

A Monotonicity Theorem for Two-dimensional Boundaries and Defects

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THE ROYAL
SOCIETY

University of Oxford
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Based on

1509.02160

with

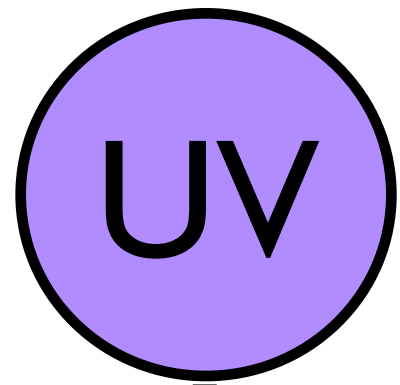
Kristan Jensen

(Stony Brook \rightarrow San Francisco State)



Outline:

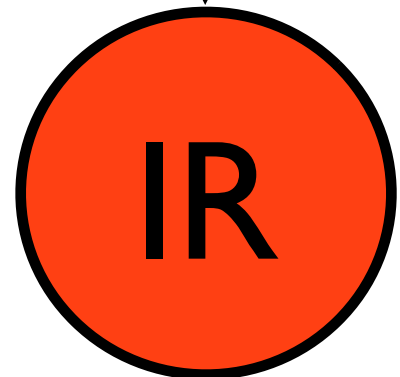
- Review: Monotonicity Theorems
- The Systems
- The Trace Anomaly
- The Proof
- Summary and Outlook



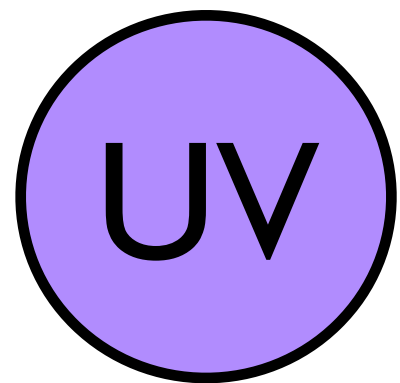
Microscopic/High-energy scales

Intuition:

The “number of degrees of freedom (DOF)”
will **DECREASE**



Macroscopic/Low-energy scales

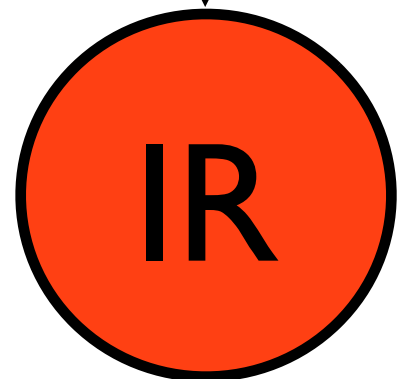


Quantum Field Theory (QFT)

Wilsonian Renormalization Group (RG)

“integrate out” DOF

Below a mass threshold,
integrate out massive DOF



massive DOF
“decouples”

Monotonicity Theorems

Make our intuition precise, for RG flows in QFT

Provide a precise way to count number of DOF

Provide rigorous proof that the number of DOF
DECREASES along RG flow

for any coupling strength!

Place stringent theoretical constraints
on what is possible in RG flows

c-theorem

2-dimensional QFT

Zamolodchikov JETP 43, 12, 565, 1986

F-theorem

3-dimensional QFT

Jafferis, Klebanov, Pufu, and Safdi 1103.1181

Casini and Huerta 1202.5650

Casini, Huerta, Myers, Yale 1506.06195

a-theorem

4-dimensional QFT

Cardy PLB 215 (1988) 749

Jack and Osborn NPB 343 (1990) 647

Komargodski and Schwimmer 1107.3987

g-theorem

2-dimensional CFT with a boundary

Affleck and Ludwig PRL 67 (1991) 161

Friedan and Konechny hep-th/0312197

The c-theorem

A.B. Zamolodchikov
JETP Vol. 43 No. 12 p. 565, 1986

for RG flows in

RENORMALIZABLE

EUCLIDEAN

QFTs in $d = 2$

Assumptions

- ① Euclidean Symmetry
(Poincaré symmetry)
- ② Locality
- ③ Reflection Positivity
(Unitarity)

1. Euclidean Symmetry

Non-dynamical background metric $g_{\mu\nu}(x)$

Action functional $S(g_{\mu\nu}, \lambda)$

Coupling constants $\lambda = (\lambda_1, \lambda_2, \dots)$

Generating functional $Z[g_{\mu\nu}, \lambda]$

1. Euclidean Symmetry

Stress-Energy Tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \ln Z[g_{\mu\nu}, \lambda]$$

$$T_{\mu\nu} = T_{\nu\mu}$$

$$g_{\mu\nu} \rightarrow \delta_{\mu\nu}$$

Translational and Rotational Symmetry

$$\partial_{\mu} T_{\mu\nu} = 0$$

1. Euclidean Symmetry

Stress-Energy Tensor

complex coordinates z, \bar{z}

$$[T_{\mu\nu}] = \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ T_{\bar{z}z} & T_{\bar{z}\bar{z}} \end{pmatrix}$$

$$T_{z\bar{z}} = T_{\bar{z}z}$$

$$T_{\mu}^{\mu} = T_{z\bar{z}} = T_{\bar{z}z}$$

2. Locality

$$S(\lambda) = \int d^2 z L(\lambda, z, \bar{z})$$

RG flow triggered by relevant local scalar operator

$$L(\lambda, z, \bar{z}) \rightarrow L(\lambda, z, \bar{z}) + \lambda_{\mathcal{O}} \mathcal{O}(z, \bar{z})$$

$$\Delta_{\mathcal{O}} < 2$$

3. Reflection Positivity

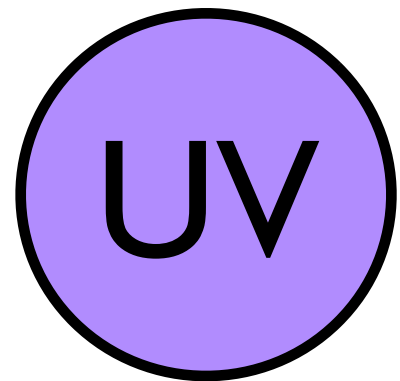
$$||\psi\rangle|^2 = \langle\psi|\psi\rangle \geq 0$$

Reflection Positivity

Two point function of local scalar operator must be non-negative

$$\langle\mathcal{O}^\dagger(x)\mathcal{O}(0)\rangle \geq 0$$

Euclidean “time evolution” preserves norm ≥ 0

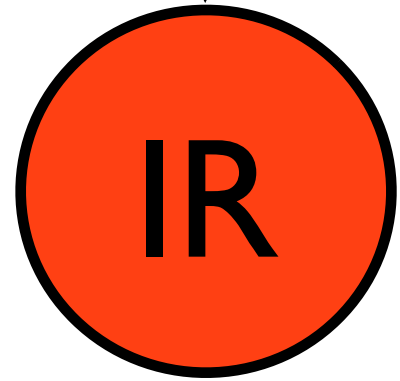


UV CFT

The c-theorem

RG flow between fixed points

Conformal Field Theories
(CFTs)



IR CFT

Conformal Field Theory

Non-dynamical background metric $g_{\mu\nu}(x)$

Conformal Transformation

Diffeomorphism

$$x^\mu \rightarrow x'^\mu(x)$$

such that

$$g_{\mu\nu}(x) \rightarrow e^{2\Omega(x)} g_{\mu\nu}(x)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow e^{2\Omega(x)} g_{\mu\nu}(x)$$

$$T_{\mu}^{\mu} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta\Omega} \ln Z$$

Conformal invariance

$$T_{\mu}^{\mu} = 0$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d > 2$$

Rotations

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$$

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

Dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

$$SO(d+1, 1)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d = 2$$

Conformal Transformation

$$z \rightarrow w(z) \qquad \bar{z} \rightarrow \bar{w}(\bar{z})$$

Conformal Field Theory

$$g_{\mu\nu}(x) \rightarrow \delta_{\mu\nu}$$

$$d = 2$$

Holomorphic and anti-holomorphic
DECOUPLE

$$\partial_{\mu} T_{\mu\nu} = 0$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{zz} = 0$$

$$T_{\mu}^{\mu} = T_{z\bar{z}} = T_{\bar{z}z} = 0$$

$$\partial_z T_{\bar{z}\bar{z}} = 0 \quad \partial_{\bar{z}} T_{zz} = 0$$

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m - n)L_{n+m} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$$SO(d + 1, 1) = SO(3, 1)$$

subgroup

$$L_{\pm 1} \text{ and } L_0$$

$$\bar{L}_{\pm 1} \text{ and } \bar{L}_0$$

Conformal Field Theory

Central Charge

Counts the number of DOF in the CFT

Add a single, free, massless, real scalar field
or single, free, massless Dirac fermion

$$c \rightarrow c + 1$$

Conformal Field Theory

Central Charge

Thermodynamic entropy

Cardy NPB 270 (186) 1986

System size L Temperature T

$$T \gg 1/L$$

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \dots$$

Conformal Field Theory

Central Charge

Entanglement Entropy (EE)

Holzhey, Larsen, Wilczek hep-th/9403108

Calabrese + Cardy hep-th/0405152

Short-distance cutoff a Interval of length ℓ

$$S_{\text{EE}} = \frac{c}{3} \ln \frac{2\ell}{a} + \dots$$

The c-theorem

$$\langle T_{zz}(z, \bar{z}) T_{zz}(0, 0) \rangle = \frac{F(z\bar{z})}{z^4}$$

$$\langle T_{\mu}^{\mu}(z, \bar{z}) T_{zz}(0, 0) \rangle = \frac{G(z\bar{z})}{z^3 \bar{z}}$$

$$\langle T_{\mu}^{\mu}(z, \bar{z}) T_{\mu}^{\mu}(0, 0) \rangle = \frac{H(z\bar{z})}{z^2 \bar{z}^2}$$

Fixed point $\implies F = c/2 \quad G = 0 \quad H = 0$

The c-theorem

$$\langle T_{zz}(z, \bar{z}) T_{zz}(0, 0) \rangle = \frac{F(z\bar{z})}{z^4}$$

$$\langle T_{\mu}^{\mu}(z, \bar{z}) T_{zz}(0, 0) \rangle = \frac{G(z\bar{z})}{z^3 \bar{z}}$$

$$\langle T_{\mu}^{\mu}(z, \bar{z}) T_{\mu}^{\mu}(0, 0) \rangle = \frac{H(z\bar{z})}{z^2 \bar{z}^2}$$

