Lifshitz Hydrodynamics

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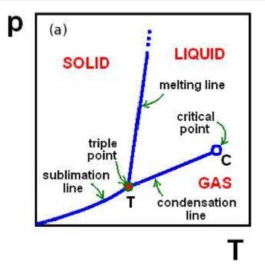
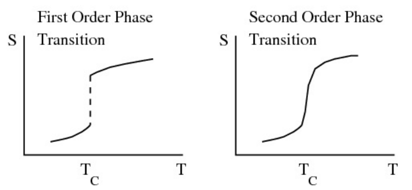
Overview

Phase Transition

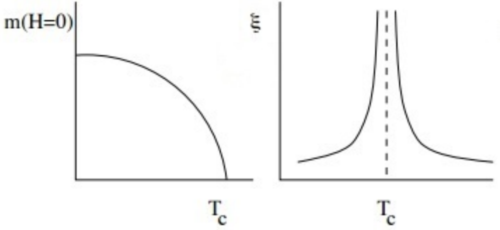
- In a phase transition the system undergoes a symmetry change.
- Discontinuous Phase Transition
  - Release of heat (latent heat)
  - The thermodynamic quantities (internal energy, entropy, enthalpy, volume etc.) are discontinuous.
- Continuous Phase Transition
  - The phase transition is continuous across the transition temperature (or other transition parameter).
  - The thermodynamic quantities are continuous, but their first derivatives are discontinuous.





Critical Point

- A Critical Point is the end point of a phase equilibrium curve.
- At the critical point the correlation length diverges.
- Critical exponents describe the behaviour of physical quantities near continuous phase transitions.



$$m\left(T, H \to 0^+\right) \propto \begin{cases} 0 & T > T_c, \\ |t|^\beta & T < T_c \end{cases}$$

Lifshitz Scaling

- The exponent which describes the behaviour of the relaxation time in the vicinity of the critical temperature is called "the dynamic critical exponent",

$$\tau \sim \xi^z, \ (\xi \text{ is the correlation length})$$

- The result is an anisotropic scaling between time and space - Lifshitz scaling symmetry,

$$\boxed{t \to \lambda^z t, \quad x^i \to \lambda x^i}$$

- Known values of z:

$$\begin{array}{ll} {}^4\text{He} & z = 3/2, \\ \text{FeF}_2 & z = 2, \\ \text{Xenon} & z = 3, \\ \text{Fe} & z = 5/2. \end{array}$$

Lifshitz algebra

- In a Lifshitz theory there are 3 rotational generators J_i, 4 translational generators P_μ and one dilation generator D,

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \quad [D, P_t] = z P_t, \quad [D, P_i] = P_i.$$

- Because Lifshitz symmetry treats time and space differently, it breaks Lorentz boosts, resulting in the breaking of the symmetric stress tensor,

$$\boxed{T^{0i} \neq T^{i0}}$$

- We still maintain a rotational symmetry $T^{ij} - T^{ji} = 0$.

Lifshitz Field Theory

- The analogous of a free scalar field for Lifshitz $z = 2$ theory is,

$$\mathcal{L} = \int d^2xdt \left((\partial_t \phi)^2 + \kappa \left(\nabla^2 \phi \right)^2 \right).$$

- This theory has a line of fixed point parametized by κ.
- Arises at finite temperature multicritical points in the phase diagrams of known materials.
- The correlation function

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle \sim \frac{1}{|x_1 - x_2|^{\pi/\sqrt{\kappa}}}$$

- The algebraic decay of the correlation is a sign of scale invariance at a quantum critical point.

Lifshitz ward identities

- Ward trace identity for the stress energy tensor:

$$zT^0{}_0 + \delta^j{}_i T^i{}_j = 0$$

- Identifying the energy density $\epsilon = -T^0_0$, the pressure $p = T^i_i$ (no sum)
- For a neutral fluid they scale,

$$\epsilon \sim p \propto T^{\frac{z+d}{z}}, \quad s \sim T^{d/z}$$

- We can also find the **equation of state**, $z\epsilon = dp$.

