Lifshitz & hyperscaling violating holography with arbitrary critical exponent z>1

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Based on: 1403.xxxx [W. Chemissany, I. Papadimitriou]
1106.4826 [I. Papadimitriou]
1007.4592 [I. Papadimitriou]

Unified approach to the holographic map

- The main motivation of this work is the desire for a universal algorithmic construction of the holographic dual to any gravitational theory
- Within the gauge/gravity duality such a universal algorithm does exist and there are feasible ways to generalize it beyond supergravity
- The input of this algorithm is the gravitational Lagrangian (or equations of motion – cf. Vasiliev's theory) and possibly a choice of boundary conditions
- The output is the full holographic dictionary, i.e. spectrum of operators, sources, 1-point functions, free energy etc.
- The central objects are a complete integral of the radial Hamilton-Jacobi equation and the symplectic 2-form of the theory

These two objects:

- determine the boundary terms that render the variational problem at infinity well posed and the on-shell action UV finite
- lead to a very efficient way of *deriving* the general asymptotic expansions – cf. Fefferman-Graham expansions, and hence identifying the sources and 1-point functions of the dual operators (symplectic form)
- provide 'fake superpotentials' that turn the second order equations of motion into first order BPS-like equations
- provide a complete set of asymptotic WKB wavefunctions of quantum gravity

Lifshitz & hyerscaling holography

- Holographic description of quantum critical points and QFTs exhibiting hyperscaling violation
- Geometries suffer from IR pathologies – not relevant here
- These backgrounds can emerge in the IR or some intermediate energy scale starting with some other UV completion – e.g. AdS in the same or higher dimensions
- Here we will focus on the case where these geometries are considered as the UV. Otherwise we can develop the holographic dictionary in whatever UV completion these geometries emerge from

Related work for Lifshitz

- [Ross & Saremi '09] (Einstein-Proca, counterterms by hand, only linearized fluctuations)
- [Ross '11] (Einstein-Proca, vielbein formalism, counteterms derived using dilatation operator method [I.P. & Skenderis '04])
- [Mann & McNees '11] (Einstein-Proca, counterterms put by hand)
- [Baggio, de Boer & Holsheimer '11] (Einstein-Proca-scalar with no mixing , only constant sources, incomplete source/vev map)
- [Chemissany, Geissbühler, Hartong & Rollier '12] ($d = 4$ $z = 2$, Scherk-Schwarz reduction from 5d axion-dilaton model [I.P. '11])
- [Korovin, Skenderis & Taylor '13] ($z = 1 + \epsilon$)
- [Christensen, Hartong, Obers & Rollier '13] ($d = 4$ $z = 2$, Scherk-Schwarz reduction from 5d axion-dilaton model [I.P. '11])
- [Andrade & Ross '13] (Einstein-Proca, linear metric fluctuations)
- Related work also in Hořava gravity

Outline

1. Lifshitz & Hyperscaling Violation
   - Geometrization of scaling fixed points
   - The model
   - Lifshitz & hvLf solutions
2. Hamilton-Jacobi approach to the holographic dictionary
   - Relativistic axion-dilaton system
   - Covariant HJ algorithm for non-relativistic systems
3. New solutions
4. Concluding remarks

- The Lifshitz metric is

$$ds^2_{d+2} = \ell^2 u^{-2}\left(du^2 - u^{-2(z-1)}dt^2 + dx^a dx^a\right)$$

with dynamical exponent $z \neq 1$

- This metric is invariant under the scaling transformation

$$x^a \to \lambda x^a\ ,\ t \to \lambda^z t\ ,\ u \to \lambda u$$

- The null energy condition

$$T_{\mu\nu}k^\mu k^\nu \geq 0, \quad k^\mu k_\mu = 0$$

requires

$$z \geq 1$$

Hyperscaling violating Lifshitz

- Hyperscaling refers to the property that the free energy and other thermodynamic quantities scale with temperature by their naive dimension. e.g. $S \sim T^{(2-\theta)/z}$, where θ is the hyperscaling violating parameter
- The Hyperscaling violating Lifshitz metric is [Huijse, Sachdev & Swingle '11]

$$ds^2_{d+2} = \ell^2 u^{-2(d-\theta)/d} \left(du^2 - u^{-2(z-1)} dt^2 + dx^a dx^a \right)$$

  with dynamical exponent $z \neq 1$ and hyperscaling violation exponent $\theta \neq 0$
- This metric has the scaling property that under

$$x^a \to \lambda x^a \ , \ t \to \lambda^z t \ , \ u \to \lambda u$$

  the metric transforms as

$$ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2}$$

- In order to interpret the space spanned by the spatial transverse coordinates as a boundary of the hvLf space then we must require that

$$\theta < d$$

- For $\theta > 0$ the metric develops a curvature singularity as $u \to 0$
- For $\theta = 0$ the metric reduces to Lifshitz geometry
- For $\theta < 0$ we can bring this metric to the canonical form

$$ds^2 = dr^2 + \gamma_{ij} dx^i dx^j$$

by the coordinate change

$$\frac{r}{\ell} = \frac{d}{\theta} u^{\theta/d}, \quad \theta < 0$$

- After this coordinate transformation the components of the induced metric γ_{ij} are

$$\gamma_{tt} = -r^{2\left(1+\frac{dz}{|\theta|}\right)}, \quad \gamma_{ta} = 0, \quad \gamma_{ab} = r^{2\left(1+\frac{d}{|\theta|}\right)}$$

$$S = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+2}x \sqrt{-g} \left(R[g] - \alpha \partial_\mu \phi \partial^\mu \phi - Z(\phi) F^2 - W(\phi) A^2 - V(\phi) \right)$$

- Preserve $U(1)$ gauge symmetry via the Stückelberg mechanism:

$$A_\mu \to B_\mu = A_\mu - \partial_\mu \omega$$

  such that under a $U(1)$ transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \quad \omega \to \omega + \Lambda$$

- Weyl transformation:

$$g \to e^{2\xi\phi} g$$

$$\boxed{S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi}\left(R[g] - \alpha_\xi\partial_\mu\phi\partial^\mu\phi - Z_\xi(\phi)F^2 - W_\xi(\phi)B^2 - V_\xi(\phi)\right)}$$

$$\begin{aligned}
\alpha_\xi &= \alpha - d(d+1)\xi^2 \\
Z_\xi(\phi) &= e^{-2\xi\phi}Z(\phi) \\
W_\xi(\phi) &= W(\phi) \\
V_\xi(\phi) &= e^{2\xi\phi}V(\phi)
\end{aligned}$$

$$V_\xi = V_o e^{2(\rho+\xi)\phi}, \qquad Z_\xi = Z_o e^{-2(\xi+\nu)\phi}, \qquad W_\xi = W_o e^{2\sigma\phi}$$

