to external metric perturbations around flat space $g_{(0)\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$ of a fluid in equilibrium:

$$\epsilon(x) = \bar{\epsilon} + \delta\epsilon(x) \;, \quad u^\mu(x) = (1, \underline{v}) \left(-g_{(0)tt} - 2g_{(0)ti}v^i - g_{(0)ij}v^i v^j\right)$$

- convenient to use $\delta\epsilon(x)$ and $\underline{v}(x)$ as fluid variables
  - ➢ equation of motion: $\nabla_\mu \langle T^{\mu\nu}\rangle [\delta\epsilon, \underline{v}; h_{\rho\sigma}] = 0$

  with boundary condition $\delta\epsilon(x) = \underline{v}(x) = 0$ for $h_{\mu\nu}(x) = 0$

4. New Kubo formulae

Simple sources & responses

- focus on external metric perturbations that preserve residual SO(2): $h_{\mu\nu} = h_{\mu\nu}(t,z)$
- the particular subset $\{h_{xy}(t,z), h_{tx}(z), h_{ty}(z), h_{xz}(t), h_{yz}(t)\}$ is found not to source any fluid fluctuations to linear order
  - ➢ no sound waves which excite theory-specific matter
  - ➢ on-shell $\delta\epsilon(h)\ ,\ \underline{v}(h) = \mathcal{O}(h^2)$
- focus on response of transverse tensor component $\langle T^{xy}\rangle$ whose constitutive relation (--> off-shell) is independent of $(\delta\epsilon, v^z)$ and (v^x, v^y)
  - ➢ on-shell:

$$\langle T^{xy}\rangle [h] = \bar{T}^{xy} + \frac{\partial \bar{T}^{xy}}{\partial h} h + \frac{1}{2}\frac{\partial^2 \bar{T}^{xy}}{\partial h^2} h^2 + \mathcal{O}\left(h^3, \partial^3\right)$$

4. New Kubo formulae

Explicitly: in the presence of $\{h_{xy}(t,z), h_{tx}(z), h_{ty}(z), h_{xz}(t), h_{yz}(t)\}$:

$$\begin{aligned}
\langle T^{xy}\rangle [h] = & \left[-\bar{p} - \eta\,\partial_t - \frac{\kappa}{2}\partial_z^2 + \left(\eta\,\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\partial_t^2\right] h_{xy}(t,z) \\
& + \Bigg[\bar{p}\,h_{xz}\,h_{yz} + \eta\,(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\,h_{yz}) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz} \\
& \quad + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\,h_{yz}\right)\Bigg] \\
& + \left[-\bar{p}\,h_{tx}\,h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\,h_{ty}\right)\right] \\
& + \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right](\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}) + \mathcal{O}(h^3, \partial^3)
\end{aligned}$$

4. New Kubo formulae

Explicitly:

[Baier, Romatschke, Son, Starinets, Stephanov '08]

$$\langle T^{xy}\rangle [h] = \left[-\bar{p} - \eta\,\partial_t - \frac{\kappa}{2}\partial_z^2 + \left(\eta\,\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\partial_t^2\right] h_{xy}(t,z)$$

$$+ \left[\bar{p}\,h_{xz}\,h_{yz} + \eta\left(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\,h_{yz}\right) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz}\right.$$

$$\left. + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\,h_{yz}\right)\right]$$

$$+ \left[-\bar{p}\,h_{tx}\,h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\,h_{ty}\right)\right]$$

$$+ \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right]\left(\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}\right) + \mathcal{O}(h^3, \partial^3)$$

4. New Kubo formulae

Explicitly: in the presence of $\{h_{xy}(t,z), h_{tx}(z), h_{ty}(z), h_{xz}(t), h_{yz}(t)\}$:

$$\langle T^{xy}\rangle [h] =$$

for conformal fluids ($\kappa^* = 0$): [Moore, Sohrabi '10]

$$+\left[\bar{p}\, h_{xz}\, h_{yz} + \eta\,(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\, h_{yz}) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz} + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\, h_{yz}\right)\right]$$

$$+\left[-\bar{p}\, h_{tx}\, h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\, h_{ty}\right)\right]$$

$$+\left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right](\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}) + \mathcal{O}(h^3, \partial^3)$$

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Explicitly: in the presence of $\{h_{xy}(t,z), h_{tx}(z), h_{ty}(z), h_{xz}(t), h_{yz}(t)\}$:

$$\begin{aligned}\langle T^{xy}\rangle[h] = &\left[-\bar{p} - \eta\,\partial_t - \frac{\kappa}{2}\partial_z^2 + \left(\eta\,\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\partial_t^2\right] h_{xy}(t,z) \\ &+ \left[\bar{p}\, h_{xz}\, h_{yz} + \eta\,(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\, h_{yz}) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz}\right. \\ &\left. + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\, h_{yz}\right)\right] \\ &+ \left[-\bar{p}\, h_{tx}\, h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\, h_{ty}\right)\right] \\ &+ \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right](\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}) + \mathcal{O}(h^3, \partial^3)\end{aligned}$$

- gives us access to 5 independent 2nd-order coefficients:

$$\eta\tau_\pi + \kappa^*\,, \quad \kappa\,, \quad \lambda_1 + \kappa^*/2\,, \quad \lambda_2\,, \quad \lambda_3 - 2\kappa^*$$

  - combinations of 5 conformal coefficients plus κ^* (no other non-conformal coefficients appear!)
  - include $H = 2\,(\eta\tau_\pi + \kappa^*) - 4\,(\lambda_1 + \kappa^*/2) - \lambda_2$

4. New Kubo formulae

- one can read off Kubo formulae by comparing this with the expansion of $\langle T^{xy}\rangle\,[h]$ in terms of retarded correlators:

$$\langle T^{\mu\nu}(x=0)\rangle\,[h] = G^{\mu\nu}(0) - \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} G^{\mu\nu,\rho\sigma}(p) h_{\rho\sigma}(p) + \frac{1}{8}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4p}{(2\pi)^4} G^{\mu\nu,\rho\sigma,\kappa\lambda}(q,p) h_{\rho\sigma}(q) h_{\kappa\lambda}(p) + \mathcal{O}(h^3)$$