Reflection Positivity $\Rightarrow H \geq 0$

The c-theorem

c-function

$$C \equiv 2F - G - \frac{3}{8}H$$

Fixed point $\Rightarrow C = c$

The c-theorem

c-function

$$C \equiv 2F - G - \frac{3}{8}H$$

$$\partial_\mu T_{\mu\nu} = 0$$

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}\bar{z}} = 0$$

Multiply by $T_{zz}(0,0)$ and $T_{\bar{z}\bar{z}}(0,0)$

and take $\langle \dots \rangle$

The c-theorem

c-function

$$C \equiv 2F - G - \frac{3}{8}H$$

$$r \equiv \sqrt{z\bar{z}}$$

$$r \frac{\partial F}{\partial r} + \frac{1}{4} \left(r \frac{\partial G}{\partial r} - 3G \right) = 0$$

$$r \frac{\partial G}{\partial r} - G + \frac{1}{4} \left(r \frac{\partial H}{\partial r} - 2H \right) = 0$$

eliminate G

The c-theorem

c-function

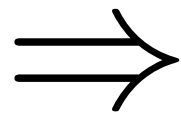
$$C \equiv 2F - G - \frac{3}{8}H$$

$$r \equiv \sqrt{z\bar{z}}$$

$$r \frac{\partial C}{\partial r} = -\frac{3}{2}H \leq 0$$

The c-theorem

$$\frac{\partial C}{\partial r} \leq 0$$



$$C_{UV} \geq C_{IR}$$

Strong form

Weak form

Other Proofs

Holography

Freedman, Gubser, Pilch, Warner hep-th/9904017

Myers and Sinha 1006.1263, 1011.5819

Null Energy Condition

Strong Form

Entanglement Entropy

Casini and Huerta hep-th/0405111

Strong Sub-Additivity

Strong Form

Trace Anomaly Matching

Schwimmer and Theisen 1011.0696

Komargodski and Schwimmer 1107.3987 Komargodski 1112.4538

Reflection Positivity

Weak form

Generalizations?

higher dimensions?

QFTs without Euclidean symmetry?

What if $d_{UV} \neq d_{IR}$?

What if the relevant operator is not a scalar?

What if the fixed points have Lifshitz scaling?

$t \rightarrow \lambda^z t$ $\vec{x} \rightarrow \lambda \vec{x}$ $z =$ dynamical exponent

What if $z_{UV} \neq z_{IR}$?

non-local and/or non-unitary QFTs?

Generalizations?

higher dimensions?

QFTs without Euclidean symmetry?

What if $d_{UV} \neq d_{IR}$?

What if the relevant operator is not a scalar?

What if the fixed points have Lifshitz scaling?

$t \rightarrow \lambda^z t$ $\vec{x} \rightarrow \lambda \vec{x}$ $z =$ dynamical exponent

What if $z_{UV} \neq z_{IR}$?

non-local and/or non-unitary QFTs?

The F-theorem

Jafferis, Klebanov, Pufu, and Safdi 1103.1181

Casini and Huerta 1202.5650

Casini, Huerta, Myers, and Yale 1506.06195

for RG flows in

RENORMALIZABLE

EUCLIDEAN SYMMETRIC

LOCAL

QFTs in $d = 3$

The F-theorem

Put fixed point CFT on S^3

$$F \equiv -\ln Z_{S^3}^{\text{ren.}}$$

$$F_{\text{UV}} \geq F_{\text{IR}}$$

Entanglement Entropy

Casini and Huerta 1202.5650

Casini, Huerta, Myers, and Yale 1506.06195

Strong Sub-Additivity

Strong Form

The F-theorem

Put fixed point CFT on S^3

$$F \equiv -\ln Z_{S^3}^{\text{ren.}}$$

$$F_{\text{UV}} \geq F_{\text{IR}}$$

Holography

Freedman, Gubser, Pilch, Warner hep-th/9904017

Myers and Sinha 1006.1263, 1011.5819

Null energy Condition

Strong Form

The a-theorem

Cardy PLB 215 (1988) 749

Jack and Osborn NPB 343 (1990) 647

Komargodski and Schwimmer 1107.3987

for RG flows in

RENORMALIZABLE

EUCLIDEAN SYMMETRIC

LOCAL

QFTs in $d = 4$

Trace Anomaly

CFT in any d

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_{\mu}^{\mu} = 0$$

Trace Anomaly

CFT in any d

Non-trivial $g_{\mu\nu}$

Quantum Effects
Break Conformal Invariance

$$T_{\mu}^{\mu} \neq 0$$

Trace Anomaly

$$d = 4$$

$$T_{\mu}^{\mu} = a E - c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Euler density

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

Weyl tensor

$$W_{\mu\nu\rho\sigma}$$

central charges a and c

The a-theorem

$$a_{UV} \geq a_{IR}$$

Trace Anomaly Matching

Schwimmer and Theisen 1011.0696

Komargodski and Schwimmer 1107.3987 Komargodski 1112.4538

Reflection Positivity

Weak form

Holography

Freedman, Gubser, Pilch, Warner hep-th/9904017

Myers and Sinha 1006.1263, 1011.5819

Null energy Condition

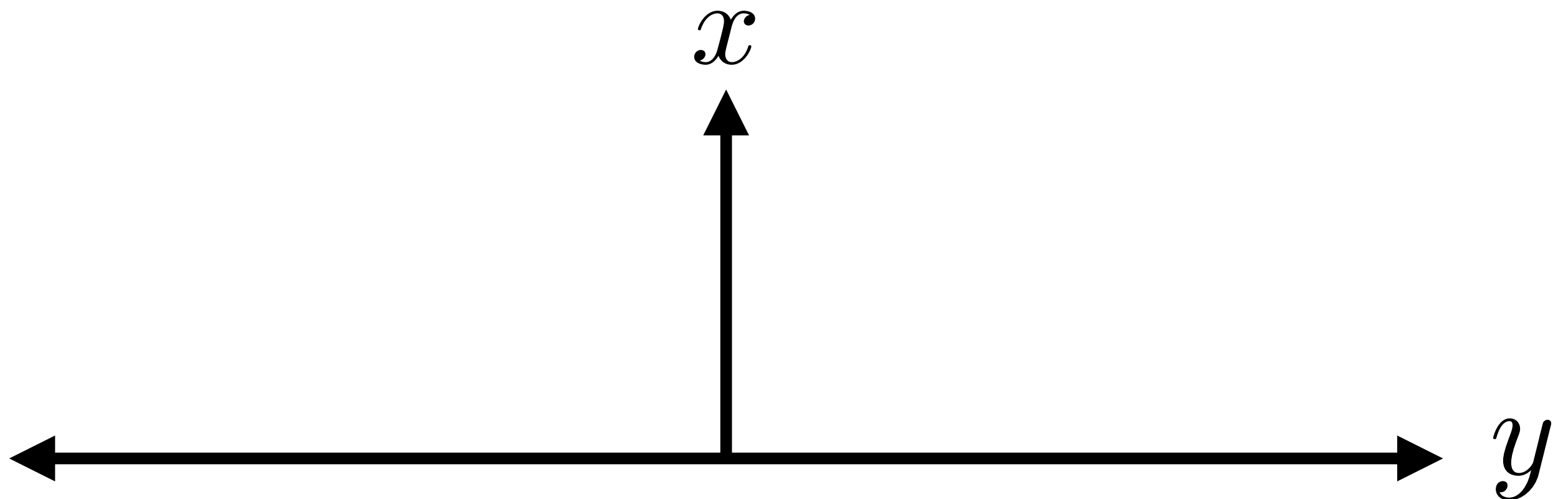
Strong Form

The g-theorem

Affleck and Ludwig PRL 67 (1991) 161

Friedan and Konechny hep-th/0312197

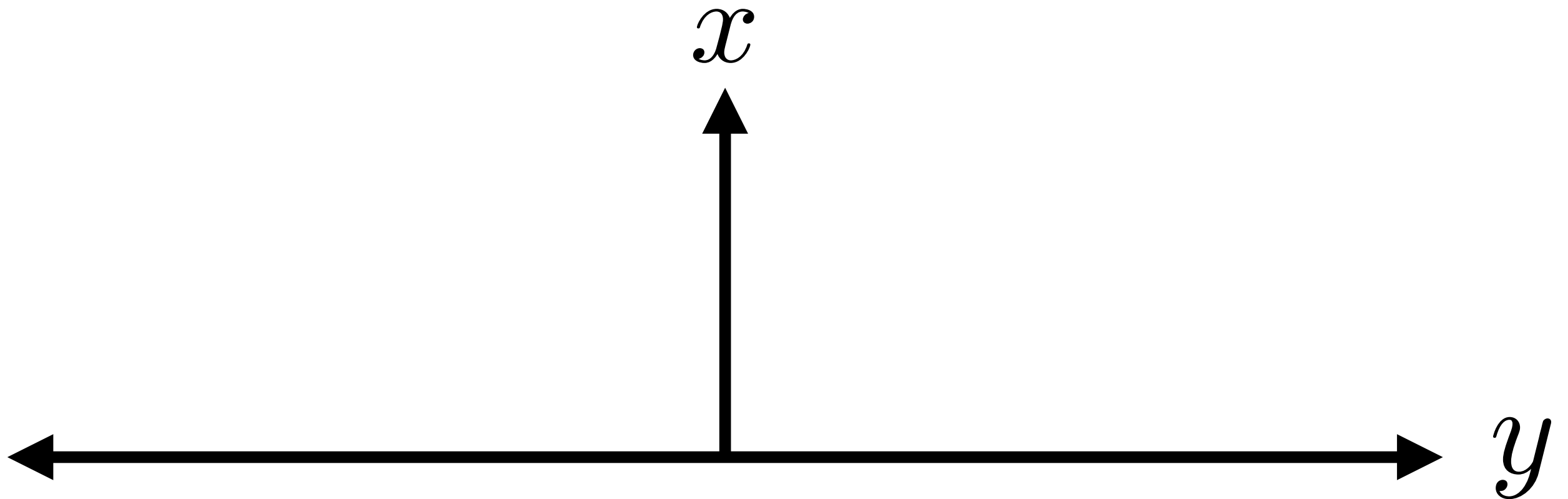
Local, reflection-positive CFT in $d = 2$
on a space with a boundary

