Thermalization of two point functions in the AdS/CFT duality

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Outline

1) CFT "quench" = Bulk gravitational collapse (use an AdS_4 example)

2) A quick review of different correlators (retarded, Wightman, relation to occupation numbers)

3) The calculation of AdS_3-Vaidya correlators

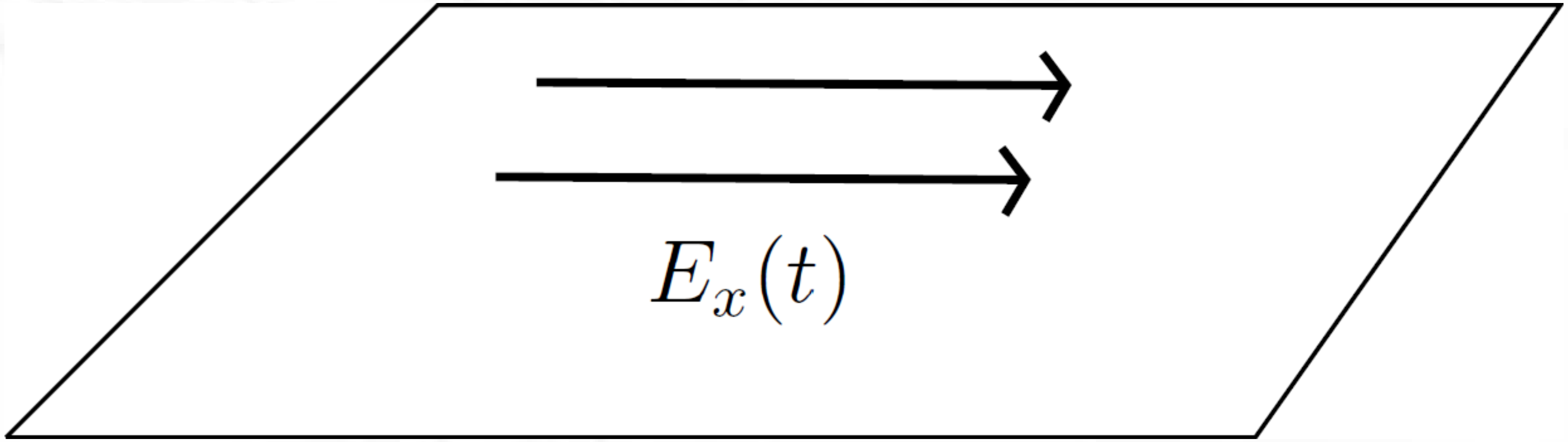
4) Results and their explanation

5) Non-equilibrium occupation numbers

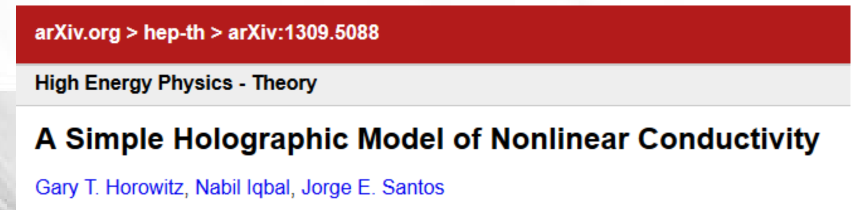
6) Future directions and conclusions

1. Quench = Gravitational collapse

- Consider a 2+1 CFT, which has a conserved current J_μ . Start from the vacuum state, and turn on a time dependent external electric field E_x, which is coupled to the current



- In the gravity dual, this means that we turn on a time dependent electric field near the boundary and see how the field will evolve


arXiv.org > hep-th > arXiv:1309.5088

High Energy Physics - Theory

A Simple Holographic Model of Nonlinear Conductivity

Gary T. Horowitz, Nabil Iqbal, Jorge E. Santos

1. Quench = Gravitational collapse

- Relevant part of the gravity action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \Big(R + 6 - F_{\mu\nu} F^{\mu\nu} + \ldots \Big)$$

- Equations of motion

$$\partial_\mu (\sqrt{|g|} g^{\mu\lambda} g^{\nu\sigma} F_{\lambda\sigma}) = 0$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 3 g_{\mu\nu} = T_{\mu\nu} = 2 F_{\mu\lambda} F_\nu^\lambda - \frac{1}{2} g_{\mu\nu} F^2$$

- Boundary conditions: Asymptotically AdS with electric field
- Ansatz

$$F_{xv} \neq 0$$

$$ds^2 = g_{vv}(z,v) dv^2 + 2 g_{zv}(z,v) dv dz + g_{xx}(z,v) dx^2 + \frac{1}{z^2} dy^2$$