Example: only turn on the plane wave $h_{xy}(t,z) = \epsilon\, H^{(b)}_{xy} e^{-i\omega t + iqz}$

$$\Longrightarrow \langle T^{xy}(x=0)\rangle\,[h] = \left[-\bar{p} + i\,\omega\,\eta + \frac{\kappa}{2}q^2 - \left(\eta\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\omega^2\right]\epsilon\, H^{(b)}_{xy} + \mathcal{O}(\epsilon^3, \partial^3)$$

$$\stackrel{!}{=} -G^{xy,xy}(\omega,q)\,\epsilon\, H^{(b)}_{xy} + \mathcal{O}(\epsilon^3)$$

$$\Longrightarrow \eta = i\ \partial_\omega G^{xy,xy}|_{(\omega,\underline{q})=0}$$

4. New Kubo formulae

In fact, all 5 coefficients

$$\eta\tau_\pi + \kappa^* \,, \quad \kappa \,, \quad \lambda_1 + \kappa^*/2 \,, \quad \lambda_2 \,, \quad \lambda_3 - 2\kappa^*$$

can be measured by turning on plane waves for $\{h_{xz}(t), h_{yz}(t)\}$, $\{h_{tx}(z), h_{ty}(z)\}$, and $\{h_{ty}(z), h_{xz}(t)\}$, one after another.

$$\begin{aligned}
\langle T^{xy}\rangle [h] = & \left[-\bar{p} - \eta\,\partial_t - \frac{\kappa}{2}\partial_z^2 + \left(\eta\,\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\partial_t^2\right] h_{xy}(t,z) \\
& + \left[\bar{p}\,h_{xz}\,h_{yz} + \eta\,(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\,h_{yz}) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz}\right. \\
& \quad \left. + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\,h_{yz}\right)\right] \\
& + \left[-\bar{p}\,h_{tx}\,h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\,h_{ty}\right)\right] \\
& + \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right]\left(\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}\right) + \mathcal{O}(h^3, \partial^3)
\end{aligned}$$

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$$\begin{aligned}
\langle T^{xy}\rangle [h] = &\left[-\bar{p} - \eta\,\partial_t - \frac{\kappa}{2}\partial_z^2 + \left(\eta\,\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\partial_t^2\right] h_{xy}(t,z) \\
&+ \left[\bar{p}\,h_{xz}\,h_{yz} + \eta\,(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\,h_{yz}) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz}\right. \\
&\quad \left. + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\,h_{yz}\right)\right] \\
&+ \left[-\bar{p}\,h_{tx}\,h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\,h_{ty}\right)\right] \\
&+ \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right](\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}) + \mathcal{O}(h^3, \partial^3)
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& + \left[\bar{p}\,h_{xz}\,h_{yz} + \eta\,(h_{xz}\,\partial_t h_{yz} + \partial_t h_{xz}\,h_{yz}) + \left(\lambda_1 - \eta\,\tau_\pi - \frac{\kappa^*}{2}\right)\partial_t h_{xz}\,\partial_t h_{yz}\right. \\
& \quad \left. + \left(\frac{\kappa}{2} - \eta\,\tau_\pi - \kappa^*\right)\left(h_{xz}\,\partial_t^2 h_{yz} + \partial_t^2 h_{xz}\,h_{yz}\right)\right] \\
& + \left[-\bar{p}\,h_{tx}\,h_{ty} + \left(\frac{\lambda_3}{4} - \frac{\kappa^*}{2}\right)\partial_z h_{tx}\,\partial_z h_{ty} - \frac{\kappa}{2}\left(h_{tx}\,\partial_z^2 h_{ty} + \partial_z^2 h_{tx}\,h_{ty}\right)\right] \\
& + \left[\frac{1}{2}\eta\,\tau_\pi - \frac{\lambda_2}{4} + \frac{\kappa^*}{2}\right]\left(\partial_z h_{tx}\,\partial_t h_{yz} + \partial_z h_{ty}\,\partial_t h_{xz}\right) + \mathcal{O}(h^3, \partial^3)
\end{aligned}$$

4. New Kubo formulae

We obtained the following new Kubo formulae:

$$\kappa = \partial_{q_z}^2 G^{xy,tx,ty}(q,p)\Big|_{q=p=0} \ ,$$

$$\eta\tau_\pi + \kappa^* = \frac{\kappa}{2} + \frac{1}{2}\,\partial_{q_0}^2 G^{xy,xz,yz}(q,p)\Big|_{q=p=0} \ ,$$

$$\lambda_1 + \frac{\kappa^*}{2} = (\eta\tau_\pi + \kappa^*) - \partial_{q_0}\partial_{p_0} G^{xy,xz,yz}(q,p)\Big|_{q=p=0} \ ,$$

$$\lambda_2 = 2\,(\eta\tau_\pi + \kappa^*) - 4\,\partial_{q_0}\partial_{p_z} G^{xy,tx,xz}(q,p)\Big|_{q=p=0} \ ,$$

$$\lambda_3 - 2\kappa^* = -4\,\partial_{q_z}\partial_{p_z} G^{xy,tx,ty}(q,p)\Big|_{q=p=0} \ .$$

5. Applying Kubo in holography

5. Applying Kubo in holography

- want to compute transport coefficients of strongly coupled field theories with holographic gravity duals

<u>Strategy</u>

- perturb external field-theory metric $g_{(0)\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$
  - presribe corresponding AdS-boundary value of dual dynamical bulk metric g_{mn}
- solve Einstein's equations perturbatively in
  - ...momenta (hydro gradient expansion)
  - ...sources ($\mathcal{O}(h^2)$ sufficient for 3-point functions)
- extract field-theory stress tensor $\langle T^{xy}\rangle\,[h]$ from dual gravity solution according to holographic dictionary
- compare with effective hydro result for $\langle T^{xy}\rangle\,[h]$ to read off transport coefficients