Landau-Fermi liquid theory

- The Landau theory of Fermi liquids (Landau 1957) describes interacting fermions in most metals at low temperatures.
- Replace the complexities by **weakly interacting quasi particles**.
- Therefore, some properties of an interacting fermion system are very similar to those of the Fermi gas.
- Important examples of Fermi liquid theory that has been successfully applied are, electrons in most metals and Liquid He-3.
- An important result of Landau-Fermi liquid theory is,

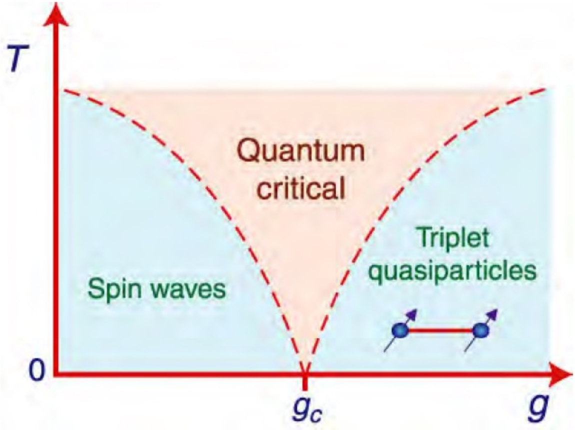
$$\rho \sim T^2.$$

Non-Fermi liquid

- Heavy fermion compounds and other materials including high T_c superconductors have a metallic phase (dubbed as strange metal) whose properties cannot be explained within the ordinary Landau-Fermi liquid theory.
- In this phase some quantities exhibit universal behaviour such as the resistivity, which is linear in the temperature T. (For example: 2D Graphene)
- Such universal properties are believed to be the consequence of quantum criticality (Coleman:2005,Sachdev:2011).
- A quantum critical point is a special class of continuous phase transition that takes place at absolute zero.

QCP

Phase transitions at zero temperature are driven by quantum fluctuations.



Motivation

- At the quantum critical point there is a Lifshitz scaling (Hornreich:1975,Grinstein:1981) symmetry.
- Systems with ordinary critical points have a hydrodynamic description with transport coefficients whose temperature dependence is determined by the scaling at the critical point (Hohenberg:1977).
- Quantum critical systems also have a hydrodynamic description, e.g. conformal field theories at finite temperature.
- At quantum critical regime the hydrodynamic description will be appropriate if the characteristic length of thermal fluctuations $\ell_T \sim 1/T^{1/z}$ is much smaller than the size of the system $L >> \ell_T$ and both are smaller than the correlation length of quantum fluctuations $\xi >> L >> \ell_T$.

Hydrodynamics

- Hydrodynamics is an effective theory of low energy dynamics of conserved charges, which remain after integrating out high energy degrees of freedom.

Relativistic Hydrodynamics characterization

The effective degrees of freedom are,

- Energy density ϵ
- Pressure p
- Entropy density s
- Relativistic fluid velocity u^{μ} $(u^{\mu}u_{\mu} = -1)$
- Chemical potential μ
- Particle number density/charge density q

The equations that connect between those degrees of freedom are:

- First law of thermodynamics $d\epsilon = Tds + \mu dn$
- The equation of state $\epsilon = f(p)$
- Conservation laws $\partial_{\nu}T^{\mu\nu} = 0, \quad \partial_{\mu}J^{\mu} = 0$

Relativistic Hydrodynamics - The Currents

- The conserved currents are the one point functions, $\langle T^{\mu\nu}\rangle, \langle J^{\mu}\rangle$.
- The currents are built from the thermodynamical d.o.f. $\Rightarrow$ "Constitutive Relations".
- For example, The ideal (zeroth order) stress energy tensor is,

$$T^{\mu\nu} = (\epsilon + p)\, u^{\mu} u^{\nu} + p\eta^{\mu\nu}$$

- At the rest frame we have: $T^{0}_{\ 0} = -\epsilon$ and $T^{i}_{\ j} = p\delta^{i}_{\ j}$