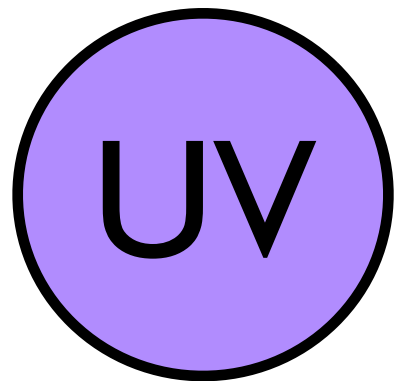


The g-theorem

Conformal boundary conditions

Boundary CFT
(BCFT)





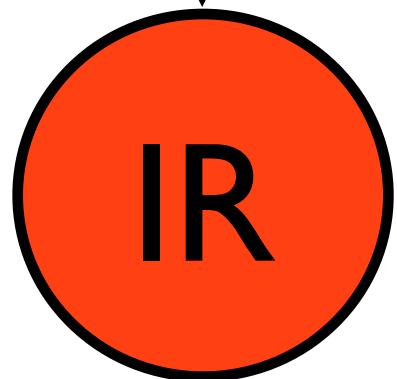
UV BCFT

CFT with boundary condition “ α ”

Boundary RG flow

$$S(\lambda)_{\text{BCFT}}^{\text{UV}} \rightarrow S(\lambda)_{\text{BCFT}}^{\text{UV}} + \int dx dy \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y)$$

$$\Delta_{\mathcal{O}} < 1$$



IR BCFT

CFT with boundary condition “ β ”

Boundary RG flow

$$S(\lambda)_{\text{BCFT}}^{\text{UV}} \rightarrow S(\lambda)_{\text{BCFT}}^{\text{UV}} + \int dx dy \delta(x) \lambda_{\circ} \mathcal{O}(y)$$

Bulk theory remains conformal

$$T_{\mu\nu} = [T_{\mu\nu}]_{\text{bulk}} + \delta(x) [T_{\mu\nu}]_{\partial}$$

Invariance under diffeomorphisms along the boundary

$$[T_{\mu\perp}]_{\partial} = [T_{\perp\mu}]_{\partial} = [T_{\perp\perp}]_{\partial} = 0$$

$$\partial_{\mu} [T_{\mu\nu}]_{\partial} \propto [T_{\perp\perp}]_{\text{bulk}}$$

Boundary RG flow

$$S(\lambda)_{\text{BCFT}}^{\text{UV}} \rightarrow S(\lambda)_{\text{BCFT}}^{\text{UV}} + \int dx dy \delta(x) \lambda_{\circ} \mathcal{O}(y)$$

Bulk theory remains conformal

$$T_{\mu\nu} = [T_{\mu\nu}]_{\text{bulk}} + \delta(x) [T_{\mu\nu}]_{\partial}$$

$$[T_{\mu}^{\mu}]_{\text{bulk}} = 0$$

$$T_{\mu}^{\mu} = \delta(x) [T_{\mu}^{\mu}]_{\partial} \neq 0$$

The g-theorem

Put fixed point BCFT on a hemisphere

$$g_\alpha \equiv Z_\alpha^{\text{ren.}}$$

“Boundary entropy”

$$\ln g_\alpha$$

Counts DOF localized at boundary

The g-theorem

Thermodynamic entropy

Affleck and Ludwig PRL 67 (1991) 161

System size L Temperature T

$$T \gg 1/L$$

$$S_{\text{thermo}} = \frac{\pi}{3} cLT + \ln g_{\alpha} + \dots$$

The g-theorem

Entanglement Entropy (EE)

Calabrese + Cardy hep-th/0405152

Interval including the boundary

$$S_{\text{EE}} = \frac{c}{6} \ln \frac{2\ell}{a} + \ln g_\alpha + \dots$$

$$\ln g_\alpha = S_{\text{EE}}^{\text{BCFT}} - \frac{1}{2} S_{\text{EE}}^{\text{CFT}}$$

The g-theorem

Affleck and Ludwig PRL 67 (1991) 161

Friedan and Konechny hep-th/0312197

Define a thermodynamic g-function

$$S_{\text{thermo}} = \frac{\pi}{3}cLT + S_{\partial}(T)$$

Euclidean symmetry, locality, reflection positivity

$$\frac{\partial S_{\partial}}{\partial T} \geq 0 \quad \Rightarrow \quad g_{\text{UV}} \geq g_{\text{IR}}$$

Strong form

Weak form

Generalizations?

Higher-dimensional g-theorems?

Proposals

Yamaguchi

hep-th/0207171

Takayanagi et al.

1105.5165, 1108.5152, 1205.1573

Estes, Jensen, O'B., Tsatis, Wrase

1403.6475

Gaiotto

1403.8052

Many tests in particular examples

No proofs until now!

GOAL

Prove a g -theorem for
Local, reflection-positive BCFT in $d = 3$

Proposals

Nozaki, Takayanagi, Ugajin	1205.1573
Estes, Jensen, O'B., Tsatis, Wrase	1403.6475

Trace Anomaly Matching

Schwimmer and Theisen	1011.0696
Komargodski and Schwimmer	1107.3987
Komargodski	1112.4538

Examples

Graphene with a boundary

Critical Ising model in $d = 3$ with a boundary

M-theory: M2-branes with a boundary

String theory: various brane intersections

Holographic BCFTs

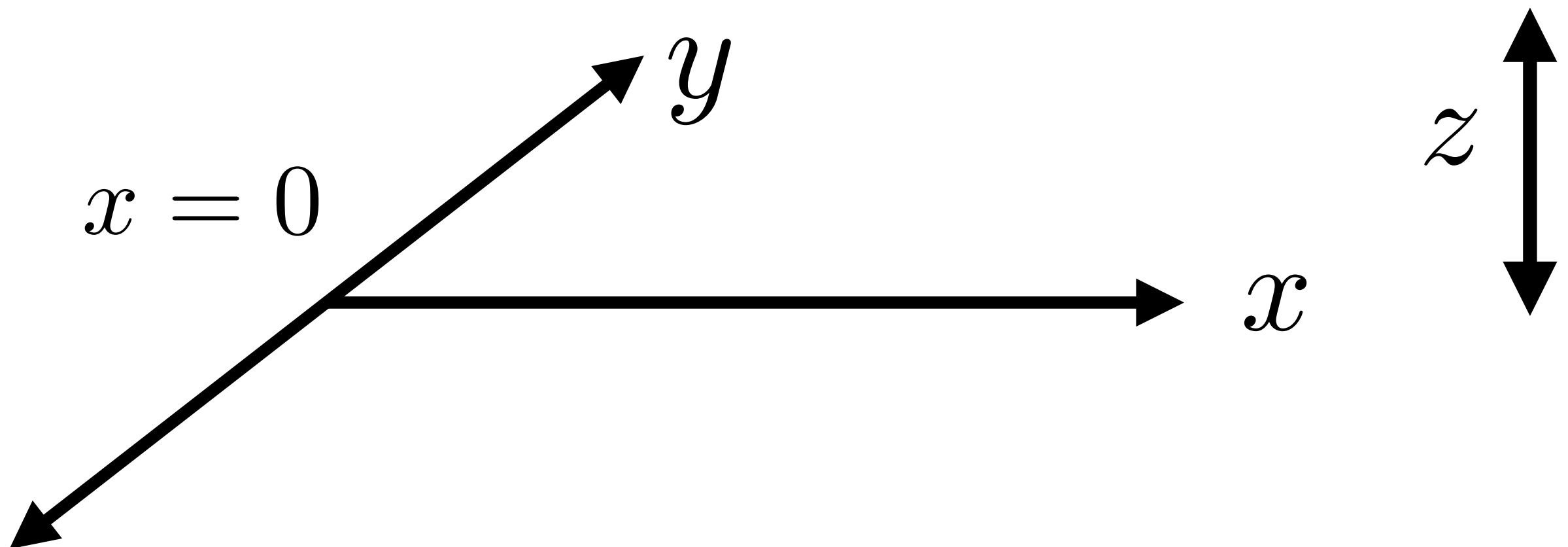
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The Systems

BCFT in $d = 3$

With a planar boundary



The Systems

$$SO(d + 1, 1) = SO(4, 1)$$

Rotations

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$$

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

Dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

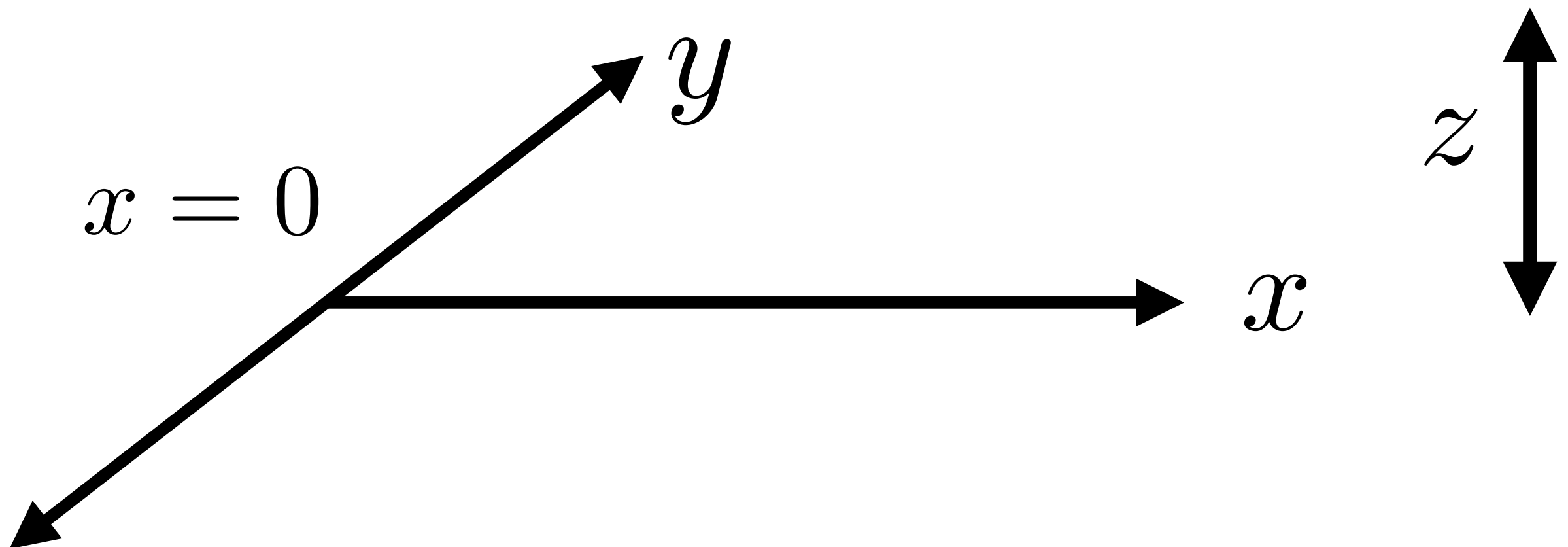
Broken to subgroup that preserves $x = 0$