5. Applying Kubo in holography

Compuation of $\langle T^{\mu\nu} \rangle$ from gravity dual

- global charges agree in dual theories
  - in particular

$$\underbrace{\mathcal{T}^{\mu\nu}} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S^{\text{on-shell}}_{\text{gravity}}}{\delta \gamma_{\mu\nu}} \propto \underbrace{\langle T^{\mu\nu} \rangle}$$

quasi-local gravity stress tensor

field-theory stress tensor

induced AdS-boundary metric

off-shell:

$$\delta S^{\text{off-shell}}_{\text{gravity}} = -\frac{1}{16\pi G_N} \int \mathrm{d}^5 x \sqrt{-g}\, \mathrm{EOM}^{mn} \delta g_{mn} + \frac{1}{2} \int_{\partial AdS_5} \mathrm{d}^4 x \sqrt{-\gamma}\, \mathcal{T}^{\mu\nu} \delta \gamma_{\mu\nu}$$

- to obtain $\mathcal{T}^{xy}$ up to $\mathcal{O}(h^2)$ in the boundary perturbation we only need to solve EOM^{xy} up to $\mathcal{O}(h^2)$ included

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6. A class of non-conformal holographic models

6. A class of non-conformal holographic models

- holographic RG flows triggered by a scalar operator of dimension Δ=3
- field theory:
  relevant deformation of UV fixed point by $\int \mathrm{d}^4x \sqrt{-g_{(0)}}\, \Lambda\, O$
- dual gravity bulk:

$$S_{\text{gravity}} = \frac{1}{16\pi G_N} \int \mathrm{d}^5x \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right)$$

with potentials of the form

$$V = \frac{1}{L^2} \left[\underbrace{-12}_{\text{cosmological constant}} \underbrace{- \frac{3}{2}\phi^2}_{\text{mass term}} \underbrace{+ \mathcal{O}(\phi^4)}_{\text{left unspecified}} \right]$$

cosmological constant — mass term — left unspecified

$m^2 L^2 = \Delta\,(\Delta - 4)$

and bulk scalar field $\phi \xrightarrow{\zeta \to 0} \Lambda\, \zeta + \ldots$

6. A class of non-conformal holographic models

Remark: restriction to (Δ=3)-operators because counterterms required for holographic renormalisation are known.

Black-brane backgrounds

$$ds^2 = e^{2A(u)}\left[-f(u)\mathrm{d}t^2 + \mathrm{d}\underline{x}^2\right] + \frac{L^2}{4u^2 f(u)}\mathrm{d}u^2$$

field-theory directions

convenient radial coordinate, $u \in (0,1)$

- solutions depend on single parameter Λ/T
- common UV fixed point $\Lambda/T = \phi = 0$: pure AdS black brane

$$e^{2A} \longrightarrow \frac{(\pi T L)^2}{u}\,, \qquad f \longrightarrow 1 - u^2$$

7. Solving Einstein's equations

7. Solving Einstein's equations

Goal

Compute response of $\langle T^{xy} \rangle$ (encoded in on-shell bulk metric) to field-theory metric perturbations and compare with hydro result

- find solutions of bulk metric fluctuations sourced by field-theory metric perturbations

  …around arbitrary black-brane background solutions

s. t. expressions for $\langle T^{xy} \rangle$ are valid for

- any operator with Δ=3 (dual $V(\phi)$ arbitrary beyond mass)
- any temperature T/Λ

7. Solving Einstein's equations

Solving the xy-component of Einstein's equations, sourced by the three dual field-theory metric perturbations $\{h_{xz}(t), h_{yz}(t)\}$, $\{h_{tx}(z), h_{ty}(z)\}$, and $\{h_{ty}(z), h_{xz}(t)\}$,

- ...to 2nd-order in sources $\mathcal{O}(h^2)$
- ...to 2nd-order in momenta $\mathcal{O}(\partial^2)$

involves 24 functions in the bulk metric.

- we found analytic solutions for 19 and explicit integral expressions for another 4
  (given in terms of arbitrary black-brane background solutions)
  - 1 unknown function
    ...which however (as it turns out) does not enter $\langle T^{\mu\nu} \rangle$!

Note: the shear-perturbations we turn on don't source fluctuations of the bulk scalar ϕ :

- they only excite universal gravity sector

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8. Analytic results on 2^{nd}-order transport

8. Analytic results on 2nd-order transport

response of holographic stress tensor $\langle T^{\mu\nu}\rangle \propto \frac{2}{\sqrt{-\gamma}} \frac{\delta S^{\text{on-shell}}_{\text{gravity}}}{\delta \gamma_{\mu\nu}}$

- ✓ ...satisfies Ward identity $\langle T^{\mu}{}_{\mu}\rangle = \langle O\rangle\, \Lambda + \text{anomaly}$
- ✓ ...takes expected hydro from (non-trivial check on global solutions!)
- ✓ reproduces η/s = 1/4π

➢ yields explicit expressions for the five 2nd-order coefficients!