1. Quench = Gravitational collapse

- There is only one non-trivial component of Maxwell's equations

$$\partial_z(\sqrt{|g|}g^{xx}g^{zv}F_{xv}) = 0$$

$$F_{xv} = \frac{E_x(v)}{\sqrt{|g|}g^{xx}g^{zv}}$$

- From the AdS/CFT dictionary we can already identify E_x as the electric field in the boundary theory
- One non-vanishing stress energy tensor component

$$T_{vv} = 2z^2E_x(v)^2$$

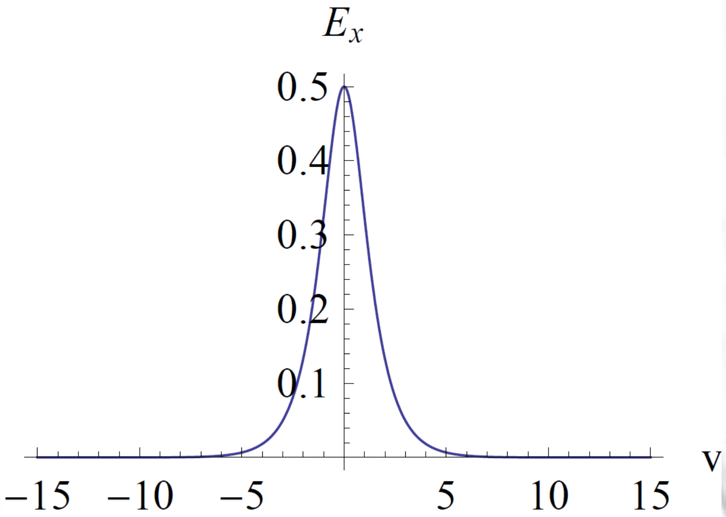
1. Quench = Gravitational collapse

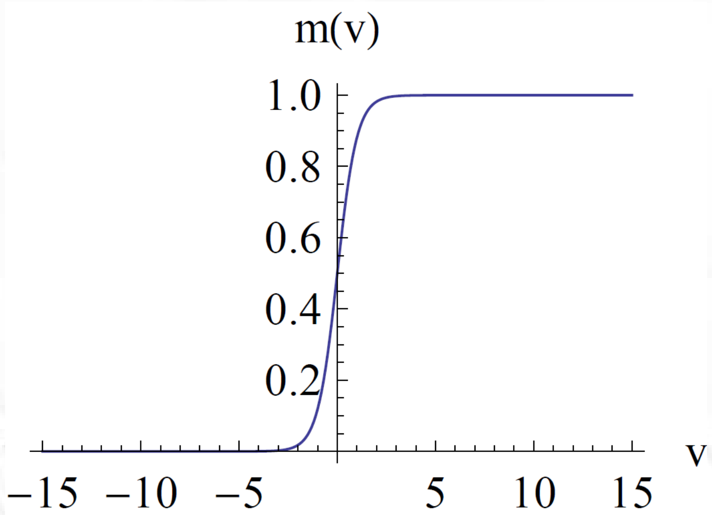
- Solving the Einstein's equations with the asymptotically AdS boundary conditions gives the metric (Vaidya spacetime)