Relativistic Hydrodynamics - The Currents

- Hydrodynamic $\Rightarrow$ the degrees of freedom are time and position dependent: $\epsilon\left(x^{\nu}\right)$, $p\left(x^{\nu}\right)$, etc...
- The d.o.f change "slowly" in time and space. (The Hydrodynamics regime)
- Derivative expansion.
- Every term is suppressed by the previous one.

$$\mathcal{O} = \sum_{i=0} \mathcal{O}_i\left(\partial^i\right), \quad \left(\mathcal{O} = T^{\mu\nu}, J^{\mu}\right)$$

- The currents,

$$T^{\mu\nu} = \left(\epsilon + p\right) u^{\mu} u^{\nu} + p\eta^{\mu\nu} + \Pi^{\mu\nu},$$
$$J^{\mu} = qu^{\mu} + \nu^{\mu}.$$

- $\Pi^{\mu\nu}$, ν^{μ} sub-leading terms, contain derivatives.

Relativistic Hydrodynamics - Entropy Current

- Define an entropy density current, S^μ.
- Demand it to locally satisfy the second law of thermodynamics,

$$\boxed{\partial_\mu S^\mu \geq 0}.$$

- Two kinds of constrains:
- Inequality constraints $\partial_\mu S^\mu \geq 0 \Rightarrow$ inequality constraints among the Transport coefficients, for example, $\eta \geq 0$.
- Equality constraints $\partial_\mu S^\mu = 0 \Rightarrow$ equality relations between the Transport coefficients.

Lifshitz Hydrodynamics

$$\boxed{t \to \lambda^z t, \quad x^i \to \lambda x^i}$$

- The Lifshitz generators are: 3 rotational generators J_i, 4 translational generators P_μ and one dilation generator D.
- We **do not** have the 3 boosts generators, which results in an antisymmetric stress energy tensor!

$$\boxed{u_\mu T^{\mu\nu} P_\nu^\alpha \neq P_\nu^\alpha T^{\nu\mu} u_\mu}$$

- We still maintain a rotational symmetry, $P_\mu^\alpha T^{\mu\nu} P_\nu^\beta = P_\nu^\beta T^{\nu\mu} P_\mu^\alpha$
- The new physics will come from the antisymmetric part in the stress energy tensor.
- The most general stress energy tensor,

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + pP^{\mu\nu} + \pi_S^{(\mu\nu)} + \mathbf{u}^{\boldsymbol{\mu}} \mathbf{V}^{\boldsymbol{\nu}}.$$

Lifshitz Hydrodynamics - Parity Breaking Sector 3+1 dim

- Consider a **charged** (J^{μ}) Lifshitz fluid in a **3+1** dimensions subject to external electric E_{μ} and magnetic B_{μ} fields.
- If we turn on the external sources the conserved currents are no longer conserved

$$\partial_{\mu} T^{\mu\nu} = F^{\mu\lambda} J_{\lambda}, \quad \partial_{\mu} J^{\mu} = C E^{\mu} B_{\mu}$$

- C is the coefficient of the triangular anomaly.
- The single derivative parity breaking sector has two pseudovectors,

$$\omega^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_{\nu} \partial_{\rho} u_{\sigma},$$
$$B^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_{\nu} F_{\rho\sigma}.$$

Lifshitz Hydrodynamics 3+1

- The constitutive relations for the single derivative parity breaking terms are,

$$V^{\mu}_{A,\not{P}} = -T\beta_{\omega}\omega^{\mu} - T\beta_{B}B^{\mu},$$
$$\nu^{\mu}_{\not{P}} = \xi_{\omega}\omega^{\mu} + \xi_{B}B^{\mu}.$$