8. Analytic results on 2nd-order transport

$$\kappa = -\frac{2}{f_b} Y^{(2tz)}_{(1,1)} sT \,,$$

$$\eta\,\tau_\pi + \kappa^* = \frac{1}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y^{(1t)}_2 - Y^{(2tz)}_{(1,1)}\right) sT \,,$$

$$\lambda_1 + \frac{\kappa^*}{2} = \frac{1}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y^{(1t)}_2 - Y^{(2tz)}_{(1,1)} + Y^{(2tt)}_{(1,1)}\right) sT \,,$$

$$\lambda_2 = \frac{2}{f_b}\left(\frac{1}{32\pi^2 T^2} + Y^{(1t)}_2 + Y^{(2tz)}_{(1,1)}\right) sT \,,$$

$$\lambda_3 - 2\kappa^* = \frac{4}{f_b} Y^{(2zz)}_{(1,1)} sT \,.$$

normalisable modes of bulk metric fluctuations

$$f \xrightarrow{u\to 0} 1 + f_b u^2$$

- natural units: s/T (prop. to d.o.f., mass dim. 2)
- $\left\{Y^{(1t)}_2, Y^{(2tt)}_{(1,1)}, Y^{(2zz)}_{(1,1)}, Y^{(2tz)}_{(1,1)}\right\}$ given by integrals over background
  - depend on operator details (dual potential) and T/Λ
- 5 transport coefficients depend on 4 $\left\{Y^{(1t)}_2, Y^{(2tt)}_{(1,1)}, Y^{(2zz)}_{(1,1)}, Y^{(2tz)}_{(1,1)}\right\}$
  - 1 independent combination!

8. Analytic results on 2nd-order transport

1 independent combination: $\tilde{H} \equiv 2\eta\tau_\pi - 2(\kappa - \kappa^*) - \lambda_2 = 0$

- obeyed by all holographic RG flows triggered by (Δ=3)-operator at infinite coupling, at any value of T/Λ (provided they admit black-brane solutions)

coefficients entering $\tilde{H}$ previously computed for

$\tilde{H} = 0$

- N=4 at infinite coupling [Baier et al. '08][Bhattacharyya et al. '08]
- non-conformal dual of Chamblin-Reall [Bigazzi,Cotrone '08]

$\tilde{H} \neq 0$

- finite coupling corrections for N=4
  [Penincasa,Buchel '05][Buchel '08][Buchel,Paulos '08][Grozdanov,Starinets. '14]

8. Analytic results on 2nd-order transport

Proof that H=0 incl. leading non-conformal corrections

$$H = 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = \left(\underbrace{Y_2^{(1t)} + Y_{(1,1)}^{(2tt)}} + \frac{1}{32\pi^2 T^2} \right) \frac{4}{f_b}$$

$$f \xrightarrow{u \to 0} 1 + f_b u^2$$

- integral over background fields *A* and *f* (dependence on ϕ cancels!)

background EOMs: solve *A* in terms of *f*

integral over *f* and its derivatives only

linearise around fixed point (pure AdS black brane)

simple functions of u

simplifies to $\int_0^1 \mathrm{d}w \left[P(u)\delta f''(u) + Q(u)\delta f'(u) + \left(Q'(u) - P''(u) \right) \delta f(u) \right]$

- integration by parts yields H=0

9. Numerical results

9. Numerical results

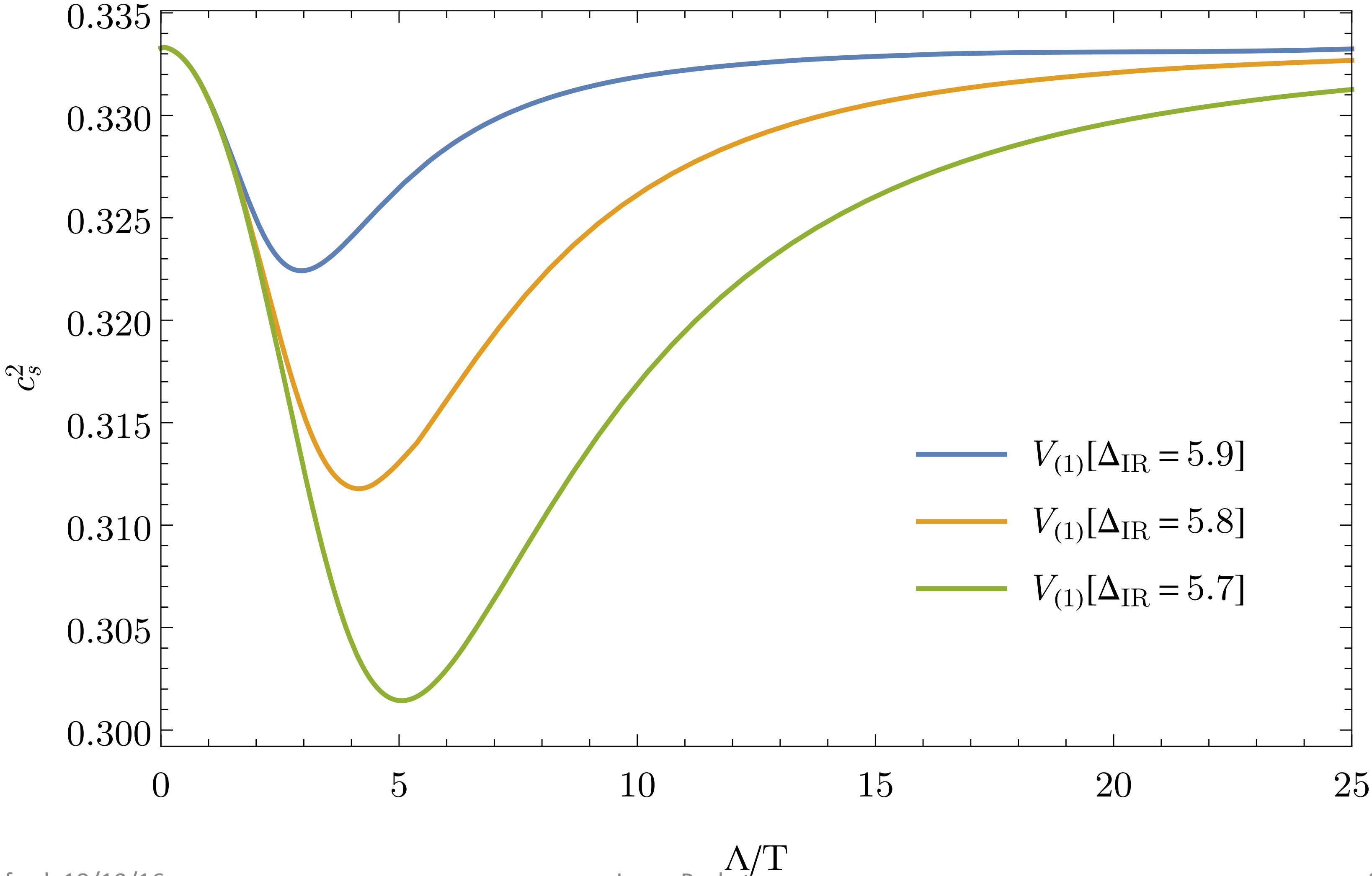
Does H vanish beyond leading non-conformal corrections?

