

Anomalies & Transport

(K. Landsteiner : Seminar Oxford 11-03-2014)

-) elementary derivation of anomaly
-) CME from energy conservation
-) Family of anomalous transport via Kubo formulas
-) Hydrodynamics
-) Anomaly & energy conservation \rightarrow cov vs. cons current
-) ~~Kubo formulae & consistency~~
-) Weyl semi-metal & physical interpretation of consistent current

Elementary derivation of Anomaly

Weyl fermion $i\sigma^M \partial_\mu \Psi = 0$

in magnetic field $A_y = B_z x$

$$[i\sigma^0 \partial_t + i\sigma^4 (\partial_y - iB_z x) + i\sigma^x \partial_x + i\sigma_z \partial_z] \Psi = 0$$

$x = \bar{x} + \frac{B}{B}$

harmonic oscillator

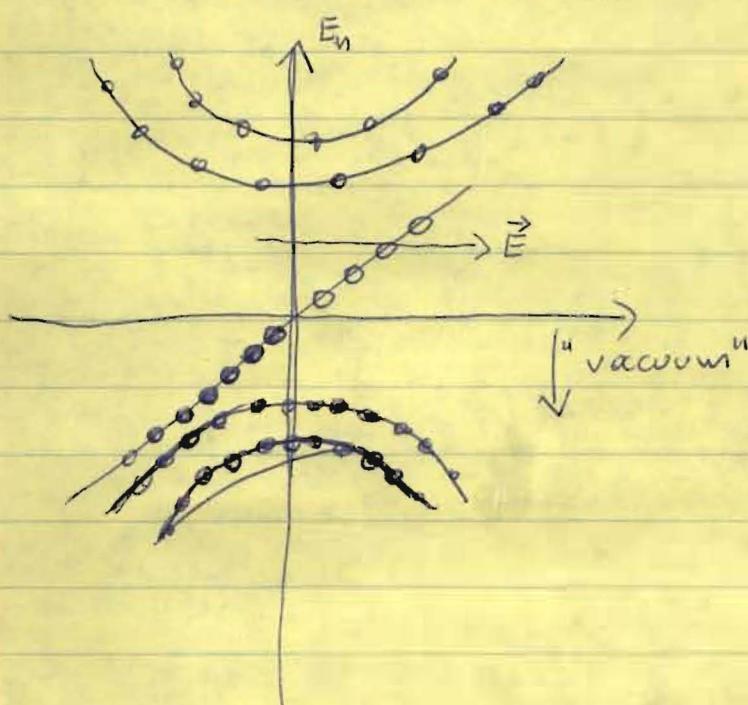
$$E_n = \pm \sqrt{\lambda B(n + \frac{1}{2}) + B\sigma_3 + B_{\perp}^2}$$

Landau Levels

$$n=0 \quad \sigma_3 = -1$$

$$E = B_{\perp}$$

pos. energies



Normal-ordered

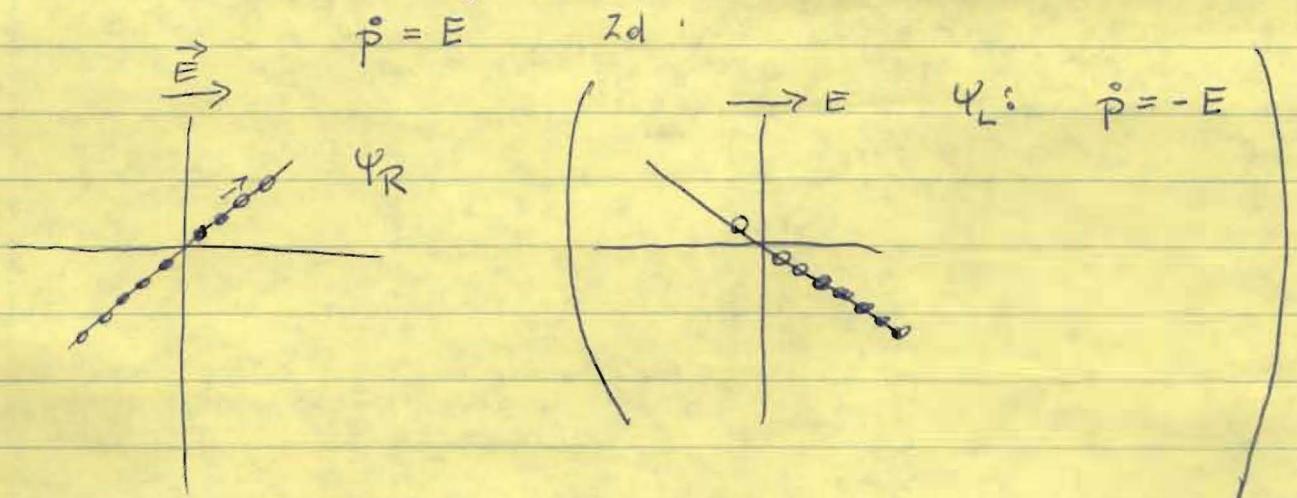
operators are blind
to negative energy
states "Dirac-sea"