The Systems

Rotations

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$$

Broken to rotations in (y, z)

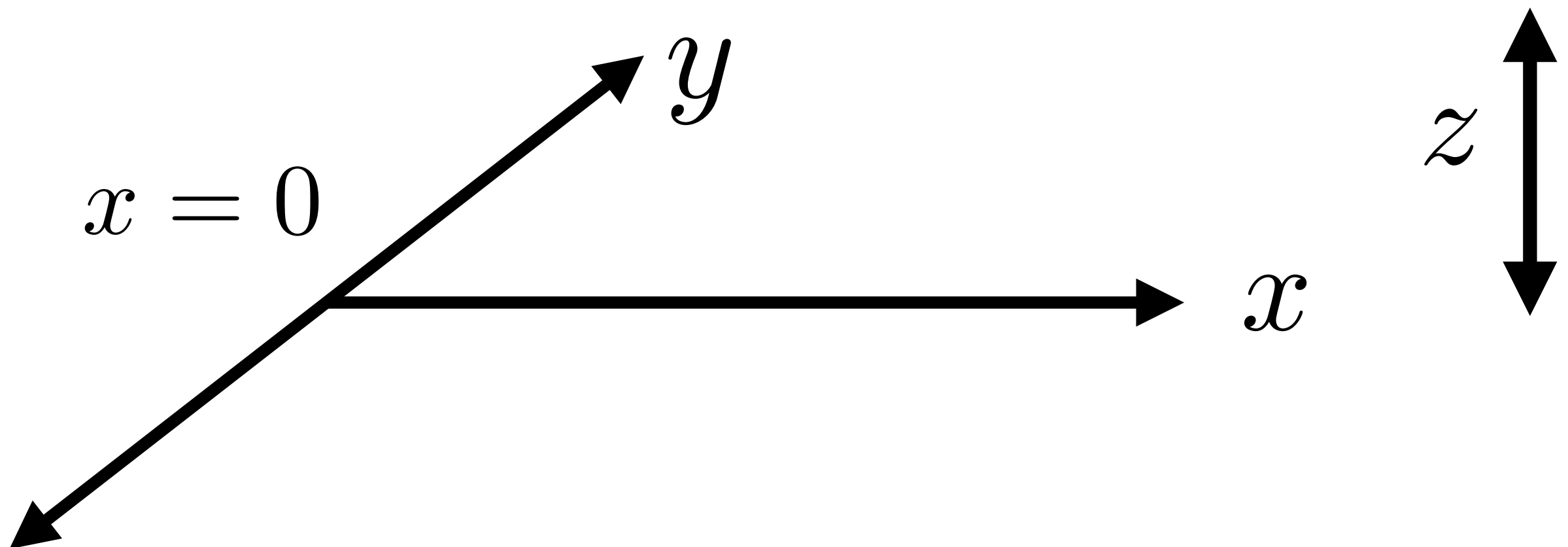


The Systems

Translations

$$x^\mu \rightarrow x^\mu + c^\mu$$

Broken to translations along (y, z)