- consider two specific families of (Δ=3)-operators (bulk potentials)
  - construct numerical black-brane backgrounds (method developed in [Gubser,Nellore '08])
  - compute the 4 $Y_j^{(a)}$ (numerical integrals over background)
  - plug result into expressions for transport coefficients

- <u>Check of results:</u>
  leading non-conformal correction to background & transport only depends on mass term in bulk potential (close to UV fixed point: scalar ϕ small)
  - leading correction common to all flows
  - we could determine the leading backreaction analytically

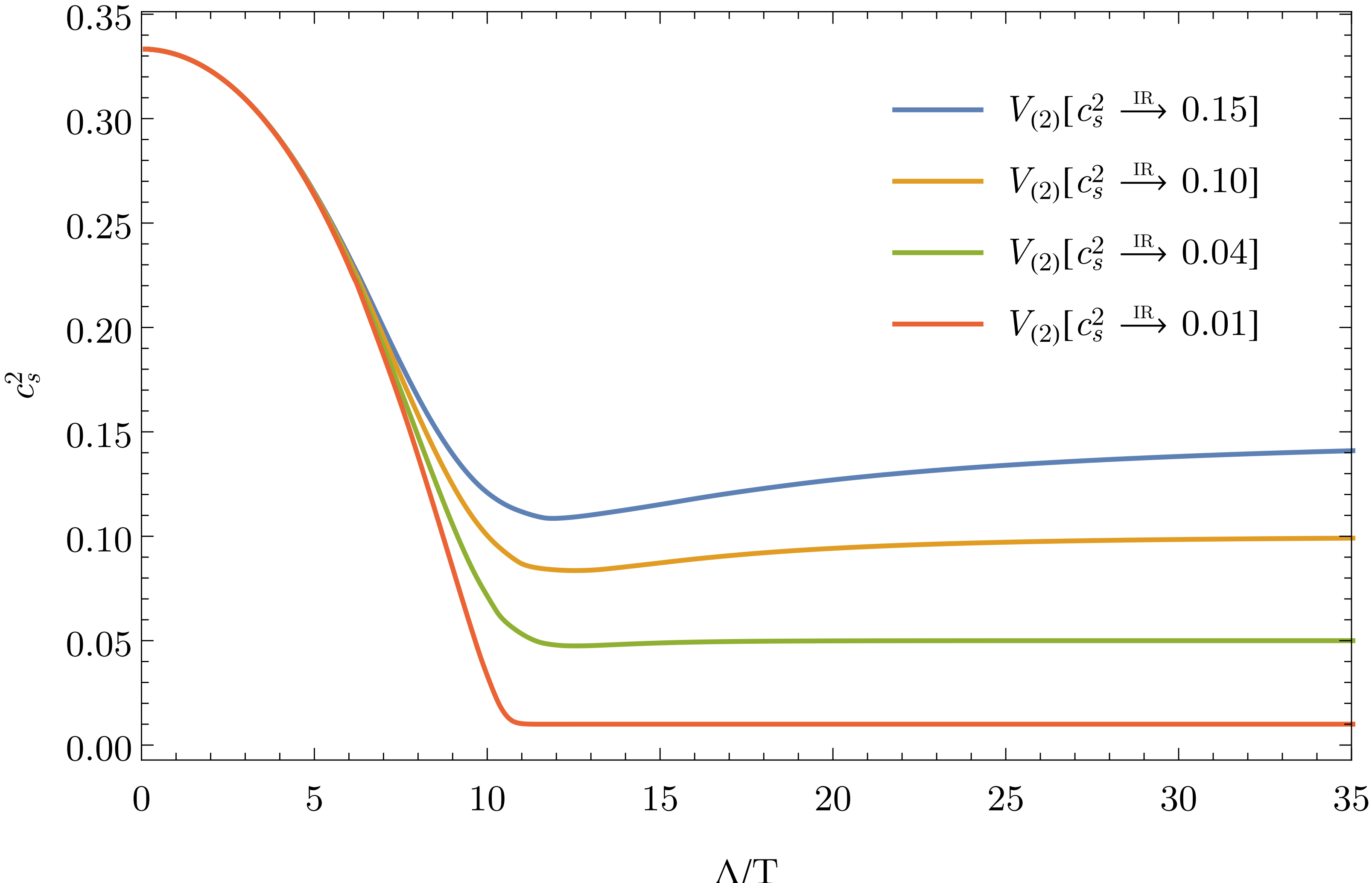
9. Numerical results

1st family of bulk potentials: RG flows to an IR fixed point



9. Numerical results

2nd family: RG flows to non-conformal (Chamblin-Reall) IR



9. Numerical results

<u>Main results</u>

(we looked at around 20 parameter values for ϕ_m and γ , and around 40 temperatures for each flow)

1) method works
2) UV (high T/Λ) well described by leading backreaction of ϕ
3) within numerical accuracy ($\lesssim 10^{-5}$):

$$H = 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0$$

…even when individual coefficients deviate from their conformal values by factors of two and more

- ➢ suggests that H vanishes in holographic fluids irrespective of conformal symmetry
- ➢ further evidence that the Haack-Yarom identity H=0 may be universally satisfied by strongly coupled fluids