right-handed fermion: $n=0, \sigma_3 = +1$ dim chiral fermion

$n=1, \sigma_3 = 1$ $n=0, \sigma_3 = +1$ see degeneracy

:

switch on (adiabatically) an electric field: Lorentz-force



How many states are there: $d_n = \frac{dP}{2\pi} \cdot \# \text{ (degeneracy of LLL)}$
 $\frac{B}{2\pi^2} \text{ (Landau)}$

$$\Rightarrow \boxed{\frac{du}{dt} = \frac{1}{4\pi^2} \frac{dP}{dt} B = \frac{\pm 1}{4\pi^2} \vec{E} \cdot \vec{B}}$$

chiral anomaly

"Hilbert-Hotel argument"
 relativistically $\boxed{\partial_u f_{R,L}^H = \frac{\pm 1}{4\pi^2} \vec{E} \cdot \vec{B}}$ (Nielsen, Niemi 1983)

- Anomaly reduced to Q.M.
- relies on exactness of Hilbert-Hotel argument
- axial anomaly $\frac{du_5}{dt} = \frac{1}{2\pi^2} \vec{E} \cdot \vec{B}$ $n_5 = n_R - n_L$

- modern language: Berry-phase, curvature
 wave-factors: $\langle \Psi(\vec{p}) \rangle$: (n) bundle over momentum space
 connection on the principle bundle

$$\langle \Psi(\vec{p}) | \frac{\partial}{\partial p_i} | \Psi(\vec{p}) \rangle = A_i(\vec{p}) \quad \text{"Berry" connection}$$

Berry curvature $\Omega_i = \epsilon_{ijk} \frac{\partial}{\partial p_j} A_k(\vec{p}) \approx \pm \frac{\vec{p}}{|\vec{p}|^3}$

- ⇒ Weyl-fermion = Monopole in momentum space
- topological protection of Weyl-fermion (Volovik)

more general: CMT: "Chern-insulators",
 "topological insulators"
 QHE

→ topological band theory

$$\Rightarrow \text{CME: } \partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \quad \partial_t T^{00} = \vec{E} \cdot \vec{J}$$

②

analogously: $\frac{du}{dt} = E B \frac{1}{4\pi^2} \quad dE = \mu \cdot dm$

$$\mu \frac{du}{dt} = \frac{\mu}{4\pi^2} \vec{E} \cdot \vec{B} = \vec{E} \cdot \vec{J} \Rightarrow \boxed{\vec{J} = \frac{\mu}{4\pi^2} \vec{B}}$$

chiral magnetic effect

energy current $\delta T^{0i} = \mu \delta J^{0i}$

$$T^{0i} = \frac{1}{4\pi^2} \int_0^\mu \mu \delta \mu \vec{B}^i = \left(\frac{\mu^2}{8\pi^2} \text{const} \right) \vec{B}^i$$

determination of integration constant: only LLL channel free
of antiparticles

$$\vec{J} = \frac{B}{2\pi e} \int_0^\infty \frac{dp}{2\pi} \left[u_F \left(\frac{p-\mu}{T} \right) - u_F \left(\frac{p+\mu}{T} \right) \right] =$$