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \qquad B = b(r)dt, \qquad \phi = \phi_o + \mu r,$$

$$\boxed{\begin{aligned}
&\rho = -\xi \\
&\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1} \\
&\nu = -\xi - \frac{z-\epsilon}{\mu} \\
&\sigma = \frac{z-\epsilon}{\mu} \\
&W_o = 2Z_o\epsilon(d + z + d\mu\xi - \epsilon) \\
&V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon \\
&b = \frac{\mathcal{Q}}{\epsilon Z_0}e^{\epsilon r} \\
&\mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon
\end{aligned}}$$

$$ds^2 = dr^2 - r^{2\nu_z} dt^2 + r^{2\nu_1} d\vec{x}^2, \qquad B = b(r)dt, \quad \phi = \phi_o + \mu \log r,$$

where

$$\nu_z = 1 + \frac{dz}{|\theta|}, \quad \nu_1 = 1 + \frac{d}{|\theta|}.$$

$$\boxed{\begin{aligned}
&\mu(\xi + \rho) = -1 \\
&\epsilon = \frac{\left(\alpha_\xi + d^2\xi^2\right)\mu^2 - d\xi(\nu_1 + 1)\mu - \nu_1(d + \nu_z - 1) + \nu_z(\nu_z - 1)}{\nu_z - \nu_1} \\
&\nu = -\xi - \frac{\nu_z - \epsilon}{\mu} \\
&\sigma = \frac{\nu_z - \epsilon - 1}{\mu} \\
&W_o = 2\epsilon Z_o[d(\nu_1 + \mu\xi) + \nu_z - 1 - \epsilon]e^{-2(\sigma+\xi+\nu)\phi_o}, \\
&V_o = \{\epsilon(\nu_1 - \nu_z) - d(\nu_1 + \mu\xi)[d(\nu_1 + \mu\xi) + \nu_z - 1]\}e^{-2(\rho+\xi)\phi_o} \\
&b = \frac{\mathcal{Q}}{\epsilon Z_o} r^\epsilon \\
&\mathcal{Q}^2 = \frac{1}{2} Z_o(\nu_z - \nu_1)\epsilon
\end{aligned}}$$

Relation between Lifshitz & HvLf

- The parameter ξ allows us to automatically construct hvLf solutions from Lifshitz ones and vice versa – but they solve different equations of motion
- If $(\bar{g}, \bar{A}_\mu, \bar{\phi})$ is a solution of the equations of motion with $\xi = 0$ then

$$(g^{(\xi)}, A_\mu^{(\xi)}, \phi^{(\xi)}) := (e^{-2\xi\bar{\phi}}\bar{g}, \bar{A}_\mu, \bar{\phi})$$

  solves the equations of motion with non-zero ξ – cf. non-conformal branes
- The above Lifshitz and hvLf solutions both solve the equations of motion with a given ξ, but we can relate Lifshitz solutions with one value of ξ, say ξ_L, to hvLf solutions with a different value ξ_h

Change of coordinates

- Starting with the hvLf metric and introducing the new coordinates

$$r = \lambda \frac{d}{|\theta|} e^{\frac{|\theta|}{d}\bar{r}}, \quad t = \left(\frac{|\theta|}{d}\right)^{-\nu_z} \lambda^{1-\nu_z} \bar{t}, \quad x^a = \left(\frac{|\theta|}{d}\right)^{-\nu_1} \lambda^{1-\nu_1} \bar{x}^a$$

  the metric takes the form

$$ds^2 = \lambda^2 e^{\frac{2|\theta|\bar{r}}{d}} \left(d\bar{r}^2 - e^{2z\bar{r}} d\bar{t}^2 + e^{2\bar{r}} d\vec{\bar{x}}^2\right)$$

  while the scalar is given by

$$\phi = \phi_{oL} + \mu_L \bar{r} = \phi_{oh} + \mu_h \log r$$

  where

$$\phi_{oh} = \phi_{0L} - \mu_L \frac{d}{|\theta|} \log \lambda - \mu_L \frac{d}{|\theta|} \log \frac{d}{|\theta|}, \quad \mu_h = \frac{d}{|\theta|} \mu_L$$

- It follows that if g_L is a Lifshitz metric and

$$\lambda = e^{\frac{|\theta|}{d\mu_L} \phi_{oL}}$$

  then

$$g_h = e^{\frac{2|\theta|}{d\mu_L} \phi} g_L$$

  is a hvLf metric with parameter θ.

Relation of parameters

- If g_o solves the equations of motion with $\xi = 0$, then $g_\xi = e^{-2\xi\phi} g_o$ solves the equations of motion with non-zero ξ
- Let $g_{\xi_L} = e^{-2\xi_L \phi} g_o$ be a Lifshitz metric and $g_{\xi_h} = e^{-2\xi_h \phi} g_o$ a hvLf one with hyperscaling violating parameter θ that solve the equations of motion corresponding respectively to $\xi = \xi_L$ and $\xi = \xi_h$
- It follows that the above Lifshitz and hvLf solutions are related via a local diffeomorphism and with the following map between parameters

$$\xi_L - \xi_h = \frac{|\theta|}{d\mu_L} = \frac{1}{\mu_h}, \qquad \alpha_{\xi_L} = \alpha_{\xi_h} - d(d+1)\frac{1}{\mu_h}\left(\frac{1}{\mu_h} + 2\xi_h\right)$$

$$\epsilon_L = \frac{\epsilon_h}{\nu_1 - 1}$$

$$\begin{aligned} \rho &= -\xi_L = -\xi_h - \frac{1}{\mu_h} \\ \nu &= \frac{\epsilon_L - z - \mu_L \xi_L}{\mu_L} = \frac{\epsilon_h - \nu_z - \mu_h \xi_h}{\mu_h} \\ \sigma &= -\frac{\epsilon_L - z}{\mu_L} = \frac{-\epsilon_h + \nu_z - 1}{\mu_h} \end{aligned}$$

$$\mathcal{Q}_h = \mathcal{Q}_L (\nu_1 - 1)^{(\nu_1 - 1)\epsilon_L - \nu_z + 1} e^{\left[\frac{(\nu_1 - 1)\epsilon_L - \nu_z + 1}{(\nu_1 - 1)\mu_L}\right]\phi_{oL}}$$

Generic dilaton-axion system

$$\boxed{S = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,\left(R[g] - \partial_\mu\varphi\partial^\mu\varphi - Z(\varphi)\partial_\mu\chi\partial^\mu\chi + V(\varphi)\right)}$$