$$ds^2 = \frac{1}{z^2}\Big(-(1 - m(v)z^3)dv^2 - 2dvdz + dx^2 + dy^2 \Big)$$

$$m(v) = 2\int_{-\infty}^{v} dv' E_x(v')^2$$

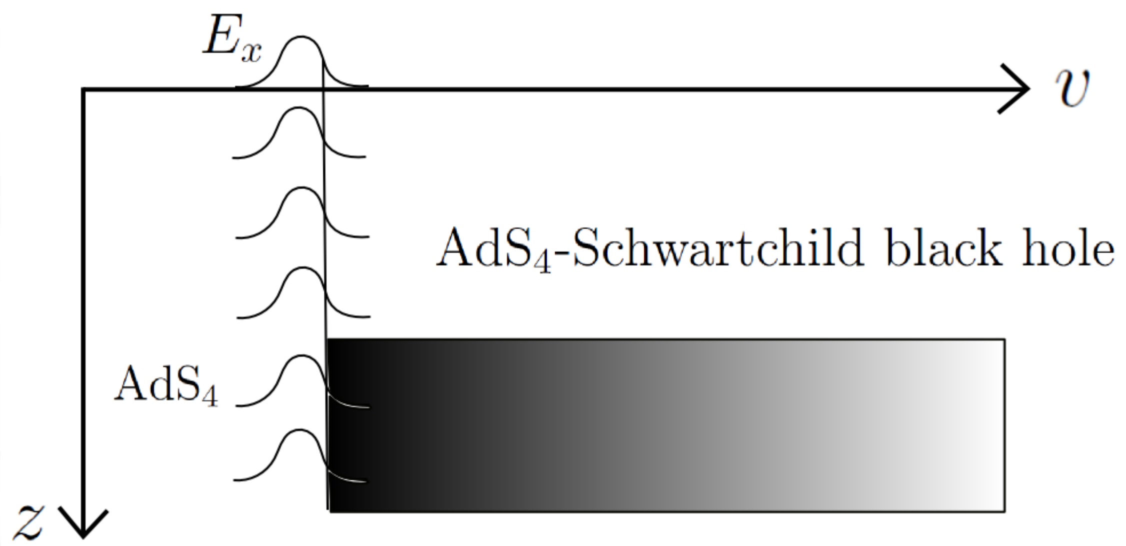
- Black hole with a time dependent mass

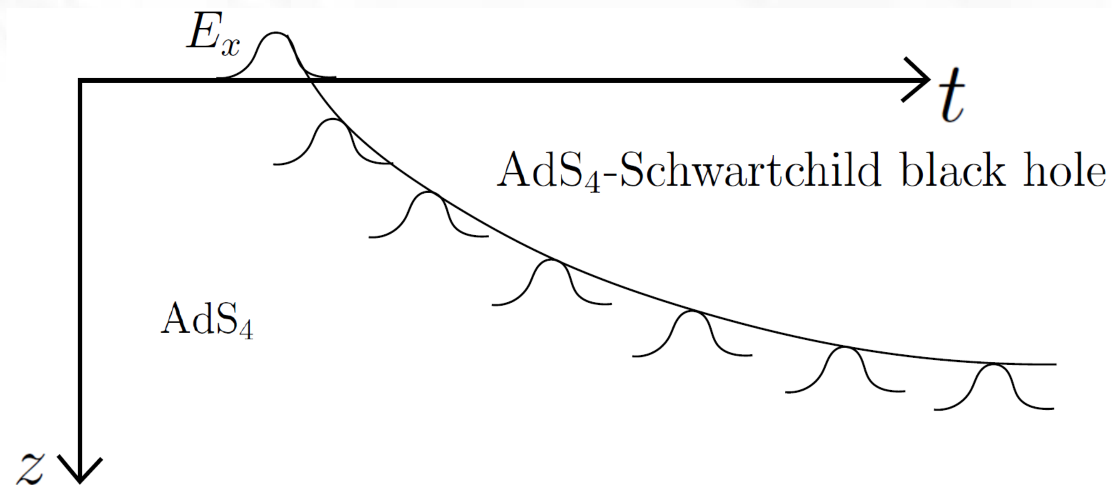




1. Quench = Gravitational collapse

- Different coordinate systems





1. Quench = Gravitational collapse

- Use the AdS/CFT dictionary to obtain CFT observables

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + z\langle T_{\mu\nu}\rangle + \dots$$

$$A_\mu = A_\mu^{bdy} + z\langle J_\mu\rangle + \dots$$

- The current expectation value is given by (Ohm's law)

$$\langle J_x(t)\rangle = \sigma E_x(t) \qquad \sigma = \frac{1}{4\pi G}$$

- The expectation value of energy density satisfies

$$\partial_t\langle T_{tt}\rangle = \rho\langle J_x\rangle^2 \qquad \text{(Joule heating)}$$

- The expectation values become stationary (and thermal) immediately when the electric field is turned off

1. Quench = Gravitational collapse

- The Vaidya spacetime generalizes to any number of dimensions

$$ds^2 = \frac{1}{z^2}\Big[-(1-m(v)z^d)dv^2 - 2dvdz + d\mathbf{x}^2\Big]$$

- One has to assume that there is some matter that gives rise to the appropriate stress energy tensor
- In the following will consider the Vaidya spacetime in 2+1 dimensional bulk as a model for a quench in a 1+1 CFT
- We will consider the case where the mass is a step function

$$m(v) = \theta(v)$$

- Want to figure out what happens to correlation functions

2. Review of correlation functions

- Retarded correlators quantify response of a system to external classical perturbations

$$S + \int d^d x \mathcal{O}(x) J(x)$$

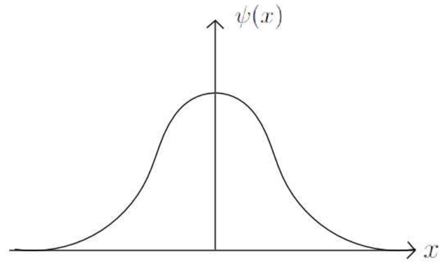
$$\delta\langle O(x)\rangle = -i \int d^d x' \, G_R(x, x') J(x')$$