$$= \frac{B}{2\pi^2} \int_0^\mu \frac{dp}{2\pi} u_F(p-\mu, T) + \int_\mu^\infty \frac{dp}{2\pi} u_F(p+\mu, T) - \int_0^\infty u_F \left(\frac{p+\mu}{T} \right) \frac{dp}{2\pi}$$

$$u_F(-x) = \frac{1}{e^{-x} + 1} = \frac{e^x}{e^x + 1} = \frac{e^x + 1 - 1}{e^x + 1} = 1 - u(x)$$

$$= \frac{B}{2\pi} \int_0^\mu \frac{dp}{2\pi} - \int_0^\mu \frac{dp}{2\pi} u_F(p+\mu, T) + \int_0^\infty \frac{dx}{2\pi} u_F(x, T) - \int_\mu^\infty u_F \left(\frac{x}{T} \right) dx$$

$$= \frac{B}{2\pi} \int_0^\mu \frac{dp}{2\pi} = \frac{\mu}{4\pi^2} B !$$

$$\vec{J}_e = (T^{0i}) = \frac{B}{2\pi} \int_0^\mu \frac{dp}{2\pi} p \left[u_F \left(\frac{p-\mu}{T} \right) + u_F \left(\frac{p+\mu}{T} \right) \right] = \left(\frac{\mu^2}{8\pi^2} + \frac{T^2}{24} \right) \vec{B}$$

Integration constant: $\frac{T^2}{24} !$

$$B \propto \vec{B} = \frac{1}{2\pi} \int_0^\mu \frac{dp}{2\pi} p + \frac{1}{2\pi^2} \int_0^\infty x u_F \left(\frac{x}{T} \right) dx$$

↓
matter

↓
thermal fluxes
around fermi
surface

Chiral Vortical Effect: $F = \vec{E} \vec{n} \times \vec{\omega} \sim 2\vec{p} \times \vec{\omega} + \Theta(\omega^2)$
 $\sim 2\vec{v} \times (\vec{E} \vec{\omega}) \rightarrow \text{Coriolis force}$

Coriolis force \sim Lorentz force $\vec{B} \rightarrow 2E\vec{\omega}$

more rigorous: gravito-magnetic field

$$ds^2 = -dt^2 + 2\vec{A}_g d\vec{x} dt + d\vec{x}^2$$

$$\rightarrow \text{geo } \vec{B}_g = \vec{\nabla} \times \vec{A}_g = 2\vec{\omega}$$

charge: energy

$$\vec{J} = \frac{\vec{B}_g}{(2\pi k)} \int_0^\infty \frac{dp}{2\pi} p \left[u\left(\frac{p-\mu}{T}\right) + u\left(\frac{p+\mu}{T}\right) \right] = \left(\frac{\mu^2}{8\pi k} + \frac{T^2}{2k} \right) \vec{B}_g \\ = \left(\frac{\mu^2}{4\pi k} + \frac{T^2}{4k} \right) \vec{\omega}$$

$$\vec{J}_e = \frac{1}{4\pi k} \int_0^\infty \frac{dp}{2\pi} p^2 \left[u\left(\frac{p-\mu}{T}\right) - u\left(\frac{p+\mu}{T}\right) \right] = \left(\frac{\mu^3}{12\pi k} + \frac{T^2}{12k} \right) \vec{B}_g$$

more first principle derivations: Kubo formulae

$$J_i = \sigma^B B_i = \sigma^B \epsilon_{ijk} \partial_j A_k$$

$$\langle J_i J_k \rangle = i P_f \sigma^B \epsilon_{ijk}$$

$$J_i = \sigma^v B_i^g = \sigma^v \epsilon_{ijk} \partial_j A_k^g$$

$$\langle J_i T_{0k} \rangle = \sigma^v i P_f \epsilon_{ijk}$$

$$\langle T_{0i} T_{0k} \rangle + \text{scall} = \sigma^v i P_f \epsilon_{ijk}$$

•) $\omega = 0$

•) $\langle JT \rangle = \langle T J \rangle$ relation between CVE + CME in \vec{J}_e

•) $\frac{d\omega}{dt} = \frac{1}{2} \omega f_I(-\omega) g^{IJ}(\omega) f_J(\omega) = 0 \rightarrow \underline{\text{dissipationless}}$!