- Anisotropic $\mathcal{N} = 4$ plasma [Mateos & Trancanelli '11]
- Improved Holographic QCD [Gursoy, Kiritsis '07]
- Non-conformal branes [Wiseman & Withers '08], [Kanitscheider, Skenderis, & Taylor '08]

Radial Hamiltonian formulation

- ADM decomposition

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Lagrangian

$$L = \tfrac{1}{2\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} N \Big(-R[\gamma] + K^i_j K^j_i - K^2 + \tfrac{1}{N^2}(\dot{\varphi} - N^i \partial_i \varphi)^2$$
$$+ \tfrac{1}{N^2} Z(\varphi)(\dot{\chi} - N^i \partial_i \chi)^2 + \partial_i \varphi \partial_i \varphi + Z(\varphi)\partial_i \chi \partial_i \chi - V(\varphi) \Big)$$

- Hamiltonian

$$H = \int_{\Sigma_r} d^d x \left(N\mathcal{H} + N_i \mathcal{H}^i \right)$$

- Constraints

$$\mathcal{H} = 2\kappa^2 \gamma^{-\frac{1}{2}} \left(\pi^i_j \pi^j_i - \frac{1}{d-1}\pi^2 + \frac{1}{4}\pi_\varphi^2 + \frac{1}{4} Z^{-1}(\varphi)\pi_\chi^2 \right)$$
$$+ \frac{1}{2\kappa^2}\sqrt{\gamma} \left(R[\gamma] - \partial_i \varphi \partial^i \varphi - Z(\varphi)\partial_i \chi \partial^i \chi + V(\varphi) \right) = 0$$
$$\mathcal{H}^i = -2D_j \pi^{ij} + \pi_\varphi \partial^i \varphi + \pi_\chi \partial^i \chi = 0$$

- Momenta from off-shell Lagrangian:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right), \quad K_{ij} = \frac{1}{2}\dot{\gamma}_{ij},$$
$$\pi_\varphi = \frac{\delta L}{\delta \dot{\varphi}} = \frac{1}{\kappa^2}\sqrt{\gamma}\dot{\varphi},$$
$$\pi_\chi = \frac{\delta L}{\delta \dot{\chi}} = \frac{1}{\kappa^2}\sqrt{\gamma}Z(\varphi)\dot{\chi}$$

- Momenta from Hamilton's principal functional $\mathcal{S}[\gamma, \varphi, \chi]$ (on-shell action):

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \quad \pi_\varphi = \frac{\delta \mathcal{S}}{\delta \varphi} \quad \pi_\chi = \frac{\delta \mathcal{S}}{\delta \chi}$$

BPS-like equations & asymptotic expansions

- Combining the two expressions for the canonical momenta leads to the first order equations:

$$\dot{\gamma}_{ij} = 4\kappa^2 \left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{kl}\gamma_{ij}\right) \frac{1}{\sqrt{\gamma}} \frac{\delta\mathcal{S}}{\delta\gamma_{kl}}$$

$$\dot{\varphi} = \kappa^2 \frac{1}{\sqrt{\gamma}} \frac{\delta\mathcal{S}}{\delta\varphi}$$

$$\dot{\chi} = \kappa^2 Z^{-1}(\varphi) \frac{1}{\sqrt{\gamma}} \frac{\delta\mathcal{S}}{\delta\chi}$$

- Given a solution $\mathcal{S}[\gamma, \varphi, \chi]$ of the Hamilton-Jacobi equation these equations:
  - are first order BPS-like equations for background solutions (cf. fake supergravity), with $\mathcal{S}$ as the (fake) superpotential
  - determine the asymptotic expansions of the fields γ_{ij}, φ and χ (cf. Fefferman-Graham expansion)

Recursive solution of the Hamilton-Jacobi equation

- The Hamilton-Jacobi equation is a functional equation for the on-shell action as a function of the induced fields on a given hypersurface Σ_r, i.e.

$$\mathcal{S} = \int_{\Sigma_r} d^d x \mathcal{L}[\gamma, \varphi, \chi]$$

- Look for a solution in the form of a covariant expansion in eigenfunctions of a suitable functional operator $\widehat{\delta}$

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \mathcal{S}_{(4)} + \cdots$$

- The choice of $\widehat{\delta}$ is not unique, but it must be chosen so that the covariant expansion is compatible with the radial asymptotic expansion of the fields

- e.g. for AlAdS asymptotics we have

$$\dot{\gamma}_{ij} \sim 2\gamma_{ij}, \quad \dot{\varphi} \sim -(d-\Delta_\varphi)\varphi, \quad \dot{\chi} \sim 0$$

- It follows that the generator of radial translations can represented asymptotically as

$$\partial_r = \int d^dx \left(\dot{\gamma}_{ij} \frac{\delta}{\delta\gamma_{ij}} + \dot{\varphi}\frac{\delta}{\delta\varphi} + \dot{\chi}\frac{\delta}{\delta\chi} \right) \sim \int d^dx \left(2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} - (d-\Delta_\varphi)\varphi\frac{\delta}{\delta\varphi} \right)$$

- This is the dilatation operator of the dual theory and it is the correct operator for the covariant expansion in the case of AlAdS asymptotics [I. P. & Skenderis '04]

Non-AlAdS asymptotics & zero order solution

- More general asymptotics can be parameterized in terms of the relations

$$\dot{\gamma}_{ij} \sim f_{ij}[\gamma, \varphi, \chi], \quad \dot{\varphi} \sim g[\gamma, \varphi, \chi], \quad \dot{\chi} \sim h[\gamma, \varphi, \chi]$$

  where f_{ij}, g, h are *algebraic* functions of γ, φ and χ

- Any asymptotic solution of this form can be obtained from a 'superpotential'

$$\boxed{\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} U(\varphi, \chi)}$$

  via the first order BPS-like equations

- $U(\varphi, \chi)$ satisfies

$$(\partial_\varphi U)^2 + Z^{-1}(\varphi)(\partial_\chi U)^2 - \frac{d}{d-1} U^2 + V(\varphi) = 0$$

- Any χ dependence of $U(\varphi, \chi)$ gives only a finite contribution to $\mathcal{S}$ and therefore we can take $U(\varphi)$ for the purpose of constructing the general asymptotic expansions

Expansion operator

- We expand $\mathcal{S}$ in eigenfunctions of an operator $\widehat{\delta}$ such that $\mathcal{S}_{(0)}$ is an eigenfunction and higher order terms are asymptotically subleading
- An operator that fulfills these criteria for arbitrary superpotential $U(\varphi)$ is

$$\widehat{\delta} := \delta_\gamma = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}}$$

- δ_γ counts powers of the induced metric γ_{ij}. e.g.