- Impose the 2^{nd} law $\partial_{\mu}S^{\mu} \geq 0$, and find equality constraint which can be solved, and we find the new transport coefficients, for $z \neq 1$,

$$\beta_B = c_B T^{\frac{2-z}{z}}, \quad \beta_{\omega} = (2c_B\bar{\mu} + c_{\omega})\, T^{\frac{2}{z}},$$
$$\xi_B = C\left(\mu - \frac{1}{2}\frac{q\mu^2}{\varepsilon+p}\right) - c_B \frac{2-z}{2-2z}\frac{q}{\varepsilon+p} T^{\frac{2}{z}},$$
$$\xi_{\omega} = C\left(\mu^2 - \frac{2}{3}\frac{q\mu^3}{\varepsilon+p}\right) + c_B\frac{z}{1-z}\left(1 - \frac{2\mu q}{\varepsilon+p}\right) T^{\frac{2}{z}}$$
$$- \frac{qT}{\varepsilon+p}\left(\frac{c_{\omega}}{1-z} + 2c_B\bar{\mu}\right) T^{\frac{2+z}{z}}.$$

Lifshitz Hydrodynamics 3+1

- For $z = 1$,

$$\beta_B = \beta_\omega = 0,$$
$$\xi_B = C\left(\mu - \frac{1}{2}\frac{q\mu^2}{\varepsilon + p}\right) - \gamma_B \frac{qT^2}{\varepsilon + p},$$
$$\xi_\omega = C\left(\mu^2 - \frac{2}{3}\frac{q\mu^3}{\varepsilon + p}\right) + 2\gamma_B T^2 - \frac{2q}{\varepsilon + p}\left(2\gamma_B \mu T^2 + \gamma_\omega T^3\right).$$

- **The same as in the relativistic case!**

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

For the application to strange metals:

- Use the Galilean description.
- Use a hydrodynamic model as an effective description to the long wavelength collective motion of the electrons.
- Look at fluids with broken Galilean boost invariance.
- Derive the constitutive relations $\Rightarrow$ taking $c \to \infty$ of the Lifshitz hydrodynamic equations.
- Group together terms proportional to factors of c and take the limit where $v << c$.

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

- The non-relativistic limit of the hydrodynamic equations.
- The current conservation equation, gives the usual continuity equation,

$$\partial_t \rho + \partial_i(\rho v^i) = 0.$$

- The non-relativistic second law $\partial_\mu S^\mu \geq 0$ gives the following constraints on the non-relativistic transport coefficients,

$$\boxed{\beta_\omega = 0}$$

$$\boxed{\beta_B - \frac{C}{T} = 0} \to \textbf{defined by the relativistic anomaly alone!}$$

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

- From the next to leading order (in c) of the conservation equation we find the Navier-Stokes equations,

$$\partial_t P^i + \partial_j \left(P^i v^j \right) + \partial^i p =$$
$$\rho \left(E^i + \epsilon^{ijk} v_j B_k \right) + \partial_j \left(\eta \sigma^{ij} + \delta^{ij} \zeta \partial_k v^k \right)$$

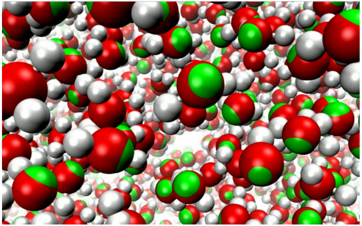
where $\sigma_{ij} = \partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v^k$ is the shear tensor.

- The momentum density is

$$P^i = \rho v^i - \alpha_a a^i - \alpha_T \partial^i T - T \beta_{\mathbf{B}} \mathbf{B^i},$$

- The term β_B allows a Chiral Magnetic Effect in a non-relativistic theory.

Parity Breaking Lifshitz Hydrodynamics 3+1 Drude Model



- The collective motion of electrons in the strange metal $\Rightarrow$ Charged fluid moving through a static medium $\Rightarrow$ produce a drag on the fluid.
- The hydrodynamic equations are

$$\partial_\mu J^\mu = C E^\mu B_\mu, \quad \partial_\mu T^{\mu\nu} = F^{\nu\sigma} J_\sigma - \lambda c \delta^{\nu i} J_i.$$

- We describe a steady state, which implies that the external fields are constant in time.