↓ work done by external source f_I

for general symmetries: T^A :

$$\vec{J}^A = \sigma_{B,E}^{AB} \vec{B}^B + 2\sigma_{\epsilon,v}^A \vec{\omega}$$

$$\vec{J}_\epsilon = \sigma_{B,E}^A \vec{B}^A + 2\sigma_{\epsilon,v} \vec{\omega}$$

$$\sigma_{B,E}^{AB} = \frac{1}{4\pi^2} d^{ABC} M_C$$

$$\sigma_{\epsilon,v}^A = \sigma_{B,E}^A = \frac{1}{8\pi^2} d^{ABC} M_B M_C + \frac{T^2}{24} b^A$$

$$\sigma_{v_E} = \frac{1}{12\pi^2} d^{ABC} M_A M_B M_C + \frac{T^2}{48} b^A M_A$$

$$d^{ABC} = \text{str}(T^A T^B T^C)_{R-L}$$

$$b^A = \text{tr}(T^A)_{R-L}$$

Hydro: Son, Susskind, Oz, Meissner, Lopatin - Tseyen, Yarom

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} - \gamma \sigma^{\mu\nu} - \zeta \Theta + Q^\mu u^\nu + Q^\nu u^\mu$$

$$j^\mu = p u^\mu + \sigma_\mu (\epsilon^\mu - p^\mu \sigma_\nu (\frac{\mu}{T})) + \delta_B^\mu B^\mu + \delta_V^\mu \omega^\mu$$

entropy current $j_S^\mu = s u^\mu - \mu \frac{v}{T}$ $\partial_\mu j_S^\mu \geq 0$
 fixes all but T^2 -descenders

$$\nabla_\mu j_A^\mu = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\lambda} \left(d^{ABC} F_{\mu\nu}^B F_{\rho\lambda}^C + \frac{1}{768\pi^2} R^A_{\mu\nu} R^B_{\rho\lambda} \right)$$

↓ ↓
 derivative covariant $O(2)$ $O(4)$!

still no "good" argument for understanding this mismatch
 (see however Loga, Tseyen, Yarom 1207.5824)

Holographic model: $S = S_{EH} + S_{GH} + S_{CS} + S_{CSK}$

$$S_{EH} = \int (R + F^2)$$

$$S_{EH} = \int K$$

$$S_{CS} = \int \lambda A_1 F_1 F + \lambda A_1 R_1 R$$

$$S_{CSK} = \int \lambda K K_1 D K_1 D K$$

K extrinsic curvature

→ holographic Kubo formulae or fluid gravity

exactly the same relations between viscosity and transport coeffs!

⇒ Correct Energy conservation?

$$\text{WI: differs } \delta A_\mu = -\partial_\mu(\epsilon A) - \epsilon^\nu F_{\nu\mu} \quad x^M \rightarrow x^M + \epsilon^\nu$$

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$$

$$W_{\text{eff}}[g_{\mu\nu}, A_\nu] \quad \delta W = \int \frac{\delta W}{\delta A_\mu} \partial_\mu x = \int -\partial_\mu f^M = \int \lambda c \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho} F_{\sigma\lambda}$$

$$\partial_\mu f^M = -c F_\lambda F$$

$$\underbrace{\epsilon_\nu \nabla_\mu T^{\mu\nu} - \epsilon_\nu F^{\nu\mu} f_\mu - \epsilon_\nu A^\lambda \epsilon^{\mu\nu\rho\lambda} F_{\mu\rho} F_{\nu\lambda}}$$

$$\Rightarrow \boxed{\partial_\mu T^{\mu\nu} = F^{\nu\mu} f_\mu + c A^\nu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} F_{\lambda\lambda}}$$

$$\text{non-covariant? } f^\mu = \frac{\delta W}{\delta A_\mu} \quad (\text{"consistent" current})$$

$$\text{define covariant current } f_{\text{cov}}^\mu = f^\mu + x \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda}$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} f_\mu^\text{cov} - x F^{\nu\mu} \epsilon_\mu^{\rho\lambda\sigma} \epsilon_\lambda^{\beta\gamma\delta} A_\beta F_{\gamma\delta} + c A^\nu \epsilon^{\mu\nu\rho\sigma} F_\rho F_\sigma$$

$$x=0 \quad \times F^{0i} \epsilon^{0j\beta} A_\beta F_{j\beta} + c A^0 \epsilon^{0j\beta} F_{0i} F_{j\beta} = 0$$

$$\Rightarrow \cancel{x} \quad x = -c$$

$$\boxed{\partial_\mu f_{\text{cov}}^\mu = -3c \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}}$$

anomaly in covariant current 3-times as big as in
consistent current?