$$\delta_\gamma \gamma_{ij} = 2\gamma_{ij}, \quad \delta_\gamma R_{ij} = 0, \quad \delta_\gamma R = -2R, \quad \delta_\gamma(\partial^i\varphi\partial_i\varphi) = -2\partial^i\varphi\partial_i\varphi$$

Linear equations for higher order terms

- Inserting the expansion of $\mathcal{S}$ in eigenfunctions of δ_γ into the Hamilton-Jacobi equation and collecting terms of the same eigenvalue we obtain a tower of linear equations for $\mathcal{L}_{(2n)}$, $n>0$

$$\boxed{U'(\varphi)\frac{\delta}{\delta\varphi}\int d^dx\mathcal{L}_{(2n)}-\left(\frac{d-2n}{d-1}\right)U(\varphi)\mathcal{L}_{(2n)}=\mathcal{R}_{(2n)},\quad n>0}$$

- The source terms are given by

$$\mathcal{R}_{(2)}=-\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma]-\partial_i\varphi\partial^i\varphi-Z(\varphi)\partial_i\chi\partial^i\chi\right),$$

$$\mathcal{R}_{(2n)}=-2\kappa^2\gamma^{-\frac{1}{2}}\sum_{m=1}^{n-1}\left(\pi_{(2m)}{}^i_j\pi_{(2(n-m))}{}^j_i-\frac{1}{d-1}\pi_{(2m)}\pi_{(2(n-m))}\right.$$
$$\left.+\frac{1}{4}\pi_{\varphi(2m)}\pi_{\varphi(2(n-m))}+\frac{1}{4}Z^{-1}(\varphi)\pi_{\chi(2m)}\pi_{\chi(2(n-m))}\right),\quad n>1$$

General solution of linear equations

- Homogeneous solution (UV finite):

$$\mathcal{L}^{hom}_{(2n)} = \mathcal{F}_{(2n)}[\gamma, \chi] e^{-(d-2n)A(\varphi)}$$

where

$$A = -\frac{1}{d-1} \int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} U(\bar{\varphi})$$

and $\mathcal{F}_{(2n)}[\gamma, \chi]$ is a local function of γ_{ij} and χ of weight $d - 2n$

- Inhomogeneous solution:

$$\boxed{\mathcal{L}_{(2n)} = e^{-(d-2n)A(\varphi)} \int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} e^{(d-2n)A(\bar{\varphi})} \mathcal{R}_{(2n)}(\bar{\varphi})}$$

Functional integration

$\mathcal{R}_{(2n)}$	$\mathcal{L}_{(2n)}$
$r_{1^m}(\varphi)t^{i_1 i_2 \dots i_m}\partial_{i_1}\varphi\partial_{i_2}\varphi\dots\partial_{i_m}\varphi$	$\fint^{\varphi}_{n,m} r_{1^m}(\bar{\varphi})t^{i_1 i_2 \dots i_m}\partial_{i_1}\varphi\partial_{i_2}\varphi\dots\partial_{i_m}\varphi$
$r_2(\varphi)t^{ij}D_i D_j\varphi$	$\fint^{\varphi}_{n,1} r_2(\bar{\varphi})t^{ij}D_i D_j\varphi$ $-\fint^{\varphi}_{n,2} U' A' \partial^2_{\bar{\varphi}}\left(\frac{1}{A'}\right)\fint^{\bar{\varphi}}_{n,1} r_2(\tilde{\varphi})t^{ij}\partial_i\varphi\partial_j\varphi$
$\left(r_{1^2 2}(\varphi)t_1^{ijkl} + s_{1^2 2}(\varphi)t_2^{ijkl}\right)\partial_i\varphi\partial_j\varphi D_k D_l\varphi$	$\fint^{\varphi}_{n,3} s_{1^2 2}(\bar{\varphi})t_2^{ijkl}\partial_i\varphi\partial_j\varphi D_k D_l\varphi$
$\left(r_{2^2}(\varphi)t_1^{ijkl} + s_{2^2}(\varphi)t_2^{ijkl}\right)D_i D_j\varphi D_k D_l\varphi$	$\left(\fint^{\varphi}_{n,2} r_{2^2}(\bar{\varphi})t_1^{ijkl} + \fint^{\varphi}_{n,2} s_{2^2}(\bar{\varphi})t_2^{ijkl}\right)D_i D_j\varphi D_k D_l\varphi$ $-2\fint^{\varphi}_{n,3} U' A' \partial^2_{\bar{\varphi}}\left(\frac{1}{A'}\right)\fint^{\bar{\varphi}}_{n,2} s_{2^2}(\tilde{\varphi})t_2^{ijkl}\partial_i\varphi\partial_j\varphi D_k D_l\varphi$

$$\fint^{\varphi}_{n,m} \equiv \left(A'\right)^m e^{-(d-2n)A}\int^{\varphi}\frac{d\bar{\varphi}}{U'}e^{(d-2n)A}\left(A'\right)^{-m}$$

Recursion procedure

$$
\begin{tikzcd}
\mathcal{R}_{(2n)} \arrow[r, "\int"] & \mathcal{L}_{(2n)} \arrow[r, "\delta"] & \{\pi_{(2n)}\} \arrow[d] \\
\{\pi_{(2n+2)}\} \arrow[d] & \mathcal{L}_{(2n+2)} \arrow[l, "\delta"'] & \mathcal{R}_{(2n+2)} \arrow[l, "\int"'] \\
\mathcal{R}_{(2n+4)} \arrow[r, "\int"] & \mathcal{L}_{(2n+4)} & \cdots
\end{tikzcd}
$$

Radial Hamiltonian formalism for massive vector-scalar theory

- Radial ADM Lagrangian:

$$
\begin{aligned}
L \quad = \quad & \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-\gamma}Ne^{d\xi\phi}\left\{R[\gamma]+K^2-K^{ij}K_{ij}+\frac{2d\xi}{N}K(\dot{\phi}-N^i\partial_i\phi)\right. \\
& -\frac{\alpha_\xi}{N^2}\left(\dot{\phi}-N^i\partial_i\phi\right)^2-\alpha_\xi\gamma^{ij}\partial_i\phi\partial_j\phi \\
& -Z_\xi(\phi)\left(\frac{2}{N^2}\gamma^{ij}(F_{ri}-N^kF_{ki})(F_{rj}-N^lF_{lj})+\gamma^{ij}\gamma^{kl}F_{ik}F_{jl}\right) \\
& \left.-W_\xi(\phi)\left(\frac{1}{N^2}\left(A_r-N^iA_i-\dot{\omega}+N^i\partial_i\omega\right)^2+\gamma^{ij}B_iB_j\right)-V_\xi(\phi)\right\}
\end{aligned}
$$