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10. Conclusion

Summary

- new Kubo formulae for $\eta\tau_\pi + \kappa^*, \;\; \kappa, \;\; \lambda_1 + \kappa^*/2, \;\; \lambda_2, \;\; \lambda_3 - 2\kappa^*$
- focusing on holographic RG flows triggered by a (Δ=3)-operator we found:
  - $\tilde{H} = 2\eta\tau_\pi - 2\left(\kappa - \kappa^*\right) - \lambda_2 = 0$
  - $H = 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0$ when taking into account leading non-conformal corrections
  - numerical evidence that $H = 0$ beyond leading corrections

10. Conclusion

Outlook

1) include sound perturbations of metric
   - ➢ possible to compute all 15 2nd-order transport coefficients
   - caveat: sound waves excite theory-specific matter content
     - ➢ universal behaviour unlikely
   - but: can check constraints from entropy current!
2) try to generalise proof that H=0
3) investigate consequences of results for entropy current
   - NB:

$$H = 2\eta\tau_\pi - 4\lambda_1 - \lambda_2 = 0 \quad \& \quad \tilde{H} = 2\eta\tau_\pi - 2\left(\kappa - \kappa^*\right) - \lambda_2 = 0$$

$$\Longleftrightarrow H = 0 \quad \& \quad 2\lambda_1 = \kappa - \kappa^*$$

   - ➢ perfect, conformal fluids satisfy $H = 0 \quad \& \quad 2\lambda_1 = \kappa$
   - ➢ entropy production in quantum fluids generally suppressed at strong coupling?

Thank you!

Backup Slides

5. Applying Kubo in holography

Gravity/Gauge Duality

5d Einstein gravity ⟺ strongly coupled 4d QFT
…in asymptotically AdS — …with UV fixed point

+ higher derivative corrections — + finite coupling corrections
+ quantum corrections — + finite central charge corrections

black-brane solutions — thermal equilibrium in flat space

AdS boundary — UV

↓ *extra holographic coordinate* — ↓ *RG flow*

horizon — IR

5. Applying Kubo in holography

"Lagrangian" Holographic Dictionary

5d Einstein gravity $\Longleftrightarrow$ strongly coupled 4d QFT

near the AdS boundary $\zeta \longrightarrow 0$ *:*

- dynamical bulk metric

$$\frac{L^2}{\zeta^2}\left(\mathrm{d}\zeta^2 + g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \ldots\right)$$

"boundary value" prescribed by…

- external field-theory metric

$$g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

- scalar field ϕ *with mass* $m^2L^2 = \Delta\left(\Delta - 4\right)$

$$\phi(\zeta, x) = \left(\Lambda(x)\,\zeta\right)^{4-\Delta} + \ldots$$

"boundary value" prescribed by…

- scalar operator $O(x)$ *of dimension Δ*

$$\underbrace{\int \mathrm{d}^4x \sqrt{-g_{(0)}}\,\Lambda(x) O(x)}_{\text{deformation of UV}}$$

5. Applying Kubo in holography

"Hamiltonian" Holographic Dictionary

- global charges agree in dual theories
  - in particular

$$\underbrace{\mathcal{T}^{\mu\nu}} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S^{\text{on-shell}}_{\text{gravity}}}{\delta \gamma_{\mu\nu}} \propto \underbrace{\langle T^{\mu\nu} \rangle}$$

quasi-local gravity stress tensor — field-theory stress tensor

induced AdS-boundary metric

off-shell:

$$\delta S^{\text{off-shell}}_{\text{gravity}} = -\frac{1}{16\pi G_N} \int \mathrm{d}^5x \sqrt{-g}\, \mathrm{EOM}^{mn} \delta g_{mn} + \frac{1}{2} \int_{\partial AdS_5} \mathrm{d}^4x \sqrt{-\gamma}\, \mathcal{T}^{\mu\nu} \delta\gamma_{\mu\nu}$$

  - to obtain $\mathcal{T}^{xy}$ up to $\mathcal{O}(h^2)$ in the boundary perturbation we only need to solve EOM^{xy} up to $\mathcal{O}(h^2)$ included

5. Applying Kubo in holography

Strategy

...to compute transport coefficients in holographic theories:

- perturb external field-theory metric $g_{(0)\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$
  - presribe corresponding AdS-boundary value of dual dynamical bulk metric g_{mn}
- solve Einstein's equations perturbatively in
  - ...momenta (hydro gradient expansion)
  - ...sources ($\mathcal{O}(h^2)$ sufficient for 3-point functions)
- extract field-theory stress tensor $\langle T^{xy} \rangle [h]$ from dual gravity solution according to holographic dictionary
- compare with effective hydro result for $\langle T^{xy} \rangle [h]$ to read off transport coefficients

7. Solving Einstein's equations

e.g. turn on field-theory metric perturbation

$$g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}\underline{x}^2 + \epsilon\left(H_{ty}^{(b)}\, e^{ip_z z}\, 2\mathrm{d}t\mathrm{d}y + H_{xz}^{(b)}\, e^{-iq_0 t}\, 2\mathrm{d}x\mathrm{d}z\right)$$

perturbative expansion in sources

- bulk metric:

$$g_{mn}\mathrm{d}x^m \mathrm{d}x^n = \underbrace{g_{mn}^{(0)}\mathrm{d}x^m \mathrm{d}x^n}_{\text{background}} + \underbrace{\epsilon g_{\mu\nu}^{(1)}\mathrm{d}x^\mu \mathrm{d}x^\nu}_{\text{sourced fluctuation}} + \underbrace{\epsilon^2 g_{\mu\nu}^{(2)}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{O}(\epsilon^3)}_{\text{dynamical backreaction}}$$