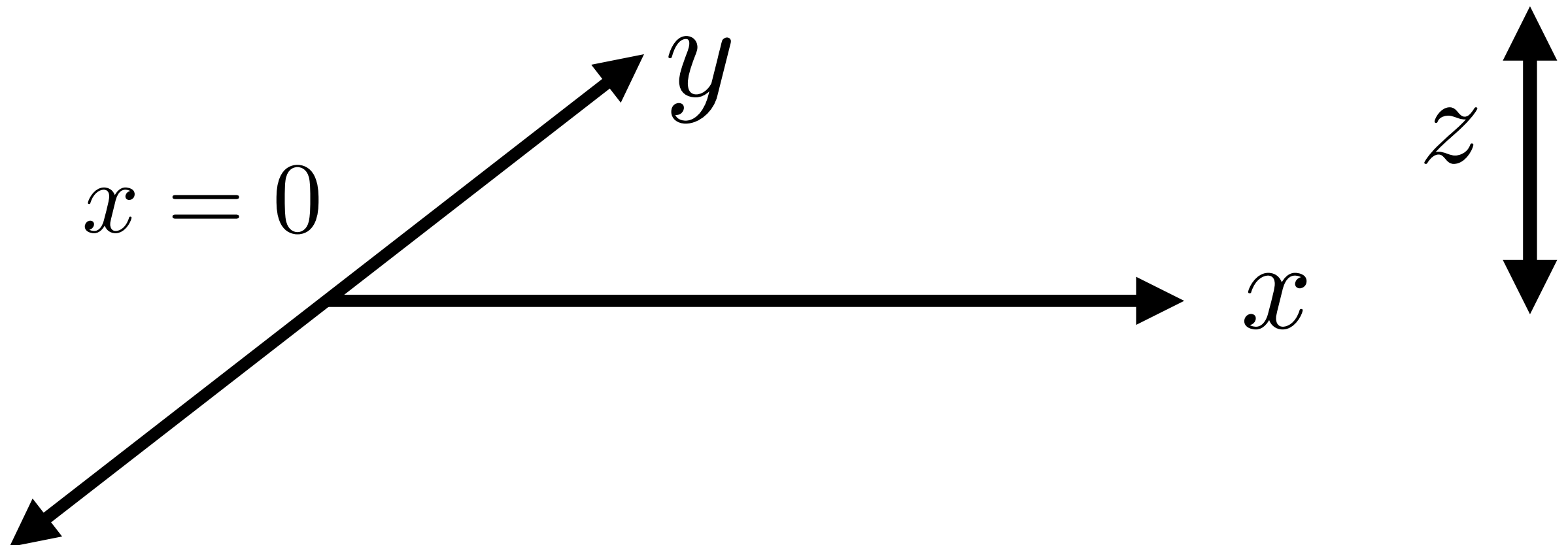


The Systems

Dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

Unbroken

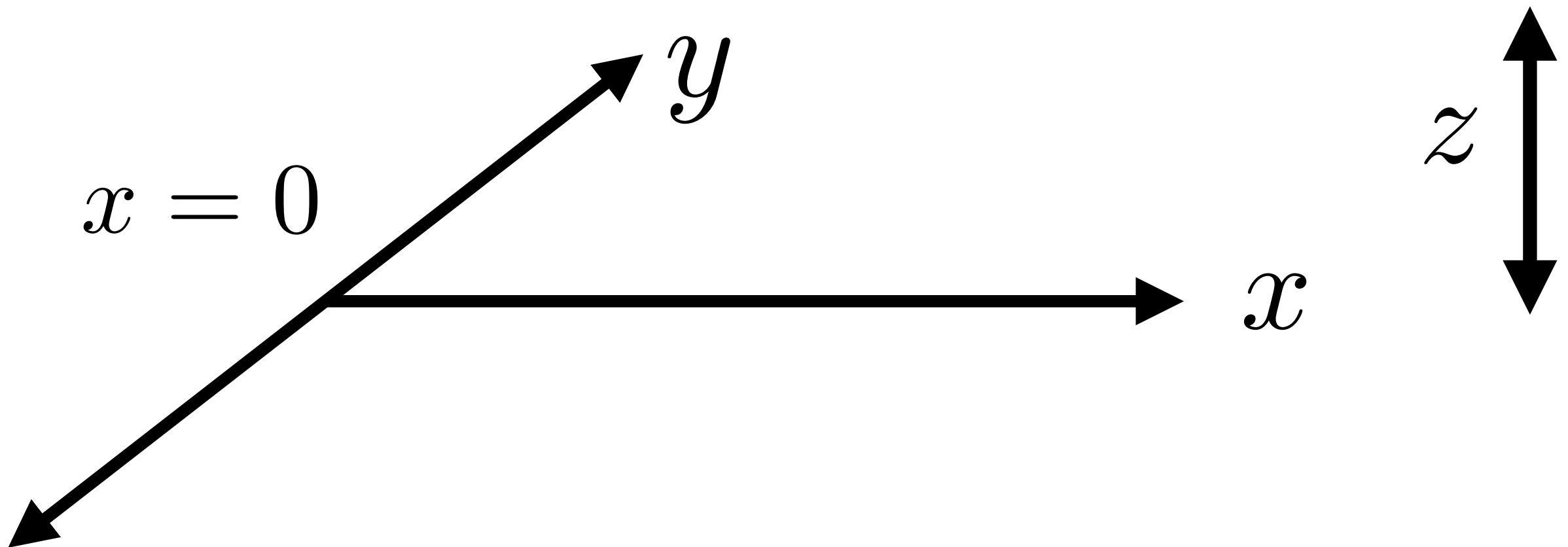


The Systems

Special Conformal

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

Broken to $b^x = 0$

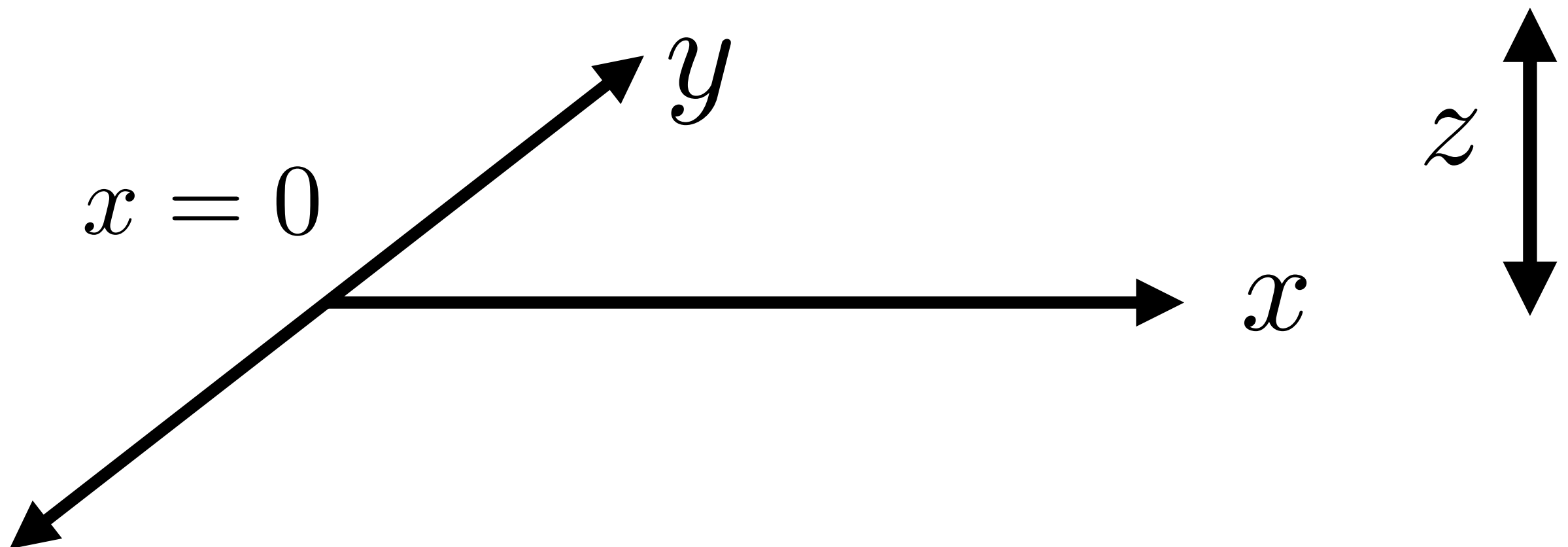


The Systems

$$SO(d + 1, 1) \rightarrow SO(d, 1)$$

$$SO(4, 1) \rightarrow SO(3, 1)$$

NOT the full $d = 2$ conformal group



The Systems

Boundary RG Flows

$$S(\lambda)_{\text{BCFT}}^{\text{UV}} \rightarrow S(\lambda)_{\text{BCFT}}^{\text{UV}} + \int dx dy dz \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

scalar \mathcal{O} with $\Delta_{\mathcal{O}} < 2$

$$T_{\mu\nu} = [T_{\mu\nu}]_{\text{bulk}} + \delta(x) [T_{\mu\nu}]_{\partial}$$

Invariance under diffeomorphisms along the boundary

$$[T_{\mu x}]_{\partial} = [T_{x\mu}]_{\partial} = [T_{xx}]_{\partial} = 0$$

$$\partial_{\mu} [T_{\mu\nu}]_{\partial} \propto [T_{xx}]_{\text{bulk}}$$

The Systems

Boundary RG Flows

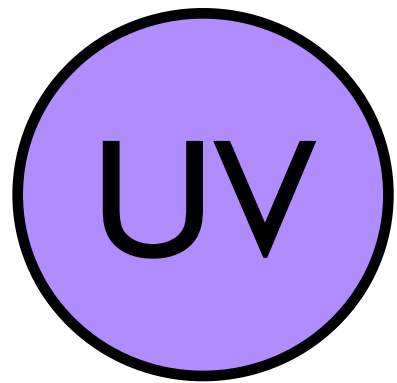
$$S(\lambda)_{\text{BCFT}}^{\text{UV}} \rightarrow S(\lambda)_{\text{BCFT}}^{\text{UV}} + \int dx dy dz \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

scalar \mathcal{O} with $\Delta_{\mathcal{O}} < 2$

$$T_{\mu\nu} = [T_{\mu\nu}]_{\text{bulk}} + \delta(x) [T_{\mu\nu}]_{\partial}$$

Bulk theory remains conformal

$$[T_{\mu}^{\mu}]_{\text{bulk}} = 0 \quad T_{\mu}^{\mu} = \delta(x) [T_{\mu}^{\mu}]_{\partial} \neq 0$$



UV BCFT

Single free, massless, real scalar
Neumann B.C.

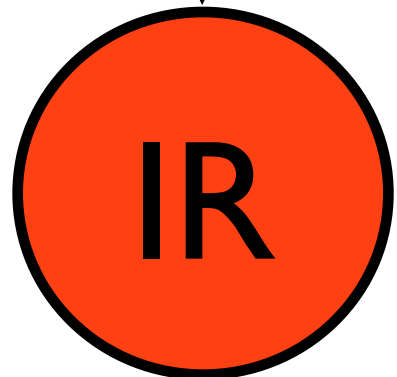
Boundary RG Flow

$$S_{\text{BCFT}}^{\text{UV}} \rightarrow S_{\text{BCFT}}^{\text{UV}} + \int d^3x \delta(x) m^2 \Phi^2(\vec{x})$$

$$\Delta_{\Phi^2} = 1$$

IR BCFT

Single free, massless, real scalar
Dirichlet B.C.



The Systems

Boundary RG Flows

$$S(\lambda)_{\text{BCFT}}^{\text{UV}} \rightarrow S(\lambda)_{\text{BCFT}}^{\text{UV}} + \int dx dy dz \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

Trace Anomaly Matching

Schwimmer and Theisen 1011.0696

Komargodski and Schwimmer 1107.3987

Komargodski 1112.4538

Reflection Positivity

Weak form

Outline:

- Review: Monotonicity Theorems
- The Systems
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Trace Anomaly

CFT in any d

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_{\mu}^{\mu} = 0$$

Trace Anomaly

CFT in any d

Non-trivial $g_{\mu\nu}$

Quantum Effects
Break Conformal Invariance

$$T_{\mu}^{\mu} \neq 0$$

Trace Anomaly

What is the general form of T_{μ}^{μ} ?

Step #1

Write down all curvature invariants
built from $g_{\mu\nu}$
with the correct dimension

$$d = 4$$

$$T_{\mu}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + \dots$$

Trace Anomaly

What is the general form of T_{μ}^{μ} ?

Step #2

Wess-Zumino consistency

$$g_{\mu\nu} \rightarrow e^{2\Omega_1} e^{2\Omega_2} g_{\mu\nu} = g_{\mu\nu} \rightarrow e^{2\Omega_2} e^{2\Omega_1} g_{\mu\nu}$$

Fixes some coefficients

$$T_{\mu}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + \dots$$

Trace Anomaly

What is the general form of T_{μ}^{μ} ?

Step #3

Add local counterterms to $S(g_{\mu\nu}, \lambda)$

Determine how they enter T_{μ}^{μ}

Fixes more coefficients

$$T_{\mu}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + \dots$$

Trace Anomaly

CFT in any d

$$d \text{ odd} \quad T_{\mu}^{\mu} = 0$$

$$d \text{ even} \quad T_{\mu}^{\mu} \neq 0$$

$$d = 2 \quad T_{\mu}^{\mu} = \frac{c}{24\pi} R$$

Trace Anomaly

$$d = 4$$

$$T_{\mu}^{\mu} = a E - c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Euler density

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

Weyl tensor

$$W_{\mu\nu\rho\sigma}$$

central charges a and c

Trace Anomaly

$$g_{\mu\nu}(x) \rightarrow e^{2\Omega(x)} g_{\mu\nu}(x)$$

$$\int d^d x \sqrt{g} T_{\mu}^{\mu} \text{ is invariant}$$

$$T_{\mu}^{\mu} = a E - c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Type A

$$\sqrt{g} E$$

Changes by a total derivative

Type B

$$\sqrt{g} W^2$$

Invariant

Trace Anomaly

BCFT in $d = 3$

$$T_{\mu}^{\mu} = [T_{\mu}^{\mu}]_{\text{bulk}} + \delta(x) [T_{\mu}^{\mu}]_{\partial}$$

$$[T_{\mu}^{\mu}]_{\text{bulk}} = 0$$

What is the general form of $[T_{\mu}^{\mu}]_{\partial}$?

Geometry of Submanifolds

“worldsheet”

$$\sigma^1, \sigma^2$$

“target space”

$$x^\mu$$

Embedding

$$x^\mu(\sigma^a)$$

Induced metric

$$\hat{g}_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}$$

$$\hat{R}_{abcd}$$

$$\hat{R}_{ab}$$

$$\hat{R}$$

Geometry of Submanifolds

Extrinsic Curvature
“Second Fundamental Form”

Gaussian Normal Coordinates

$$g_{\mu\nu} dx^\mu dx^\nu = dx^2 + \hat{g}_{ab}(x, \sigma^a) d\sigma^a d\sigma^b$$

$$K_{ab} = \frac{1}{2} \partial_x \hat{g}_{ab}(x, \sigma)$$

Mean curvature

$$K \equiv \hat{g}^{ab} K_{ab}$$

Trace Anomaly

What is the general form of $[T_{\mu}^{\mu}]_{\partial}$?