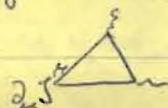
elementary triangle



$$\sim \frac{1}{96\pi^2} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \sim$$

$$\sim \frac{1}{12\pi^2} \vec{E} \cdot \vec{B}$$

putting all anomaly in one vertex



$$\sim \frac{1}{4\pi^2} \vec{E} \cdot \vec{B}$$

factor 3 is from Bose symmetrization!

Nielsen-Ninomiya argument gives covariant anomaly:

Note: CME: $\vec{J} = \frac{\mu}{4\pi^2} \vec{B} - \frac{A_0}{12\pi^2} \vec{B}$ in consistent current

CME in consistent current?

do we set $A_0 = \mu$?

What is " μ "?

Def.: 1: $\mu = A_0$. $\partial_t \rightarrow \partial_t - i\mu$ on $\overline{\Gamma}$

Def.: 2: $\Psi(t - \frac{i}{\tau}) = -e^{\frac{i\mu t}{\tau}} \Psi(t)$ boundary condition on fields

invariant statement: parallel transport of charged field along $t \rightarrow t - \frac{i}{\tau}$

A: $H \rightarrow H_0 - \mu Q$

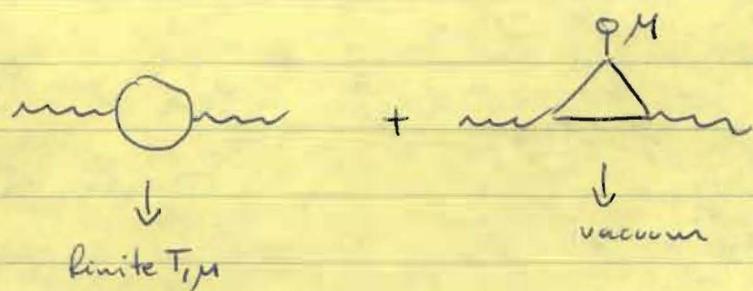
$\Psi(t - \frac{i}{\tau}) = -e^{\frac{i\mu t}{\tau}} \Psi(t)$

B: $H = H_0$

$\Psi(t - \frac{i}{\tau}) = -e^{\frac{i\mu t}{\tau}} \Psi(t)$

related by gauge-trafe: $A_0 = \partial_t(\mu t)$
 $\Psi \rightarrow e^{-i\lambda t} \Psi$

but when anomaly A, B are not gauge-equivalent



in either case.
or consistent
regardless.

•) Dirac fermion, WSM, phys. interpret.

cov. currents: J_R^μ, J_L^μ ; $\tilde{W}_{\text{eff}}[A_\mu, \tilde{A}_\mu] = W_{\text{eff}}[A_\mu, \tilde{A}_\mu] + \underbrace{\Lambda A_5^5 (x_1 F + x_2 F_5)}_{\Lambda A_5^5}$

$$\partial_\mu J_R^\mu = \frac{1}{4\pi^2} E_R B_R \quad \partial_\mu J_L^\mu = \frac{1}{4\pi^2} E_L B_L \quad \rightarrow$$

$$J_{\text{vec}} = J_R + J_L \quad J_A = J_R - J_L \quad \text{et } A_R + A_L = \frac{1}{2} A \quad A_R - A_L = \frac{1}{2} A_5$$

$$\partial_\mu J_A^\mu = \frac{1}{2\pi^2} (\vec{E} \cdot \vec{B} + \vec{E}_5 \cdot \vec{B}_5) \quad \text{backside}$$

$$\partial_\mu J_V^\mu = \frac{1}{2\pi^2} (\vec{E}_5 \cdot \vec{B} + \vec{E} \cdot \vec{B}_5) = \frac{1}{8\pi^2} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}^5 \rightarrow \rightarrow$$

not good! $J_{el}^\mu = \partial_\nu F^{\mu\nu} \Rightarrow \partial_\mu J_{el}^\mu = 0$