- Hamiltonian:

$$
\begin{aligned}
H \quad & = \quad \int d^{d+1}x\left(\dot{\gamma}_{ij}\pi^{ij}+\dot{A}_i\pi^i+\dot{\phi}\pi_\phi+\dot{\omega}\pi_\omega\right)-L \\
& = \quad \int d^{d+1}x\left(N\mathcal{H}+N_i\mathcal{H}^i+A_r\mathcal{F}\right)
\end{aligned}
$$

Constraints

$$\mathcal{H} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left\{ 2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} + \frac{1}{2\alpha}\left(\pi_\phi - 2\xi\pi\right)^2 + \frac{1}{4} Z_\xi^{-1}(\phi)\pi^i\pi_i + \frac{1}{2} W_\xi^{-1}(\phi)\pi_\omega^2 \right\} + \frac{\sqrt{-\gamma}}{2\kappa^2} e^{d\xi\phi}\left(-R[\gamma] + \alpha_\xi \partial^i\phi\partial_i\phi + Z_\xi(\phi)F^{ij}F_{ij} + W_\xi(\phi)B^iB_i + V_\xi(\phi)\right)$$

$$\mathcal{H}^i = -2D_j\pi^{ji} + F^i{}_j\pi^j + \pi_\phi\partial^i\phi - B^i\pi_\omega$$

$$\mathcal{F} = -D_i\pi^i + \pi_\omega$$

Canonical momenta

- From off-shell Lagrangian:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(K\gamma^{ij} - K^{ij} + \frac{d\xi}{N}\gamma^{ij}\left(\dot{\phi} - N^k\partial_k\phi\right)\right),$$
$$\pi^{i} = \frac{\delta L}{\delta \dot{A}_{i}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}Z_\xi(\phi)\frac{4}{N}\gamma^{ij}\left(F_{rj} - N^k F_{kj}\right),$$
$$\pi_\phi = \frac{\delta L}{\delta \dot{\phi}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(2d\xi K - \frac{2\alpha_\xi}{N}(\dot{\phi} - N^i\partial_i\phi)\right),$$
$$\pi_\omega = \frac{\delta L}{\delta \dot{\omega}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}W_\xi(\phi)\frac{2}{N}\left(\dot{\omega} - N^i\partial_i\omega - A_r + N^i A_i\right)$$

- From on-shell action:

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}, \quad \pi^{i} = \frac{\delta \mathcal{S}}{\delta A_{i}}, \quad \pi_\phi = \frac{\delta \mathcal{S}}{\delta \phi}, \quad \pi_\omega = \frac{\delta \mathcal{S}}{\delta \omega}$$

- Combining the two expressions for the momenta:

$$\dot{\gamma}_{ij} = -\frac{4\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\left(\gamma_{ik}\gamma_{jl} - \frac{\alpha_\xi + d^2\xi^2}{d\alpha} \gamma_{ij}\gamma_{kl} \right) \frac{\delta}{\delta\gamma_{kl}} - \frac{\xi}{2\alpha} \gamma_{ij} \frac{\delta}{\delta\phi} \right) \mathcal{S},$$

$$\dot{A}_i = -\frac{\kappa^2}{2} \frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} Z_\xi^{-1}(\phi) \gamma_{ij} \frac{\delta}{\delta A_j} \mathcal{S},$$

$$\dot{\phi} = -\frac{\kappa^2}{\alpha} \frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\frac{\delta}{\delta\phi} - 2\xi\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} \right) \mathcal{S},$$

$$\dot{\omega} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} W_\xi^{-1}(\phi) \frac{\delta}{\delta\omega} \mathcal{S}$$

Zero derivative solution

- The zero order solution of the HJ equation contains not transverse derivatives:

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int d^{d+1}x \sqrt{-\gamma} U(\phi, A_i A^i)$$

- Inserting this ansatz into the Hamiltonian constraint yields a PDE for $U(X, Y)$, where $X := \phi$, $Y := B_i B^i = A_i A^i$ (cf. superpotential equation)

$$\frac{1}{2\alpha} \left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 + Z_\xi^{-1}(X) Y U_Y^2$$
$$- \frac{1}{2d} \left((d+1)U + 2(d-1)Y U_Y\right)\left(U - 2Y U_Y\right) = \frac{1}{2} e^{2d\xi X} \left(W_\xi(X) Y + V_\xi(X)\right)$$

- Lifshitz asymptotics impose constraints on the asymptotic form of $U(X, Y)$

Lifshitz constraints

- Decomposition in time and spatial parts:

$$\gamma_{ij}dx^i dx^j = -(n^2 - n_a n^a)dt^2 + 2n_a dt dx^a + \sigma_{ab}dx^a dx^b, \quad A_i dx^i = a dt + A_a dx^a$$

- The first order flow equations become:

$$\begin{aligned}
\partial_r n^2 &= 4e^{-d\xi\phi}\left(-U_Y(a - n^a A_a)^2 + \left(\frac{\alpha_\xi}{2d\alpha}U + \frac{\xi}{2\alpha}U_X - \frac{\alpha_\xi + d^2\xi^2}{d\alpha}YU_Y\right)n^2\right)\\
\dot{n}_a &= 4e^{-d\xi\phi}\left(U_Y a A_a + \left(\frac{\alpha_\xi}{2d\alpha}U + \frac{\xi}{2\alpha}U_X - \frac{\alpha_\xi + d^2\xi^2}{d\alpha}YU_Y\right)n_a\right)\\
\dot{\sigma}_{ab} &= 4e^{-d\xi\phi}\left(U_Y A_a A_b + \left(\frac{\alpha_\xi}{2d\alpha}U + \frac{\xi}{2\alpha}U_X - \frac{\alpha_\xi + d^2\xi^2}{d\alpha}YU_Y\right)\sigma_{ab}\right)\\
\dot{a} &= -e^{-d\xi\phi}Z_\xi^{-1}(\phi)U_Y a\\
\dot{A}_a &= -e^{-d\xi\phi}Z_\xi^{-1}(\phi)U_Y A_a\\
\dot{\phi} &= -\frac{1}{\alpha}e^{-d\xi\phi}\left(U_X - (d+1)\xi U + 2\xi Y U_Y\right)
\end{aligned}$$

- Requiring that $U(X,Y)$, via these flow equations, leads to asymptotically Lifshitz bahaviour, i.e.