Schwimmer + Theisen 0802.1017

See also:

Berenstein, Corrado, Fischler, Maldacena hep-th/9809188

Graham + Witten hep-th/9901021

Henningson + Skenderis hep-th/9905163

Gustavsson hep-th/0310037, 0404150

Asnin 0801.1469

Trace Anomaly

$$[T_{\mu}^{\mu}]_{\partial} = c_1 \hat{R} + c_2 \left(K_{ab} K^{ab} - \frac{1}{2} K^2 \right)$$

Boundary “central charges”

c_1 and c_2

Trace Anomaly

$$g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu}$$

$$[T_{\mu}^{\mu}]_{\partial} = c_1 \hat{R} + c_2 \left(K_{ab} K^{ab} - \frac{1}{2} K^2 \right)$$

Type A

Type B

$$\sqrt{\hat{g}} \hat{R} \rightarrow \sqrt{\hat{g}} \left[\hat{R} - 2\nabla^2 \Omega \right]$$


$$\sqrt{\hat{g}} \left(K_{ab} K^{ab} - \frac{1}{2} K^2 \right)$$

Changes by a total derivative

Invariant

Trace Anomaly

$$[T_{\mu}^{\mu}]_{\partial} = c_1 \hat{R} + c_2 \left(K_{ab} K^{ab} - \frac{1}{2} K^2 \right)$$

$$\int d^2 \sigma \sqrt{\hat{g}} \left(K_{ab} K^{ab} - \frac{1}{2} K^2 \right)$$


Physics

“Rigid String” Action

Polyakov, NPB 268 (1986) 406

Mathematics

Willmore functional

Trace Anomaly

GOAL

$$[T_{\mu}^{\mu}]_{\partial} = c_1 \hat{R} + c_2 (K_{ab} K^{ab} - \frac{1}{2} K^2)$$

Boundary RG Flow

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

Nozaki, Takayanagi, Ugajin

1205.1573

Estes, Jensen, O'B., Tsatis, Wrase

1403.6475

Trace Anomaly

GOAL

$$[T_{\mu}^{\mu}]_{\partial} = c_1 \hat{R} + c_2 \left(K_{ab} K^{ab} - \frac{1}{2} K^2 \right)$$

$$\partial_{\mu} [T_{\mu\nu}]_{\partial} \propto [T_{xx}]_{\text{bulk}}$$

We can't just copy Zamolodchikov's proof!

Trace Anomaly

GOAL

$$[T_{\mu}^{\mu}]_{\partial} = c_1 \hat{R} + c_2 (K_{ab} K^{ab} - \frac{1}{2} K^2)$$

Trace Anomaly Matching

Schwimmer and Theisen 1011.0696

Komargodski and Schwimmer 1107.3987

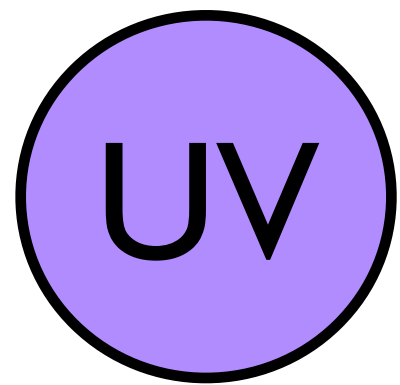
Komargodski 1112.4538

Reflection Positivity

Weak form

Outline:

- Review: Monotonicity Theorems
- The Systems
- The Trace Anomaly
- The Proof
- Summary and Outlook



The Proof

Trace Anomaly Matching

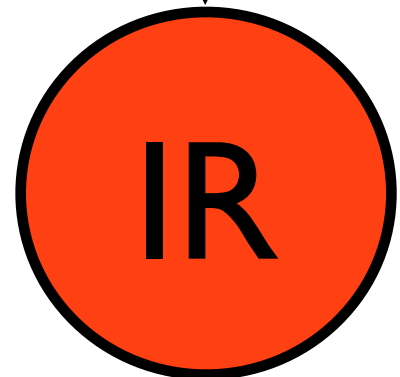
Schwimmer and Theisen 1011.0696

Komargodski and Schwimmer 1107.3987

Komargodski 1112.4538

local, reflection-positive QFT in any d

RG flow between fixed point CFTs



Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

$$g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu} \quad \tau \rightarrow \tau + \Omega$$

Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

$$\lambda_{\mathcal{O}} \mathcal{O}(x) \rightarrow e^{(\Delta_{\mathcal{O}} - d)\tau(x)} \lambda_{\mathcal{O}} \mathcal{O}(x)$$

Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

$$S(\lambda)_\tau = \int d^d x \sqrt{g} L(\lambda, \vec{x})_\tau$$

$$= \int d^d x \sqrt{g} \left[L(\lambda, \vec{x})_{\tau=0} + \tau [T_\mu^\mu]_{\tau=0} + \mathcal{O}(\tau^2) \right]$$

Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

Trace Anomaly Matching

d even

Dilaton

Integrate out massive DOF

Obtain effective action

$$S_{\text{eff}} \equiv -\ln Z$$

Regular and local in \mathcal{T}

Expand in \mathcal{T}

$$S_{\text{eff}} = S_{\text{eff}}^{\mathcal{T}=0} + S_{\text{eff}}^{\mathcal{T}}$$

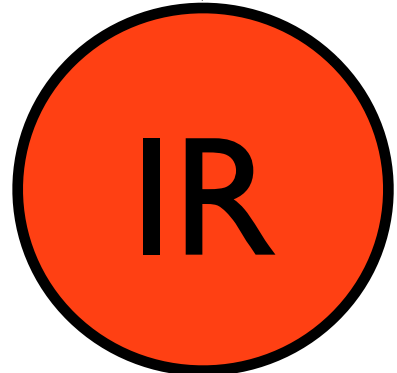
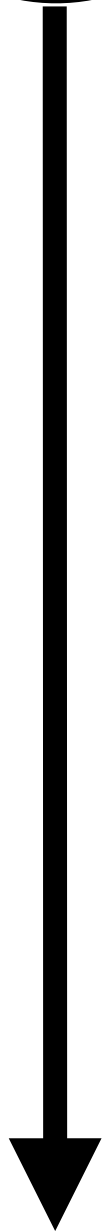
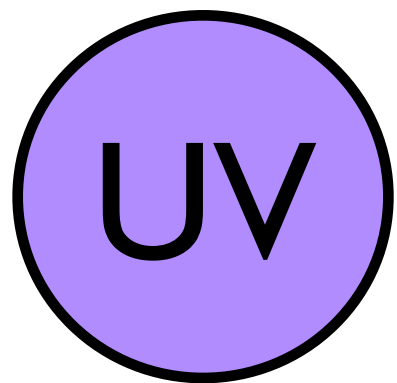
Dilaton

$$g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu} \quad \tau \rightarrow \tau + \Omega$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - \frac{\delta}{\delta \Omega} \ln Z$$

Expand in \mathcal{T}

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0} + \frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega}$$



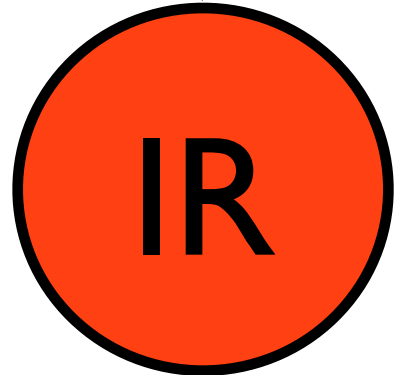
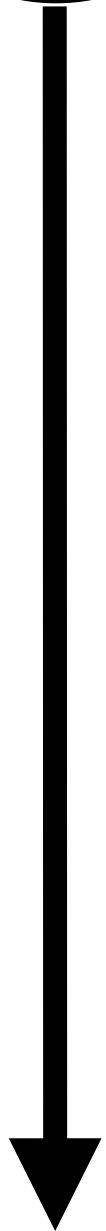
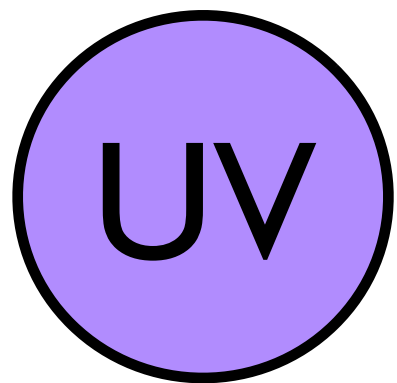
$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - [T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} = 0$$

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$$\tau = 0$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - [T_{\mu}^{\mu}]_{\tau=0} \neq 0$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} = 0$$



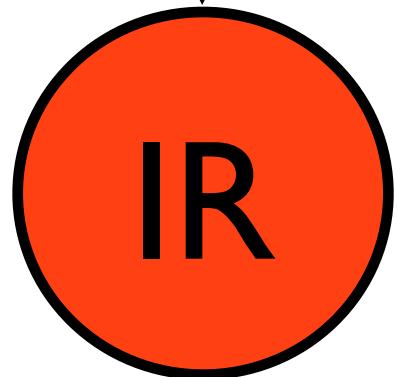
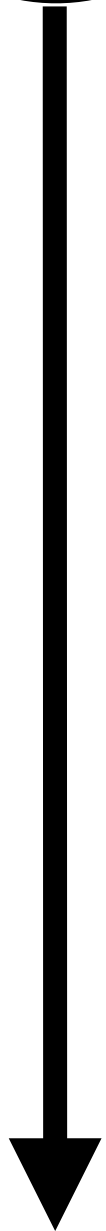
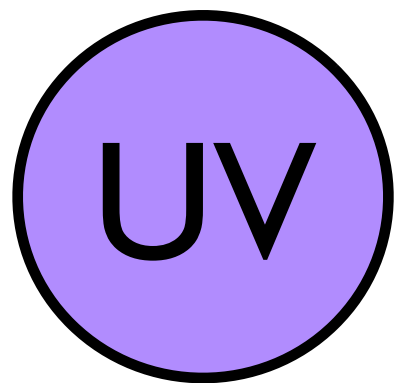
$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - [T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} = 0$$

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$$\tau \neq 0$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - [T_{\mu}^{\mu}]_{\tau=0} + \frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = 0$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = - [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} = 0$$



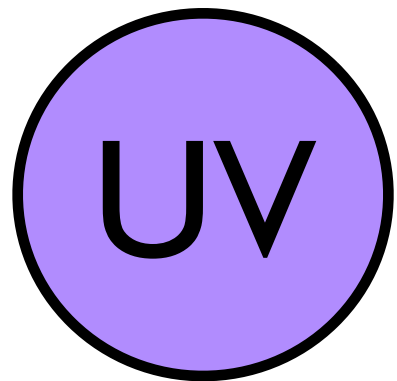
$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} \neq 0$$

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

$$\tau = 0$$

$$[T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} \neq [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}}$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} \neq 0$$

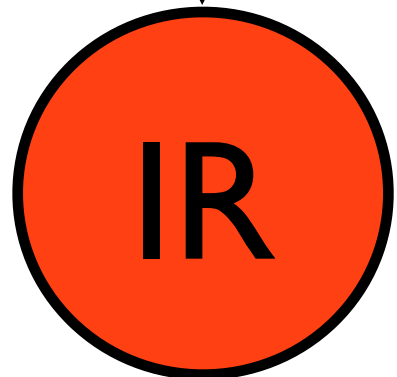


$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} \neq 0$$

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

$$\tau \neq 0$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0} + \frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} \neq 0$$



$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} + \frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = -\sqrt{g} [T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} \neq 0$$

Dilaton

Trace Anomaly Matching

Schwimmer and Theisen 1011.0696

At the IR fixed point, S_{eff}^{τ} must produce

$$\frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = -\sqrt{g} \left[\left[T_{\mu}^{\mu} \right]_{\tau=0}^{\text{UV}} - \left[T_{\mu}^{\mu} \right]_{\tau=0}^{\text{IR}} \right]$$

Dilaton

Boundary RG Flow

$$S_{\text{BCFT}}^{\text{UV}} \rightarrow S_{\text{BCFT}}^{\text{UV}} + \int dx dy dz \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

$$\lambda_{\mathcal{O}} \mathcal{O}(y, z) \rightarrow e^{(\Delta_{\mathcal{O}} - 2)\tau(y, z)} \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

\mathcal{T} is localized at the boundary,
and depends only on boundary coordinates!