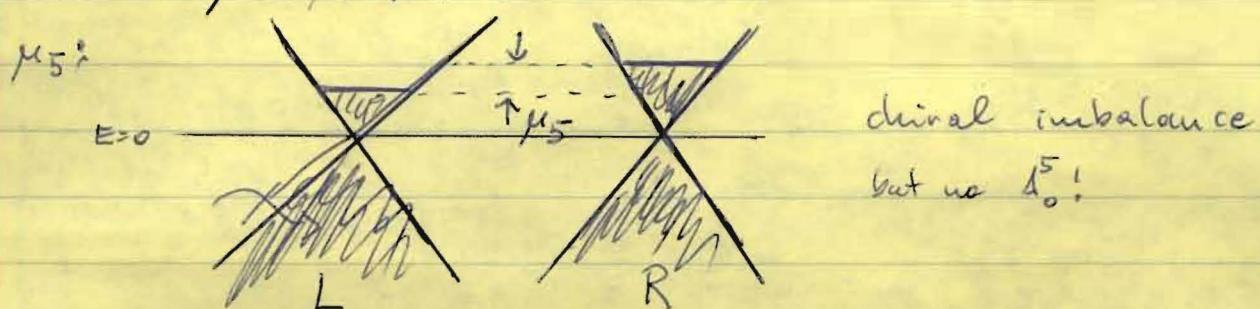
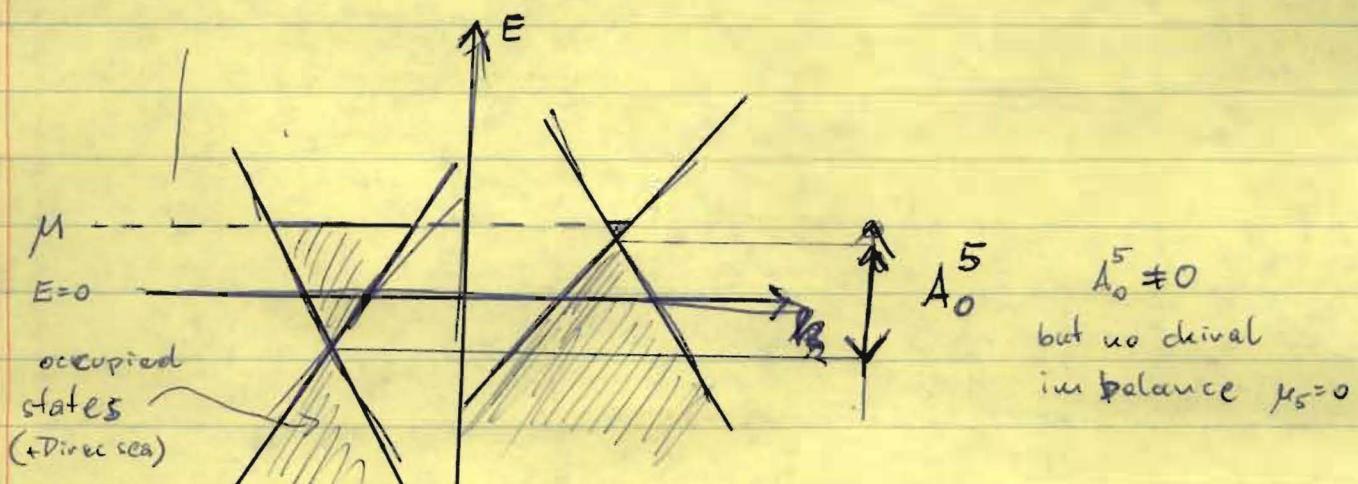
electric current: $J_{el}^\mu = J_{\text{cov},\nu}^\mu - \frac{1}{4\pi^2} \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda}$

CME $\vec{J}_{el} = \left(\frac{\mu_R - \mu_L}{4\pi^2} - \frac{A_R^0 - A_L^0}{4\pi^2} \right) \vec{B} = 0 \text{ if } \frac{A_R^0 - A_L^0}{4\pi^2} = \mu_{R,L}$