- Wightman functions quantify fluctuations in the quantum state of the system (e.g. occupation numbers of states)

$$G_+(x_1, x_2) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \qquad G_-(x_1, x_2) = \langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle$$

- E.g. harmonic oscillator

$$\psi(x) = Ce^{-x^2/2} \qquad \Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = \int dx \, x^2 |\psi(x)|^2 = \frac{1}{2}$$



2. Review of correlation functions

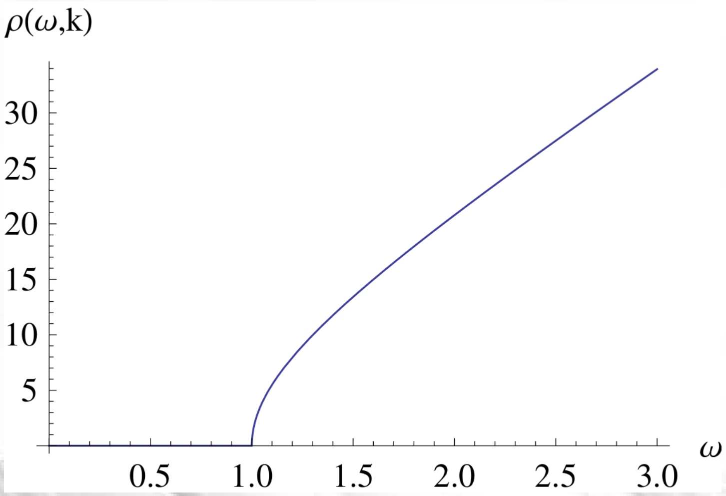
- In thermal equilibrium the Wightman functions (fluctuations) and the retarded correlators (response) are related by the fluctuation dissipation relation

$$G_{-}(\omega, k) = 2n(\omega)\mathrm{Re}\, G_R(\omega, k)$$

$$n(\omega) = \frac{1}{e^{\beta\omega} - 1}$$

- The density of states

$$\rho(\omega, k) = 2\mathrm{Re}\, G_R(\omega, k)$$



3. Details of the calculation

- In the following will consider a scalar field in AdS_3/Vaidya which is dual to some scalar CFT operator

$$S = -\frac{1}{2}\int d^d x\sqrt{-g}\Big(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\Big) \qquad m^2 = -3/4$$

- Will use the “extrapolate dictionary” to obtain boundary theory correlators from the bulk correlators

$$\langle\mathcal{O}(x)\mathcal{O}(x')\rangle = \lim_{z\to 0} z^{-2\Delta}\langle\phi(x,z)\phi(x',z)\rangle$$

- Most of the time will talk about the bulk correlator, the limit to boundary does not do anything special

3. Details of the calculation

- To obtain the bulk Wightman function, we use the Heisenberg equation of motion

$$(\Box - m^2)\phi(X) = 0$$

from which it follows that

$$(\Box_1 - m^2)\langle\phi(X_1)\phi(X_2)\rangle = 0$$
$$(\Box_2 - m^2)\langle\phi(X_1)\phi(X_2)\rangle = 0$$

- The problem is to solve this 6 dimensional PDE with the initial data that the correlator at early times agrees with the known AdS vacuum correlator

$$\langle\phi(X_1)\phi(X_2)\rangle_{AdS} = \frac{1}{|X_1 - X_2|} - \frac{1}{|X_1 - I(X_2)|}$$

3. Details of the calculation

- Some technical points:

1) Since we have spatial translational symmetry, we can perform a spatial Fourier transform → 2x2 dimensional PDE's

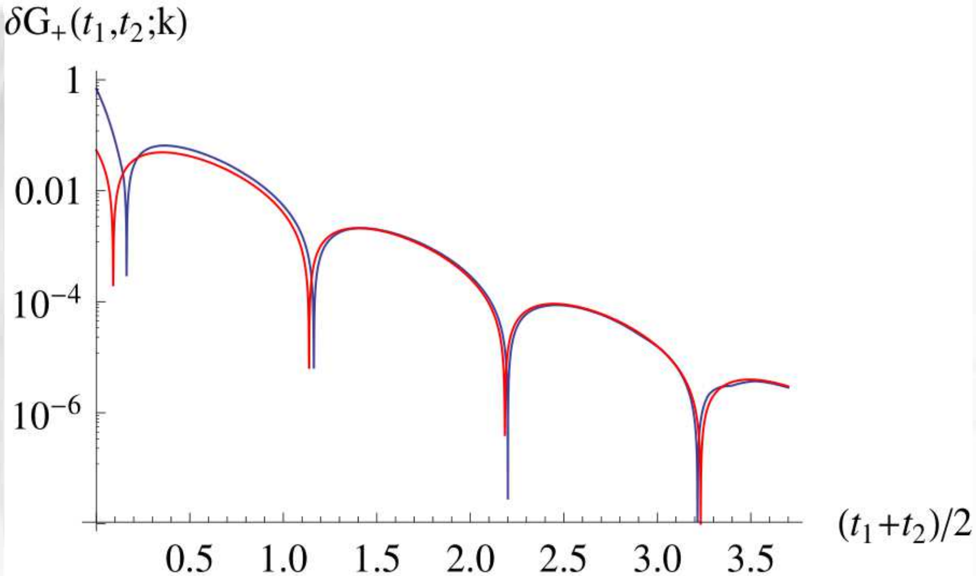
2) The initial data has singularities at short distances → Makes the numerical calculation a bit challenging