UV

$$[T_{\mu}^{\mu}]_{\partial}^{\text{UV}} = c_1^{\text{UV}} \hat{R} + c_2^{\text{UV}} (K_{ab} K^{ab} - \frac{1}{2} K^2)$$

$$T_{\mu}^{\mu} = [T_{\mu}^{\mu}]_{\text{bulk}} + \delta(x) [T_{\mu}^{\mu}]_{\partial}$$

Boundary RG Flow

$$S_{\text{BCFT}}^{\text{UV}} \rightarrow S_{\text{BCFT}}^{\text{UV}} + \int dx dy dz \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

$$[T_{\mu}^{\mu}]_{\text{bulk}} = 0$$

IR

$$[T_{\mu}^{\mu}]_{\partial}^{\text{IR}} = c_1^{\text{IR}} \hat{R} + c_2^{\text{IR}} (K_{ab} K^{ab} - \frac{1}{2} K^2)$$

The Proof

Strategy

Komargodski and Schwimmer 1107.3987

Komargodski 1112.4538

In $S_{\text{eff}}^{\mathcal{T}}$ write the $\mathcal{T} \nabla^2 \mathcal{T}$ term in TWO WAYS

①

trace anomaly matching

coefficient $\propto c_1^{\text{UV}} - c_1^{\text{IR}}$

②

Reflection Positivity \implies coefficient ≥ 0

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

The Proof

$$S_{\text{eff}}^{\tau} = -\int d^3x \delta(x) \sqrt{g} \tau \left[(c_1^{\text{UV}} - c_1^{\text{IR}}) \hat{R} + (c_2^{\text{UV}} - c_2^{\text{IR}}) (K_{ab} K^{ab} - \frac{1}{2} K^2) \right] + \mathcal{O}(\tau^2)$$

$$g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu} \quad \tau \rightarrow \tau + \Omega$$

$$\sqrt{\hat{g}} \hat{R} \rightarrow \sqrt{\hat{g}} \left[\hat{R} - 2\nabla^2 \Omega \right]$$

$$\frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = -\sqrt{g} \left[[T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} - [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} \right] + \sqrt{g} (c_1^{\text{UV}} - c_1^{\text{IR}}) 2\nabla^2 \tau$$

The Proof

$$S_{\text{eff}}^{\tau} = - \int d^3x \delta(x) \sqrt{g} \tau \left[(c_1^{\text{UV}} - c_1^{\text{IR}}) \hat{R} + (c_2^{\text{UV}} - c_2^{\text{IR}}) (K_{ab} K^{ab} - \frac{1}{2} K^2) \right]$$
$$- \int d^3x \delta(x) \sqrt{g} (c_1^{\text{UV}} - c_1^{\text{IR}}) \tau \nabla^2 \tau + \dots$$

$$\frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = - \sqrt{g} \left[[T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} - [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} \right]$$

The Proof

$$S_{\text{eff}}^{\tau} = -\int d^3x \delta(x) \sqrt{g} \tau \left[(c_1^{\text{UV}} - c_1^{\text{IR}}) \hat{R} + (c_2^{\text{UV}} - c_2^{\text{IR}}) (K_{ab} K^{ab} - \frac{1}{2} K^2) \right]$$
$$- \int d^3x \delta(x) \sqrt{g} (c_1^{\text{UV}} - c_1^{\text{IR}}) \tau \nabla^2 \tau + \dots$$



Survives the flat-space limit

$$g_{\mu\nu} \rightarrow \delta_{\mu\nu}$$

$$\frac{\delta S_{\text{eff}}^{\tau}}{\delta \Omega} = -\sqrt{g} \left[[T_{\mu}^{\mu}]_{\tau=0}^{\text{UV}} - [T_{\mu}^{\mu}]_{\tau=0}^{\text{IR}} \right]$$

The Proof

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Another form for the two-derivative term

$$Z = \int \mathcal{D}[\text{fields}] e^{-S(\lambda)_\tau}$$

$$S_\tau = \int d^3x \sqrt{g} \left[L(\lambda, \vec{x})_{\tau=0} + \tau [T_\mu^\mu]_{\tau=0} + \mathcal{O}(\tau^2) \right]$$

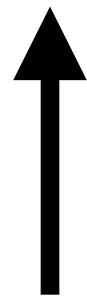
$$S_{\text{eff}} = -\ln Z = -\left\langle e^{-\int d^3x \tau(x) T_\mu^\mu(x) + \dots} \right\rangle_{\tau=0}$$

The Proof

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Another form for the two-derivative term

$$\begin{aligned} \langle e^{-\int d^3x \tau(x) T_{\mu}^{\mu}(x) + \dots} \rangle &= 1 - \int d^3x \tau(x) \langle T_{\mu}^{\mu}(x) \rangle \\ &+ \frac{1}{2} \int d^3x d^3y \tau(x) \tau(y) \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle + \dots \end{aligned}$$



Taylor expand about x

The Proof

Another form for the two-derivative term

$$\frac{1}{2} \int d^3x d^3y \tau(x) \tau(y) \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle$$

$$\supset \frac{1}{4} \int d^3x \tau(x) \partial_{\rho} \partial_{\sigma} \tau(x) \left[\int d^3y (y-x)^{\rho} (y-x)^{\sigma} \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle \right]$$

$$T_{\mu}^{\mu} = [T_{\mu}^{\mu}]_{\text{bulk}} + \delta(x) [T_{\mu}^{\mu}]_{\partial}$$

$$[T_{\mu}^{\mu}]_{\text{bulk}} = 0$$

The Proof

Another form for the two-derivative term

$$\frac{1}{2} \int d^3x d^3y \tau(x) \tau(y) \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle$$

$$\supset \frac{1}{4} \int d^3x \tau(x) \partial_{\rho} \partial_{\sigma} \tau(x) \left[\int d^3y (y-x)^{\rho} (y-x)^{\sigma} \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle \right]$$

$$\langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle = \delta(x) \delta(y) \langle [T_{\mu}^{\mu}(x)]_{\partial} [T_{\mu}^{\mu}(y)]_{\partial} \rangle$$

The Proof

Another form for the two-derivative term

$$\frac{1}{2} \int d^3x d^3y \tau(x) \tau(y) \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle$$

$$\supset \frac{1}{4} \int d^3x \tau(x) \partial_{\rho} \partial_{\sigma} \tau(x) \left[\int d^3y (y-x)^{\rho} (y-x)^{\sigma} \langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle \right]$$

Reflection Positivity

$$\langle T_{\mu}^{\mu}(x) T_{\mu}^{\mu}(y) \rangle = \delta(x) \delta(y) \langle [T_{\mu}^{\mu}(x)]_{\partial} [T_{\mu}^{\mu}(y)]_{\partial} \rangle \geq 0$$

The Proof

Another form for the two-derivative term

$$\frac{1}{2} \int d^3x d^3y \tau(x) \tau(y) \langle T_\mu^\mu(x) T_\mu^\mu(y) \rangle$$

$$\supset \frac{1}{4} \int d^3x \tau(x) \partial_\rho \partial_\sigma \tau(x) \left[\int d^3y (y-x)^\rho (y-x)^\sigma \langle T_\mu^\mu(x) T_\mu^\mu(y) \rangle \right]$$

Translation invariance along the boundary

$$\begin{aligned} & \delta(x) \int d^3y \delta(y) (y-x)^\rho (y-x)^\sigma \langle [T_\mu^\mu(x)]_\partial [T_\mu^\mu(y)]_\partial \rangle \\ &= \delta(x) \frac{1}{2} \delta^{\rho\sigma} \int d^3y \delta(y) y^2 \langle [T_\mu^\mu(0)]_\partial [T_\mu^\mu(y)]_\partial \rangle \end{aligned}$$

The Proof

Another form for the two-derivative term

$$\frac{1}{2} \int d^3x d^3y \tau(x) \tau(y) \langle T_\mu^\mu(x) T_\mu^\mu(y) \rangle$$

$$\supset \frac{1}{4} \int d^3x \tau(x) \partial_\rho \partial_\sigma \tau(x) \left[\int d^3y (y-x)^\rho (y-x)^\sigma \langle T_\mu^\mu(x) T_\mu^\mu(y) \rangle \right]$$

Reflection Positivity

$$\begin{aligned} & \delta(x) \int d^3y \delta(y) (y-x)^\rho (y-x)^\sigma \langle [T_\mu^\mu(x)]_\partial [T_\mu^\mu(y)]_\partial \rangle \\ &= \delta(x) \frac{1}{2} \delta^{\rho\sigma} \int d^3y \delta(y) y^2 \langle [T_\mu^\mu(0)]_\partial [T_\mu^\mu(y)]_\partial \rangle \geq 0 \end{aligned}$$