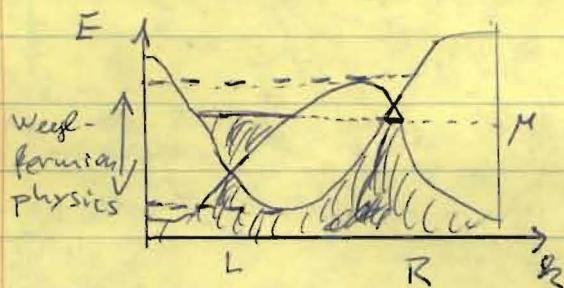
$$\mu_5 = \frac{1}{2} (\mu_R - \mu_L) \quad A_5^0 = \frac{1}{2} (A_R^0 - A_L^0)$$

what does A_5^0 do: $i[\gamma^\mu (\partial_\mu - \gamma_5^\mu \gamma_5)] \Psi = 0$

\Rightarrow shifts by $\gamma_5^\mu A_5^\mu$



Weyl-semimetal (= Graphene in 3D)



- ⇒ compact momenta in space = Brillouin zone
- ⇒ linear band touching points
- ⇒ locally described by Weyl-fermion

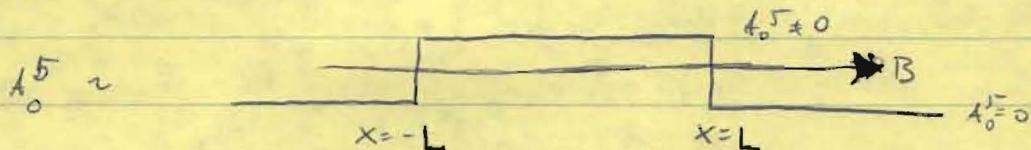
No chiral imbalance $\mu_5 = 0$ but $A_0^5 \neq 0$: tips of Weyl-cone carry at different energies

⇒ no CME in covariant current

but CS-current in consistent (=electric) current

$$\vec{j} = -\frac{A_0^5}{2\pi^2} \vec{B}$$

why? WSM is finite region in space



at $x = \pm L$: axial "electric" field $\partial_x A_0^5 = \pm E_5$

in B-field: anomaly in covariant current $\partial_t f_{cov}^0$

$$\partial_t f_{cov}^0 = \frac{1}{2\pi^2} E_5 B$$

$$x = -L$$

$$\partial_t f_{cov}^0 = -\frac{1}{2\pi^2} E_5 B$$

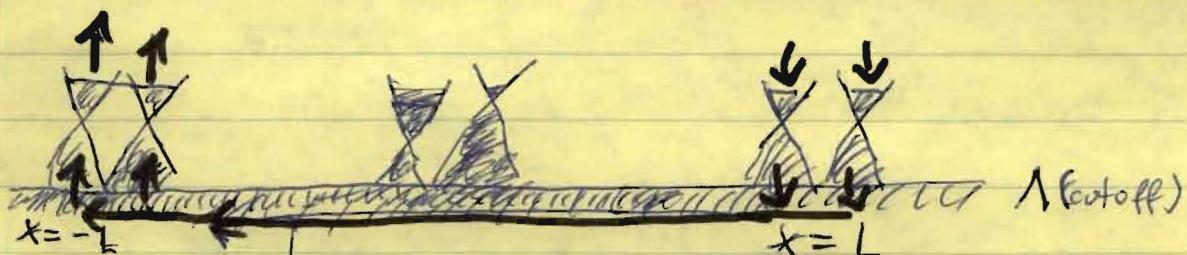
$$x = L$$

CS-current unless this an inflow of charge $\partial_t f^0 = -\vec{J}_{CS}$

covariant current: anomaly

consistent current: inflow

CS-current stems from UV-regularization \Rightarrow inflow of charge comes from states at (and beyond) cutoff



charge transport at (or beyond) the cutoff

appears as Chern-Simons current $\int_L^\infty e^{i\pi s} A_0^5 F_{0s}$

(e.g. integer quantum Hall effect!) looks like anomaly below Λ

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