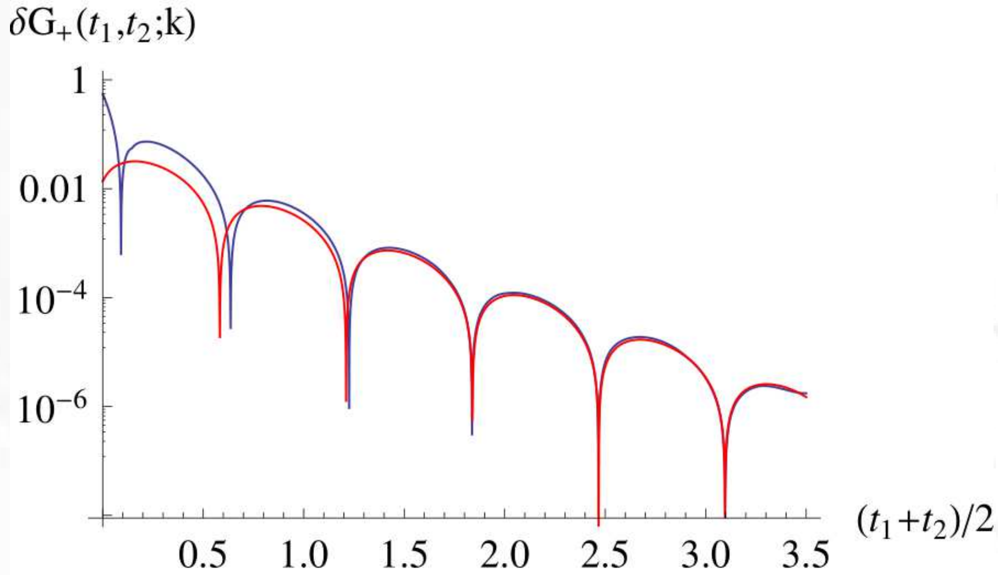
3) We solved this problem by using the method of Green's functions and reduced it to performing integrals over known functions → Integrate numerically and be careful with the short distance singularities

$$G_+(X_4, X_3)$$
$$= -i \int d\mathbf{X}_0\, d\mathbf{X}_1\, d\mathbf{X}_2 \Big(G_+^{AdS}(X_1, X_0) \overleftrightarrow{D}^{v_1} G_R^{BTZ}(X_4, X_1) \Big) \overleftrightarrow{D}^{v_0} \Big(G_R^{AdS}(X_2, X_0) \overleftrightarrow{D}^{v_2} G_R^{BTZ}(X_3, X_2) \Big)$$

4. Results and explanation

- Results:





$$G_+(t_1, t_2; k) - G_+^{(thermal)}(t_1, t_2; k) \propto e^{-i\omega_*(t_1+t_2)} + c.c.$$

$$\omega_* = k - i\Delta$$

- The Wightman function seems to approach the thermal one with a rate given by the lowest quasinormal mode

4. Results an explanation

- Quasinormal modes: Poles of the boundary retarded two point function
- They are actually also poles of the bulk retarded two point function
- Upon Fourier transforming, the poles become exponentials

$$G_R(X_1, X_2) = \sum_n c_n(z_1, z_2; k) e^{-i\omega_*^{(n)}(v_1 - v_2)}$$

- At large timelike separation the retarded correlator decays exponentially with a rate given by the lowest quasinormal mode

4. Results and explanation

- A useful fact about the retarded correlator is that it satisfies the Klein-Gordon equation with a delta function source

$$(\Box - m^2)G_R(X_1, X_2; k) = i\delta(z_1 - z_2)\delta(v_1 - v_2)$$

  and the initial condition

$$D^{v_2} G_R(X_1, X_2; k)|_{v_1 = v_2} = i\delta(z_1 - z_2)$$

- Any initial data for the Klein-Gordon equation can be propagated in time by using the retarded propagator

$$\phi(X_1; k) = -i \int dz_2\, \phi(X_2; k) D^{v_2} G_R(X_1, X_2; k)$$

arXiv.org > hep-th > arXiv:1212.6066

High Energy Physics - Theory

Thermalization of the spectral function in strongly coupled two dimensional conformal field theories