$$\begin{aligned} & n \sim e^{zr} n_{(0)}(t,x) \\ & n_a \sim e^{2r} n_{(0)a}(t,x) \\ & \sigma_{ab} \sim e^{2r} g_{(0)ab}(t,x) \\ & a \sim a_{(0)}(t,x) e^{\epsilon r} \\ & A_a \sim A_{(0)a}(t,x) e^{\epsilon r} \end{aligned}$$

- imposes the constraints:

$$Y = B_i B^i \sim Y_o(X) := -\frac{z-1}{2\epsilon} Z_\xi^{-1}(X), \quad B_a \sim 0$$

- or equivalently

$$\boxed{Y = B_i B^i \sim Y_o(X) B_i \sim B_{oi} = (B_{ot}, B_{oa}) := \left(n\sqrt{\frac{z-1}{2\epsilon} Z_\xi^{-1}(\phi)}\,,\, 0 \right)}$$

Asymptotic form of zero order solution

- Moreover, the function $U(X,Y)$ must satisfy

$$\boxed{\begin{aligned} &U(X,Y_o(X)) \sim e^{d\xi X}\left(d(1+\mu\xi)+z-1\right)\\ &U_Y(X,Y_o(X)) \sim -\epsilon e^{d\xi X} Z_\xi(X)\\ &U_X(X,Y_o(X)) \sim e^{d\xi X}\left(-\mu\alpha_\xi + d\xi(d+z)\right)\end{aligned}}$$

- Hence, the asymptotic form of the zero order solution is

$$\boxed{\mathcal{S}_{(0)} \sim \frac{1}{\kappa^2}\int_{\Sigma_r} d^{d+1}x\sqrt{-\gamma}e^{d\xi\phi}\left(d(1+\mu\xi)+\frac{1}{2}(z-1)-\epsilon Z_\xi(\phi)B_iB^i\right)}$$

- B_i must be kept explicitly in the zero order solution to reproduce the asymptotic form of π^i via the flow equations
- The Lifshitz constraints means that we set some of the sources of the full theory to zero – sources of irrelevant operators with respect to the Lifshitz theory
- How the Lifshitz constraints are relaxed in subleading orders depends on a complete integral $U(X, Y)$ of the superpotential equation – we will mostly focus on the case where the Lifshitz constraints are violated by normalizable modes only, i.e.

$$B_i - B_{oi} = \mathcal{O}\left(e^{(\epsilon - z - d - d\mu\xi)r}\right)$$

- Our approach remains fully covariant despite the non-relativistic constraints – simplifies the solution of the HJ equation

Recursive solution of the HJ equation

- In order to solve the HJ equation iteratively, we again expand $\mathcal{S}$ formally in eigenfunctions of a suitably chosen operator $\widehat{\delta}$
- The requirements that determine the operator $\widehat{\delta}$ are:
  - the zero order solution

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int_{\Sigma_r} d^{d+1}x\sqrt{-\gamma}U(\phi, B^2)$$

    is an eigenfunction for arbitrary $U(\phi, B^2)$ and
  - the expansion in eigenfunctions of $\widehat{\delta}$ is compatible with the asymptotic expansions in the radial coordinate
- These determine

$$\widehat{\delta} := \int d^{d+1}x\left(2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} + B_i\frac{\delta}{\delta B_i}\right)$$

- We then write

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \cdots + \mathcal{S}_{(k)} + \cdots$$

  with

$$\widehat{\delta}\mathcal{S}_{(k)} = (d+1-k)\mathcal{S}_{(k)}$$

Linear recursion equations

$$\boxed{\begin{aligned}&e^{-d\xi\phi}\left\{\frac{1}{\alpha}\left(U_X-(d+1)\xi U+2\xi YU_Y\right)\frac{\delta}{\delta\phi}-4U_YB_iB_j\frac{\delta}{\delta\gamma_{ij}}\right.\\&\left.+\left[\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)+Z_\xi^{-1}U_Y\right]B_i\frac{\delta}{\delta B_i}\right\}\int d^{d+1}x\mathcal{L}_{(k)}\\&-\frac{1}{d\alpha}e^{-d\xi\phi}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)(d+1-k)\mathcal{L}_{(k)}=\mathcal{R}_{(k)},\quad k>0\end{aligned}}$$

$$\begin{aligned}\mathcal{R}_{(k)}\quad=\quad&\frac{\sqrt{-\gamma}}{2\kappa^2}e^{d\xi\phi}\left(-R[\gamma]+\alpha_\xi\partial^i\phi\partial_i\phi+Z_\xi(\phi)F^{ij}F_{ij}\right)\delta_{k,2}\\&-\frac{1}{2}\frac{\kappa^2}{\sqrt{-\gamma}}e^{-d\xi\phi}\sum_{m=1}^{k-1}W_\xi^{-1}(\phi)\pi_{\omega\,(m)}\pi_{\omega\,(k-m)}\\&-\frac{\kappa^2}{\sqrt{-\gamma}}e^{-d\xi\phi}\sum_{m=1}^{k-1}\left\{2\left(\gamma_{ik}\gamma_{jl}-\frac{1}{d}\gamma_{ij}\gamma_{kl}\right){\pi_{(m)}}^{ij}{\pi_{(k-m)}}^{kl}\right.\\&\left.+\frac{1}{4}Z_\xi^{-1}(\phi){\pi_{(m)}}^{i}\pi_{(k-m)\,i}+\frac{1}{2\alpha}\left(\pi_{\phi\,(m)}-2\xi\pi_{(m)}\right)\left(\pi_{\phi\,(k-m)}-2\xi\pi_{(k-m)}\right)\right\}\end{aligned}$$

General solution method (characteristics)

- To solve these linear equations in general we need to eliminate the functional derivatives w.r.t. γ_{ij} and B_i by means of a field redefinition

$$\begin{aligned}
&\gamma_{ij} \to \widetilde{\gamma}_{ij} = \gamma_{ij} - \vartheta(\phi, B^2) B_i B_j \\
&B_i \to v_i = \beta(\phi, B^2) B_i
\end{aligned}$$

- The functions $\vartheta(\phi, B^2)$ and $\beta(\phi, B^2)$ are determined through linear PDEs
- The recursion relations then simplify to

$$\begin{aligned}
&e^{-d\xi\phi}\frac{1}{\alpha}\left(U_X - (d+1)\xi U + 2\xi Y U_Y\right)\frac{\delta}{\delta\phi}\int d^{d+1}x \mathcal{L}_{(n)}[\widetilde{\gamma}, v, \phi] \\
&-\frac{1}{d\alpha}e^{-d\xi\phi}\left(\alpha_\xi U - 2(\alpha_\xi + d^2\xi^2) Y U_Y + d\xi U_X\right)(d+1-n)\mathcal{L}_{(n)}[\widetilde{\gamma}, v, \phi] \\
&= \mathcal{R}_{(n)}[\widetilde{\gamma}, v, \phi], \quad n > 0
\end{aligned}$$

- Only need to integrate in φ, keeping $\widetilde{\gamma}_{ij}$ and v_i fixed.