7. Solving Einstein's equations

e.g. turn on field-theory metric perturbation

$$g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}\underline{x}^2 + \epsilon\left(H^{(b)}_{ty}e^{ip_z z}\,2\mathrm{d}t\mathrm{d}y + H^{(b)}_{xz}e^{-iq_0 t}\,2\mathrm{d}x\mathrm{d}z\right)$$

perturbative expansion in sources

- bulk metric:

$$g_{mn}\mathrm{d}x^m \mathrm{d}x^n = \underbrace{g^{(0)}_{mn}\mathrm{d}x^m \mathrm{d}x^n}_{\text{background}} + \underbrace{\epsilon g^{(1)}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}_{\text{sourced fluctuation}} + \underbrace{\epsilon^2 g^{(2)}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{O}(\epsilon^3)}_{\text{dynamical backreaction}}$$

$$\Longrightarrow g^{(0)}_{mn}\mathrm{d}x^m \mathrm{d}x^n = e^{2A}\left[-f\mathrm{d}t^2 + \mathrm{d}\underline{x}^2\right] + \frac{L^2}{4u^2 f}\mathrm{d}u^2$$

Note: the shear-perturbations we turn on do not source fluctuations of the bulk scalar ϕ

- only excite universal gravity sector

7. Solving Einstein's equations

e.g. turn on field-theory metric perturbation

$$g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}\underline{x}^2 + \epsilon \left(H^{(b)}_{ty} e^{ip_z z}\, 2\mathrm{d}t\mathrm{d}y + H^{(b)}_{xz} e^{-iq_0 t}\, 2\mathrm{d}x\mathrm{d}z \right)$$

perturbative expansion in sources

- bulk metric:

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$$\Longrightarrow g^{(1)}_{mn}\mathrm{d}x^m \mathrm{d}x^n = e^{2A}\left[H^{(1z)}(u,p_z)\, H^{(b)}_{ty} e^{ip_z z}\, 2\mathrm{d}t\mathrm{d}y + H^{(1t)}(u,q_0)\, H^{(b)}_{xz} e^{-iq_0 t}\, 2\mathrm{d}x\mathrm{d}z \right]$$

- Einstein's eqs. at $\mathcal{O}(\epsilon)$ = boundary value problem for bulk functions $H^{(1z)}(u,p_z)$ & $H^{(1t)}(u,q_0)$

$H^{(1z)}(u,p_z)$:
= 1 at boundary (explicitly sourced)
= regular at horizon (static)

$H^{(1t)}(u,q_0)$:
= 1 at boundary (explicitly sourced)
= incoming-wave at horizon (time-dep.)

7. Solving Einstein's equations

e.g. turn on field-theory metric perturbation

$$g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}\underline{x}^2 + \epsilon\left(H^{(b)}_{ty}\, e^{ip_z z}\, 2\mathrm{d}t\mathrm{d}y + H^{(b)}_{xz}\, e^{-iq_0 t}\, 2\mathrm{d}x\mathrm{d}z\right)$$

perturbative expansion in sources

- bulk metric:

$$g_{mn}\mathrm{d}x^m \mathrm{d}x^n = \underbrace{g^{(0)}_{mn}\mathrm{d}x^m\mathrm{d}x^n}_{\text{background}} + \underbrace{\epsilon g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}_{\text{sourced fluctuation}} + \underbrace{\epsilon^2 g^{(2)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathcal{O}(\epsilon^3)}_{\text{dynamical backreaction}}$$

$$\Longrightarrow g^{(2)}_{xy} = e^{2A}\, H^{(2tz)}(u, q_0, p_z)\, H^{(b)}_{ty}\, e^{ip_z z}\, H^{(b)}_{xz}\, e^{-iq_0 t}$$

- xy-component of Einstein's eqs. at $\mathcal{O}(\epsilon^2)$ = boundary value problem for bulk function $H^{(2tz)}(u, q_0, p_z)$

= 0 at boundary (not explicitly sourced)
& near-horizon form dictated by $H^{(1z)}(u)$ and $H^{(1t)}(u)$

7. Solving Einstein's equations

It turns out that the 3 field-theory metric perturbations
$\{h_{xz}(t), h_{yz}(t)\}$, $\{h_{tx}(z), h_{ty}(z)\}$, $\{h_{ty}(z), h_{xz}(t)\}$
involve 5 independent bulk functions:

$$H^{(1z)}(u,q)\ , \quad H^{(1t)}(u,q)$$

$$H^{(2tz)}(u,q,p)\ , \quad H^{(2tt)}(u,q,p)\ , \quad H^{(2zz)}(u,q,p)$$

- ✓ done: perturbative expansion in sources (in ϵ)
- ➢ next: hydro gradient expansion
  - ➢ 2x $H^{(1\dots)}(u,q) = H_0(u) + q\, H_1(u) + q^2\, H_2(u) + \mathcal{O}(q^3)$
  - ➢ 3x $H^{(2\dots)}(u,q,p) = H_{(0,0)}(u) + \left[q\, H_{(1,0)}(u) + p\, H_{(0,1)}\right] + \left[q^2\, H_{(2,0)} + q\, p\, H_{(1,1)}(u)(u) + p^2\, H_{(0,2)}(u)\right] + \dots$
- ➢ 2x3 + 3x6 = 24 functions

7. Solving Einstein's equations

- 2x3 + 3x6 = 24 functions
- We found analytic solutions for 19 and explicit integral expressions for another 4
  (given in terms of arbitrary black-brane background solutions)
- 1 unknown function
  …which however (as it turns out) does not enter $\langle T^{\mu\nu} \rangle$!