V. Balasubramanian, A. Bernamonti, B. Craps, V. Keränen, E. Keski-Vakkuri, B. Müller, L. Thorlacius, J. Vanhoof

4. Results and explanation

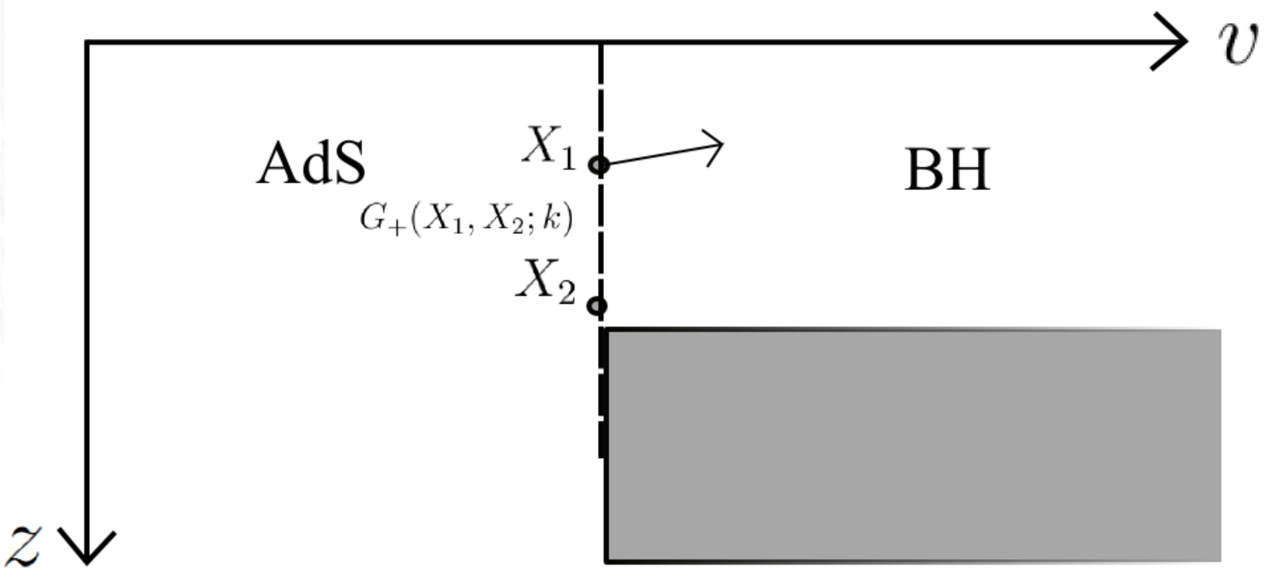
$$\phi(X_1;k) = -i\int dz_2\,\phi(X_2;k)D^{v_2}G_R(X_1,X_2;k)$$

1) Sufficiently s**mooth initial data** decays at late times with the lowest quasinormal mode

2) Initial data with **sharp features near the horizon** takes long time to reach the boundary → Starts decaying only after reflecting from the boundary

4. Results and explanation

- Apply the lessons to the Wightman function



- The initial data has singularities

$$\langle \phi(X_1)\phi(X_2)\rangle_{AdS} = \frac{1}{|X_1 - X_2|} - \frac{1}{|X_1 - I(X_2)|}$$

- Thus, the Wightman function doesn't belong to the 1) category → Good, since it shouldn't decay to zero!

4. Results and explanation

- But consider the difference

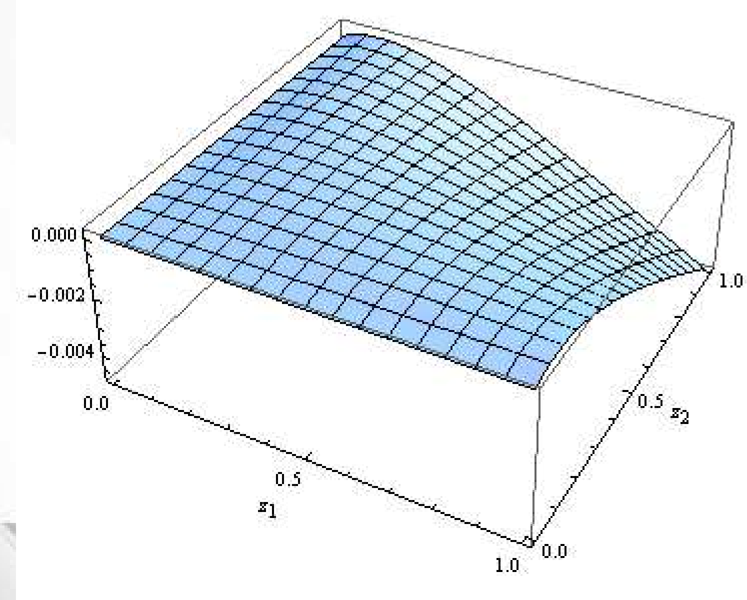
$$\delta G_+(X_1, X_2; k) = G_+(X_1, X_2; k) - G_+^{(thermal)}(X_1, X_2; k)$$