The Proof

$$\int d^3x \delta(x) \tau \nabla^2 \tau (c_1^{\text{UV}} - c_1^{\text{IR}})$$
$$= \int d^3x \delta(x) \tau \nabla^2 \tau \left[\frac{1}{8} \int d^3y \delta(y) y^2 \langle [T_{\mu}^{\mu}(0)]_{\partial} [T_{\mu}^{\mu}(y)]_{\partial} \rangle \right]$$
$$c_1^{\text{UV}} - c_1^{\text{IR}} = \left[\frac{1}{8} \int d^3y \delta(y) y^2 \langle [T_{\mu}^{\mu}(0)]_{\partial} [T_{\mu}^{\mu}(y)]_{\partial} \rangle \right]$$

The Proof

$$\int d^3x \delta(x) \tau \nabla^2 \tau (c_1^{\text{UV}} - c_1^{\text{IR}})$$
$$= \int d^3x \delta(x) \tau \nabla^2 \tau \left[\frac{1}{8} \int d^3y \delta(y) y^2 \langle [T_{\mu}^{\mu}(0)]_{\partial} [T_{\mu}^{\mu}(y)]_{\partial} \rangle \right]$$
$$c_1^{\text{UV}} - c_1^{\text{IR}} = \left[\frac{1}{8} \int d^3y \delta(y) y^2 \langle [T_{\mu}^{\mu}(0)]_{\partial} [T_{\mu}^{\mu}(y)]_{\partial} \rangle \right] \geq 0$$

Reflection Positivity

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

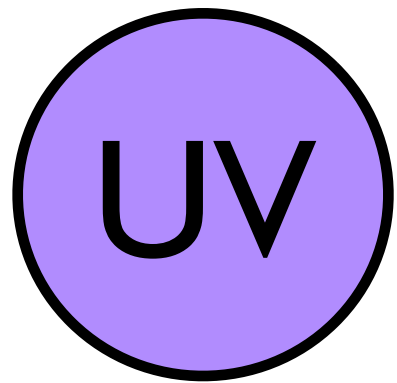
Does c_1 count DOF?

Add a $d = 2$ CFT with central charge c
at the boundary

$$24\pi c_1 \rightarrow 24\pi c_1 + c$$

c_2 unchanged

Both depend on boundary conditions of bulk fields



UV BCFT

Single free, massless, real scalar
Neumann B.C.

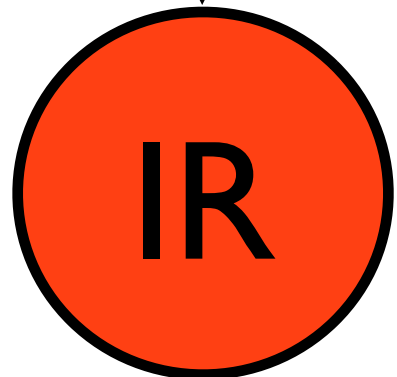
Boundary RG Flow

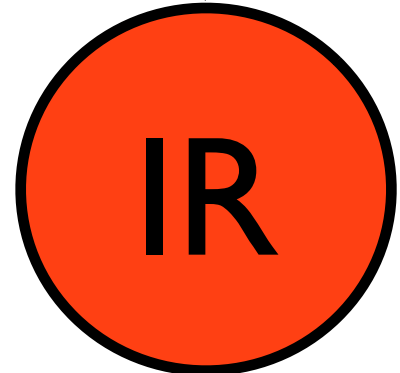
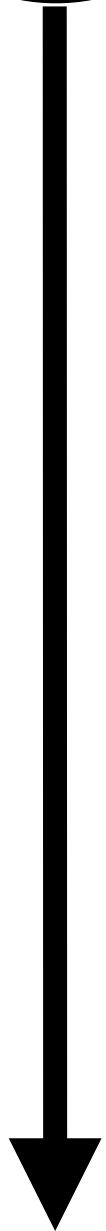
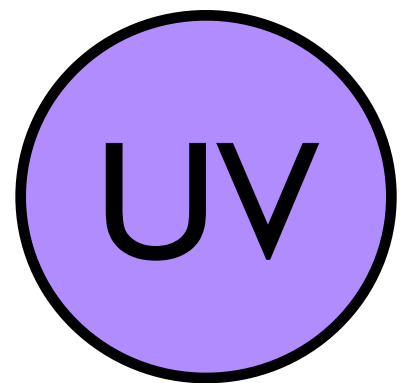
$$S_{\text{BCFT}}^{\text{UV}} \rightarrow S_{\text{BCFT}}^{\text{UV}} + \int d^3x \delta(x) m^2 \Phi^2(\vec{x})$$

$$\Delta_{\Phi^2} = 1$$

IR BCFT

Single free, massless, real scalar
Dirichlet B.C.



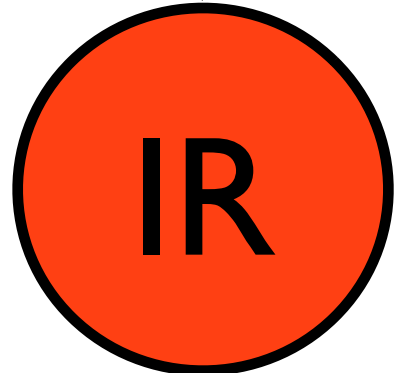
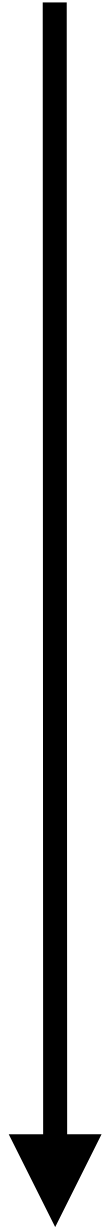
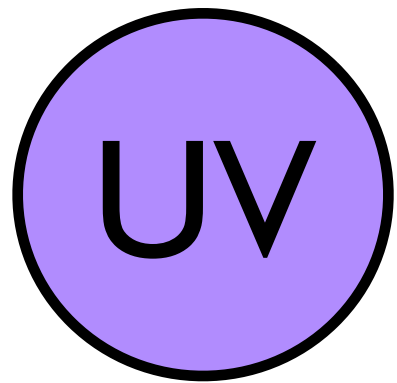


$$c_1^{UV} = \frac{1}{24\pi} \frac{1}{16}$$

Neumann B.C.

$$c_1^{IR} = -\frac{1}{24\pi} \frac{1}{16}$$

Dirichlet B.C.



$$c_1^{UV} = \frac{1}{24\pi} \frac{1}{16}$$

Neumann B.C.

$$c_1^{UV} \geq c_1^{IR}$$

Is c_1 bounded below?

$$c_1^{IR} = -\frac{1}{24\pi} \frac{1}{16}$$

Dirichlet B.C.

Defects

Local, reflection-positive CFT in any $d \geq 3$

With a two-dimensional planar defect

Conformal defect

$$SO(d+1, 1) \rightarrow SO(3, 1) \times SO(d-2)$$

conformal transformations
preserving the defect

rotations about
the defect

“Defect CFT” (DCFT)

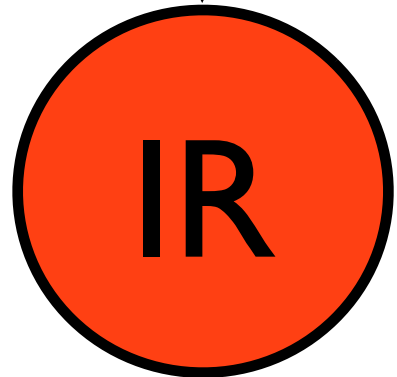
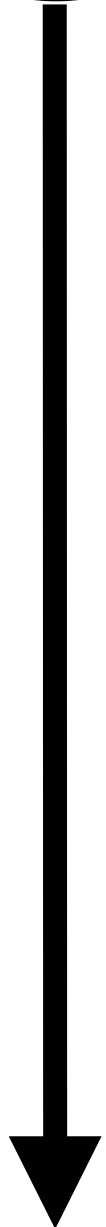
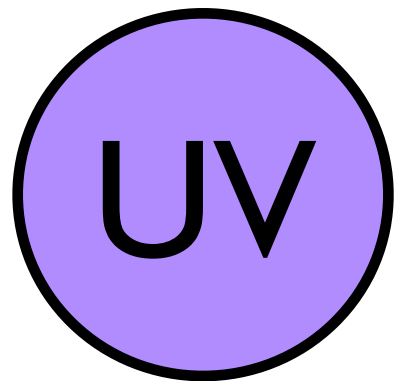
Defects

$$[T_{\mu}^{\mu}]_{\text{defect}} = c_1 \hat{R} + c_2 (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) + c_3 \hat{g}^{ac} \hat{g}^{bd} W_{abcd}$$

Defect “central charges”

c_1 c_2 c_3

$\hat{g}^{ac} \hat{g}^{bd} W_{abcd}$ is Type B



UV DCFT

Defect RG Flow

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

IR DCFT

Outline:

- Review: Monotonicity Theorems
- The Systems
- The Trace Anomaly
- The Proof
- Summary and Outlook

Summary

Local, reflection-positive BCFT in $d = 3$

Local, reflection-positive DCFT in $d \geq 3$
with two-dimensional defect

$$[T_{\mu}^{\mu}]_{\text{defect}} = c_1 \hat{R} + c_2 (K_{ab}^{\mu} K_{\mu}^{ab} - \frac{1}{2} K^{\mu} K_{\mu}) + c_3 \hat{g}^{ac} \hat{g}^{bd} W_{abcd}$$

Boundary or Defect RG Flows

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

Summary

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

Higher-dimensional g-theorem

Generalization of the weak form
of Zamolodchikov's c-theorem
to include coupling to higher-dimensional CFT

Proof used only existing ingredients!

Outlook

Immediate questions

Can we define a c_1 -function?

Is c_1 bounded below?

Other methods of proof?

What about EE? Or holography?

Examples

Graphene with a boundary

Critical Ising model in $d = 3$ with a boundary

M-theory: M2-branes with a boundary

String theory: various brane intersections

Holographic BCFTs

Outlook

Prove more boundary/defect monotonicity theorems!

Yamaguchi	hep-th/0207171
Estes, Jensen, O'B., Tsatis, Wrase	1403.6475
Gaiotto	1403.8052

Do monotonicity theorems always survive coupling to a higher-dimensional CFT?

Find a “universal” proof of monotonicity theorems?

Myers and Sinha	Giombi and Klebanov
1006.1263, 1011.5819	1409.1937

Thank You.