Solution with Lifshitz constraints implemented

- If the Lifshitz constraints are only violated by normalizable modes, then a simpler recursion algorithm is possible. Namely, in that case we need only know $\mathcal{S}$ to linear order in $B_i - B_{oi}$
- At each order k in the expansion in eigenfunctions of $\widehat{\delta}$ we expand

$$\mathcal{L}_{(k)}[\gamma(x), B(x), \phi(x)] = \mathcal{L}^0_{(k)}[\gamma(x), \phi(x)]$$
$$+ \int d^{d+1}x'(B_i(x') - B_{oi}(x'))\mathcal{L}^{1i}_{(k)}[\gamma(x), \phi(x); x'] + \cdots$$

$k = 0$ equations

- The zero order solution $\mathcal{S}_{(0)}$ is specified by the function $U(X,Y)$ which can be Taylor expanded as

$$U = U_0(\phi) + U_1(\phi)(Y - Y_o(\phi)) + U_2(\phi)(Y - Y_o(\phi))^2 + \mathcal{O}(Y - Y_o(\phi))^3$$

  where

$$Y - Y_o = 2B_o^i(B_i - B_{oi}) + (B^i - B_o^i)(B_i - B_{oi})$$

- Parameterizing the coefficients as

$$U_n = e^{(d+1)\xi\phi} Y_o^{-n} u_n(\phi)$$

  and inserting this expansion in the 'superpotential equation' for $U(X,Y)$ we get a tower of equations for the functions $u_n(\phi)$

- The lowest two equations are:

$$\frac{1}{2\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)^2 + \left(-\frac{2\epsilon}{z-1} + \frac{2(d-1)}{d}\right)u_1^2 + \frac{2}{d}u_0u_1 - \frac{d+1}{2d}u_0^2 = \frac{1}{2}V - \frac{z-1}{4\epsilon}WZ^{-1}$$

$$\left[\frac{2}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{Z'}{Z} - \frac{8\epsilon}{z-1}u_1 + \frac{4}{d}\left(u_0 + 2(d-1)u_1\right)\right]u_2$$
$$+\frac{1}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\left(u_1' + \frac{Z'}{Z}u_1\right) - \frac{2\epsilon}{z-1}u_1^2 - u_1(u_0 - 4u_1) = -\frac{z-1}{2\epsilon}WZ^{-1}$$

$k > 0$ equations

- For $k > 0$ the leading order equation is

$$
\begin{aligned}
&\left[\frac{1}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{\delta}{\delta\phi} - 4Y_o^{-1}u_1 B_{oi}B_{oj}\frac{\delta}{\delta\gamma_{ij}}\right]\int d^{d+1}x'\mathcal{L}^0_{(n)} \\
&+\left(\frac{1}{2\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{Z'}{Z} + \frac{1}{d}(u_0 + 2(d-1)u_1) - \frac{2\epsilon}{z-1}u_1\right)B_{oi}\int d^{d+1}x'\mathcal{L}^{1i}_{(n)} \\
&-(d+1-n)\left(\frac{\xi}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right) + \frac{1}{d}(u_0 - 2u_1)\right)\mathcal{L}^0_{(n)}[\gamma(x),\phi(x)] = e^{-\xi\phi}\mathcal{R}^0_{(n)}
\end{aligned}
$$

Consistency condition

- The Taylor expansion in $B_i - B_{oi}$ is not automatically consistent with the radial evolution. Demanding compatibility leads to additional equations
- The potential incompatibility arises due to two possible ways of evaluating the radial derivative of B_i:
  - Since $B_i = B_{oi}$ up to normalizable modes

$$\dot{B}_i = \dot{B}_{oi} + \text{normalizable} = \left(\frac{\dot{n}}{n} - \frac{1}{2} \frac{Z'_\xi(\phi)}{Z_\xi(\phi)} \dot{\phi} \right) B_{oi} + \text{normalizable}$$

  - But also

$$\dot{B}_i = -\frac{\kappa^2}{2} \frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} Z_\xi^{-1}(\phi) \gamma_{ij} \frac{\delta}{\delta B_j} \mathcal{S} - \partial_i \dot{\omega}$$

- Demanding that these two expressions give the same result lead to the following consistency conditions

$$\frac{2}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{Z'}{Z} - \frac{8\epsilon}{z-1}u_1 + \frac{4}{d}\left(u_0 + 2(d-1)u_1\right) = 0$$

$$\left(-2Y_o^{-1}B_{ok}B_{ol}\frac{\delta}{\delta\gamma_{kl}} + \frac{1}{2\alpha}\frac{Z'}{Z}\frac{\delta}{\delta\phi}\right)\int d^{d+1}x'\mathcal{L}^0_{(k)} + (d+1-k)\left(\frac{1}{d} - \frac{\xi}{2\alpha}\frac{Z'}{Z}\right)\mathcal{L}^0_{(k)} =$$
$$-\left[\frac{d-1}{d} + \frac{1}{4\alpha}\left(\frac{Z'}{Z}\right)^2 - \frac{\epsilon}{z-1}\right]B_{oj}\int d^{d+1}x'\mathcal{L}^{1j}_{(k)}[\gamma(x'),\phi(x');x]$$
$$+\, e^{d\xi\phi}Y_o^{-1}B_o^i D_i\left(e^{-d\xi\phi}W_\xi^{-1}(\phi)D_j\int d^{d+1}x'\mathcal{L}^{1j}_{(k-2)}[\gamma(x'),\phi(x');x]\right)$$

General solution to linear order in Lifshiz constraints at $k = 0$

$$\boxed{\begin{aligned}
&V = \frac{1}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)(u_0' - u_1') + \frac{4}{d}(u_0 - u_1)u_1 + u_0u_1 - \frac{2\epsilon}{z-1}u_1^2 - \frac{2(d+1)}{d}u_0^2\\
&W = -\frac{2\epsilon}{z-1}Z\left(\frac{1}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\left(u_1' + \frac{Z'}{Z}u_1\right) - \frac{2\epsilon}{z-1}u_1^2 - u_1(u_0 - 4u_1)\right)\\
&\frac{2}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{Z'}{Z} - \frac{8\epsilon}{z-1}u_1 + \frac{4}{d}\left(u_0 + 2(d-1)u_1\right) = 0
\end{aligned}}$$

$$\boxed{\begin{aligned}
&u_0(\phi) \sim \left(z - 1 + d(1 + \mu\xi)\right)e^{-\xi\phi}\\
&u_1(\phi) \sim \frac{1}{2}(z-1)e^{-\xi\phi}
\end{aligned}}$$