This satisfies the Klein-Gordon equation

$$(\Box - m^2)\delta G_+(X_1, X_2; k) = 0$$

and furthermore has no singularities at the initial time

- This is because the singularities are short distance singularities, which do not depend on the state of the bulk scalar



4. Results and explanation

- Thus, we can use the K-G equation to first evolve it with respect to x_1

$$\delta G_+(X_1, X_2; k) \propto f(X_1, X_2) e^{-i\omega_* v_1}$$

  This provides smooth initial data f(x_2) that we next evolve with respect to x_2 to get

$$\delta G_+(X_1, X_2; k) \propto e^{-i\omega_* v_1} e^{-i\omega_* v_2}$$

  Taking the points to the boundary, we get the desired result

- Nothing in the previous discussion relied in the fact that we are considering AdS_3-Vaidya or on the specific value of the mass → Expect the conclusion to hold for any d and m

5. Occupation numbers

- Recall the fluctuation dissipation relation in thermal equilibrium

$$G_{-}(\omega, k) = 2n(\omega)\mathrm{Re}\, G_R(\omega, k)$$

Thus, in thermal equilibrium the occupation number is given by a ratio of correlation functions

- We would like to define an effective occupation number in the non-equilibrium system similarly as a ratio of correlators
- How does one define the Fourier transforms in this case?

$$G(\omega, k) = \int dx\, dt\, e^{-i\omega t + ikx} G(x, t; 0, 0)$$

5. Occupation numbers

- One way to circumvent the problem: Wigner transform

$$t = t_1 - t_2$$

$$T = \frac{t_1 + t_2}{2}$$

$$G(x_1, t_1; x_2, t_2) = G(x_1 - x_2, t, T)$$

$$G^W(\omega, k; T) = \int dx dt \, e^{-i\omega t + ikx} G(x, t, T)$$

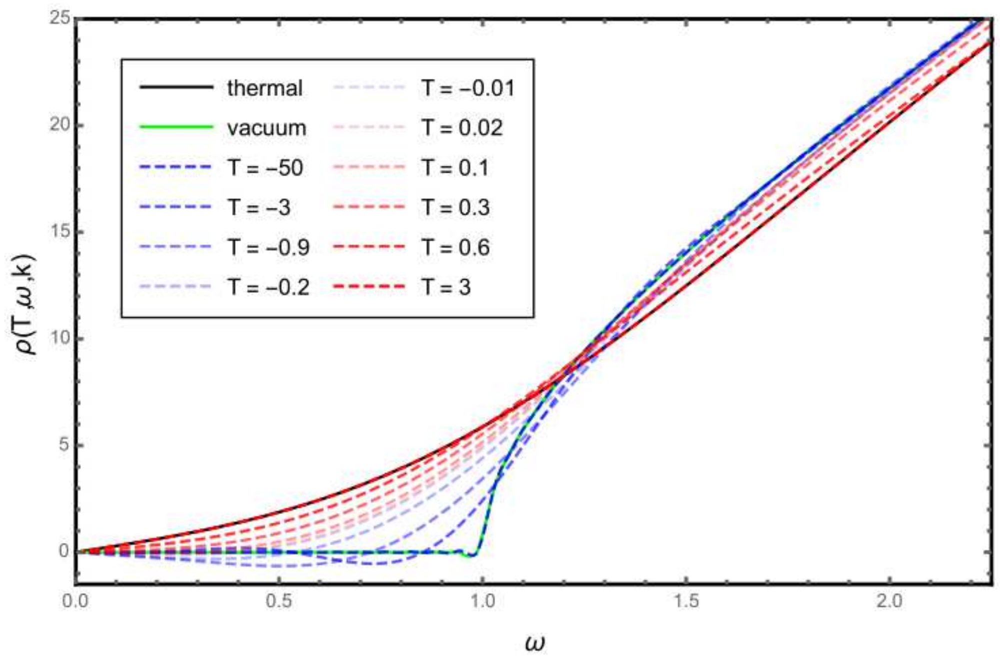
- Define an effective occupation number as