Parity Breaking Lifshitz Hydrodynamics 3+1 Drude Model

The setup:

- Constant electric field, E_x.
- Slowly varying magnetic field, $B_z(x)$.
- Solving Navier-Stokes equations order by order for the velocity.
- To leading order, the current has only a longitudinal component,

$$J_x = \frac{\rho}{\lambda} E_x \Rightarrow \text{Ohm law.}$$

Parity Breaking Lifshitz Hydrodynamics 3+1 Drude Model

- There are two kind of contributions, pointing in different directions.
  - One from the Lorenz force term,

$$J_y = -\frac{\rho}{\lambda^2} E_x B_z.$$

  - The second is due to the Chiral Magnetic term and points in the direction of the magnetic field,

$$\boxed{J_z = \left[\frac{T\beta_B}{\lambda^2}\partial_x B_z\right] E_x}.$$

This new current would be forbidden in a Galilean-invariant theory. **It can be measured in the lab!**

Parity Breaking Lifshitz Hydrodynamics 2+1

- There is no anomaly, since the number of space-time dimensions is odd.
- There are three new pseudovectors,
  - $\tilde{U}_1^\mu = \epsilon^{\mu\nu\sigma} u_\nu a_\sigma$
  - $\tilde{U}_2^\mu = \epsilon^{\mu\nu\sigma} u_\nu E_\sigma$
  - $\tilde{U}_3^\mu = \epsilon^{\mu\nu\sigma} u_\nu \left(E_\sigma - T\partial_\sigma \frac{\mu}{T}\right)$
- Add them to the antisymmetric part of the stress tensor and to the charge current,

$$V^\mu = -T\sum_{i=1}^{3} \tilde{\mu}_i \tilde{U}_i^\mu$$

$$\nu^\mu = \sum_{i=1}^{3} \tilde{\delta}_i \tilde{U}_i^\mu$$

Parity Breaking Lifshitz Hydrodynamics 2+1

- Taking $c \to \infty$ we recover the momentum density,

$$P^i = \rho v^i - \alpha_a a^i - \alpha_T \partial^i T - \beta_T \epsilon^{ij} \partial_j T - \beta_\mu \epsilon^{ij} \partial_j \mu_{NR} - \beta_E \epsilon^{ij} \left(E_j - \epsilon_{jk} v^k B \right)$$

- $\alpha' s$ even sector.
- $\beta' s$ odd sector.

Application for strange metal, setup:

- Constant electric field E_x
- Allow gradients along of the x direction of $T(x)$, $\mu(x)$, $p(x)$, $\rho(x)$.

Parity Breaking Lifshitz Hydrodynamics 2+1 - Drude model

- Small gradients $\Rightarrow$ derivative expansion of N-S order by order around constant density and velocities.
- To leading order: $J_x = \frac{\rho}{\lambda} E_x$.
- A transverse (Hall) current is also generated. The leading contribution to the velocity in the y direction is,

$$J_y = \frac{1}{\lambda^2}\left[\frac{\beta_E}{\rho}\partial_x^2 p - \beta_T \partial_x^2 T - \beta_\mu \partial_x^2 \mu_{NR}\right] E_x .$$

- This can be interpreted as an anomalous Hall effect! (A transverse current in the absence of magnetic fields).

Future Research

- What are the higher order corrections to the constitutive relations? For instance, finding the allowed second order transport coefficients in the antisymmetric part of the stress energy tensor.
- Understand why are the transport coefficients in the odd sector of the theory the same in both the relativistic theory and the Lifshitz theory?

$$Lorentz = Lifshitz(z = 1).$$

- Find a Holographic setup which produces Lifshitz hydrodynamics beyond the ideal order.
- Bulding Supersymmetric Lifshitz field theory.

The End

Thank you for your attention!