7. Solving Einstein's equations

To give an idea of what the solutions look like:

$$H^{(1t)}(u,q) = \underbrace{(1-u)^{-iq/(4\pi T)}}_{\text{incoming-wave}} \left[1 - q\,\frac{i}{4\pi T}\log\left(\frac{f(u)}{1-u}\right) + q^2 K_2^{(1t)} + \mathcal{O}(q^3)\right]$$

incoming-wave   analytic solutions

...where

$$K_2^{(1t)}(u) = \int_0^u \mathrm{d}v\,\frac{1}{v\,f(v)\,e^{4A(v)}}\int_1^v \mathrm{d}w\,w\,f(w)\,e^{4A(w)}\left(-\frac{L^2}{4f(w)}\right)\left\{\frac{1}{w^2\,f(w)\,e^{2A(w)}} + \frac{f + 2\,(1-w)\,f' - \log\left(\frac{1-w}{f}\right)\left[\frac{f}{w} + 4\,(1-w)\,A'\,f + (1-w)\,f'\right]}{(1-w)^2\,{f_H}^2\,e^{2A_H}}\right\}$$

derivative of f at horizon   value of A at horizon

Note: all results reproduce conformal expressions for $\phi \to 0$.

9. Numerical results

Two families of holographic RG flows

1st family of bulk potentials:

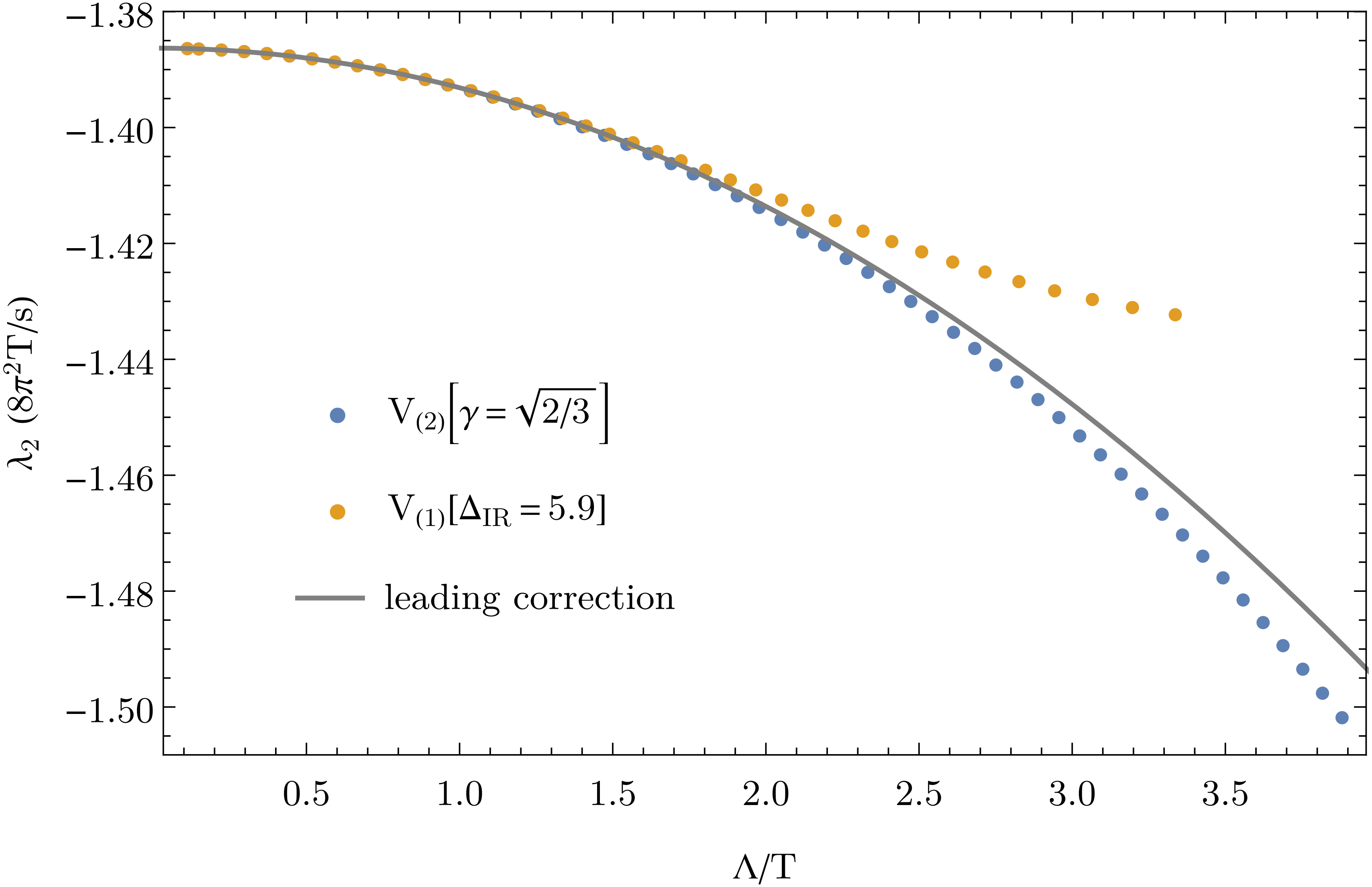
- derives from quartic superpotential $L\,W = -\frac{3}{2} - \frac{\phi^2}{8} + \frac{\phi^4}{16\phi_m^2}$
- has max at $\phi = 0$, min at $\phi = \phi_m$
  - second AdS region for $\phi \to \phi_m$ with smaller AdS radius and IR operator dimension $\Delta_{\text{IR}} = 4 + 48/\left(24 + \phi_m^2\right) \in (4, 6)$
    - dual IR fixed point

2nd family of bulk potentials:

$$V_{(2)} = \frac{1}{L^2}\left[-12 - \left(\frac{3}{2} - \frac{1}{\gamma^2}\right)\phi^2 + \frac{2}{\gamma^4}\left(1 - \cosh(\gamma\phi)\right)\right]$$

- monotonically decreasing --> non-conformal IR
- in the deep IR: $L^2\,V_{(2)} \xrightarrow{\phi\to\infty} -e^{\gamma\phi}/\gamma^4$
  - non-conformal Chamblin-Reall background (transport coefficients obtained by compactifying AdS)

9. Numerical results



9. Numerical results