General solution to linear order in Lifshiz constraints at $k > 0$

- $\mathcal{L}^0_{(k)}$:

$$\boxed{\begin{aligned}&\left[\frac{1}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{\delta}{\delta\phi} - 4Y_o^{-1}u_1 B_{oi}B_{oj}\frac{\delta}{\delta\gamma_{ij}}\right]\int d^{d+1}x'\mathcal{L}^0_{(k)}\\ &-(d+1-k)\left(\frac{\xi}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right) + \frac{1}{d}(u_0 - 2u_1)\right)\mathcal{L}^0_{(k)} = e^{-\xi\phi}\mathcal{R}^0_{(k)}\end{aligned}}$$

- $\mathcal{L}^{1j}_{(k)}$ is determined algebraically from $\mathcal{L}^0_{(k)}$

Solution method (characteristics)

- The vector field

$$v_i := s(\phi)\frac{B_{oi}}{\sqrt{-Y_o}}$$

  where

$$s(\phi) := \exp\left(2\alpha\int^{\phi}\frac{d\phi' u_1}{u_0' + \frac{Z'}{Z}u_1}\right)$$

  is constant w.r.t. the differential operator acting on $\mathcal{L}^0_{(k)}$

- The functional derivative w.r.t. the induced metric γ_{ij} can be eliminated by a change of variables

$$\gamma_{ij} \to \widetilde{\gamma}_{ij} = \gamma_{ij} - \vartheta(\phi)B_{oi}B_{oj}$$

  where

$$\vartheta = Y_o^{-1}(1 - c^2 s^2(\phi))$$

- The integration constant c must be non-zero since $\det\widetilde{\gamma} = c^2u^2\det\gamma$

Formal solution

- In terms of the new variables the equation for $\mathcal{L}^0_{(k)}$ becomes

$$\frac{1}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right)\frac{\delta}{\delta\phi}\int d^{d+1}x'\,\mathcal{L}^0_{(k)}$$
$$-\,(d+1-k)\left(\frac{\xi}{\alpha}\left(u_0' + \frac{Z'}{Z}u_1\right) + \frac{1}{d}(u_0 - 2u_1)\right)\mathcal{L}^0_{(k)} = e^{-\xi\phi}\mathcal{R}^0_{(k)}$$

- The inhomogeneous solution is

$$\boxed{\mathcal{L}^0_{(k)}[\widetilde{\gamma},\phi;v] = \alpha e^{(d+1-k)\mathcal{K}(\phi)}\int^{\phi} d\phi'\,\frac{e^{-(d+1-k)\mathcal{K}(\phi')}}{e^{\xi\phi'}\left(u_0' + \frac{Z'}{Z}u_1\right)}\mathcal{R}^0_{(k)}[\widetilde{\gamma},\phi';v]}$$

where

$$e^{\mathcal{K}(\phi)} = Z_\xi^{-1/2}s^{\frac{\epsilon}{z-1}-1}$$

- The functional integrations can be now performed as in the relativistic axion-dilaton case

- 'Superpotential' equation:

$$\frac{1}{2\alpha}\left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 + Z_\xi^{-1}(X) Y U_Y^2$$
$$-\frac{1}{2d}\left((d+1)U + 2(d-1)YU_Y\right)\left(U - 2YU_Y\right) = \frac{1}{2}e^{2d\xi X}\left(W_\xi(X)Y + V_\xi(X)\right)$$

- First order flow equations:

$$\dot{\gamma}_{ij} = 4e^{-d\xi X}\left(U_Y A_i A_j + \left(\frac{\alpha_\xi}{2d\alpha}U + \frac{\xi}{2\alpha}U_X - \frac{\alpha_\xi + d^2\xi^2}{d\alpha}YU_Y\right)\gamma_{ij}\right)$$
$$\dot{A}_i = -e^{-d\xi X} Z_\xi^{-1}(X) U_Y A_i$$
$$\dot{\phi} = -\frac{1}{\alpha}e^{-d\xi X}\left(U_X - (d+1)\xi U + 2\xi Y U_Y\right)$$
$$\dot{\omega} = 0$$

- Ansatz that separates variables:

$$U(X,Y) = \varepsilon_1 e^{d\xi X}\sqrt{e^{2\xi X}u^2(X) + \varepsilon_2 v^2(X)Y}$$

- Inserting this ansatz into the superpotential equation gives

$$\begin{aligned} v'^2 &= \alpha\varepsilon_2 W(X) \geq 0 \\ 2vv'uu' - u^2\left(2\alpha v^2 + v'^2\right) &= \alpha v^2\left(V(X) - \frac{v^2}{2}\varepsilon_2 Z^{-1}(X)\right) \\ u^2\left(u'^2 - \frac{(d+1)\alpha}{d}u^2\right) &= \alpha u^2 V(X) \end{aligned}$$

- These can be solved to obtain

$$\begin{aligned} v &= \pm\sqrt{\alpha}\int^X dX'\sqrt{\varepsilon_2 W(X')} \\ u^2 &= \begin{cases} \alpha v\vartheta^{-1}\int^X \vartheta\left(V - \frac{1}{2}v^2\varepsilon_2 Z^{-1}\right)v'^{-1}, & v' \neq 0 \\ -\frac{1}{2}\left(V - \frac{1}{2}v^2\varepsilon_2 Z^{-1}\right), & v' = 0 \end{cases} \end{aligned}$$

where

$$\vartheta(X) \equiv e^{-2\alpha\int^X \frac{v}{v'}}$$

- This class of solutions is compatible with Lifshitz asymptotics provided

$$\epsilon = z + \frac{\mu}{2}\left(\xi \pm \sqrt{\xi^2 - 4\alpha}\right), \quad \xi^2 \geq 4\alpha$$

  and

$$-\mu\xi \geq z + \frac{\alpha\mu^2}{z}$$

- Given this explicit solution for $U(X,Y)$ the first order flow equations can be integrated to obtain a multi-parameter family of solutions – cf. BPS solutions

Concluding remarks

- General recursive algorithm for solving the radial Hamilton-Jacobi equation for an Einstein-Proca-scalar theory with arbitrary scalar couplings
- Lifshitz and hvLf asymptotics can be imposed covariantly as second class constraints – they correspond to turning off certain irrelevant operators (in agreement with the results of Ross in the vielbein formalism)
- Work in progress to reproduce the results of our algorithm for Lifshitz boundary conditions using the Dirac algorithm for second class constraints
- Work in progress to deduce the asymptotic expansions and the full dictionary for a number of concrete examples