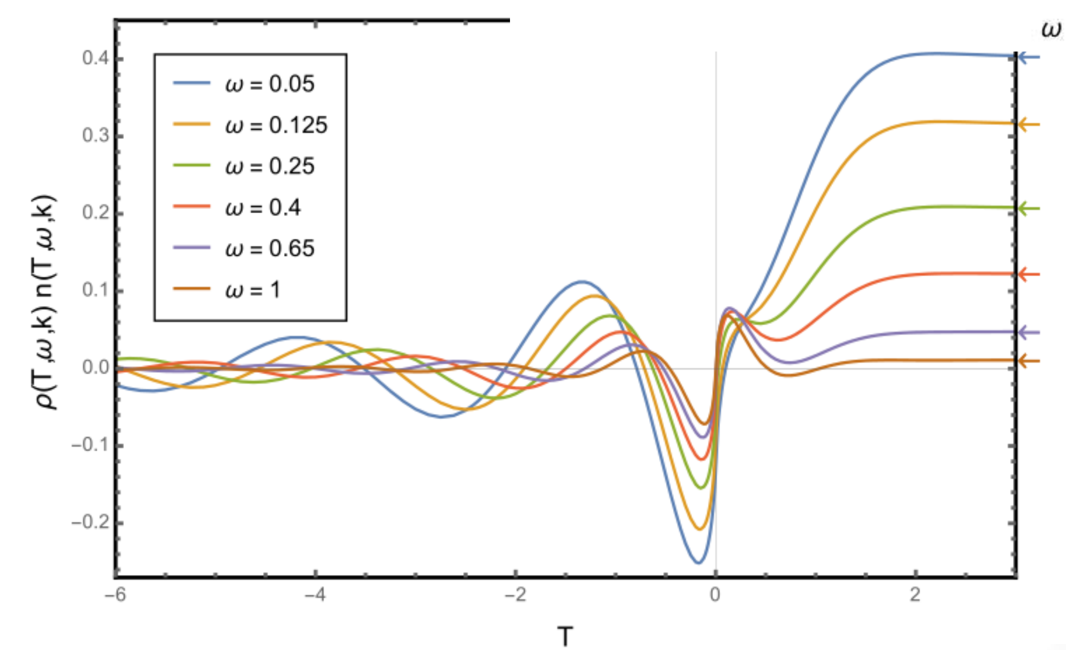
$$n(\omega, k, , T) = \frac{G_-^W(\omega, k, T)}{2\mathrm{Re}\, G_R^W(\omega, k, T)} = \frac{G_-}{\rho}$$

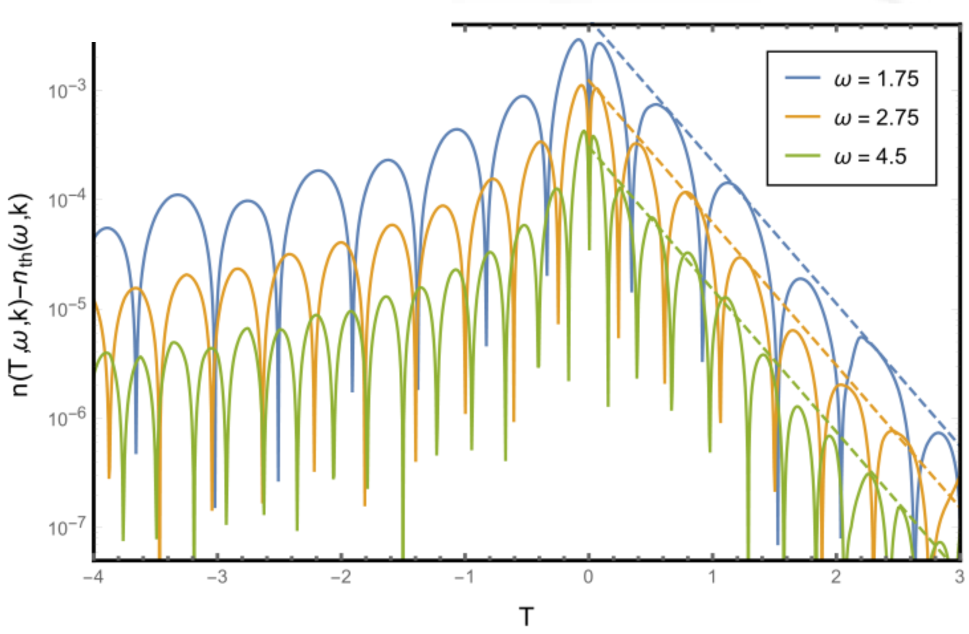
- This is well defined out of equilibrium and reduces to the Bose-Einstein distribution in equilibrium

5. Occupation numbers

- Results

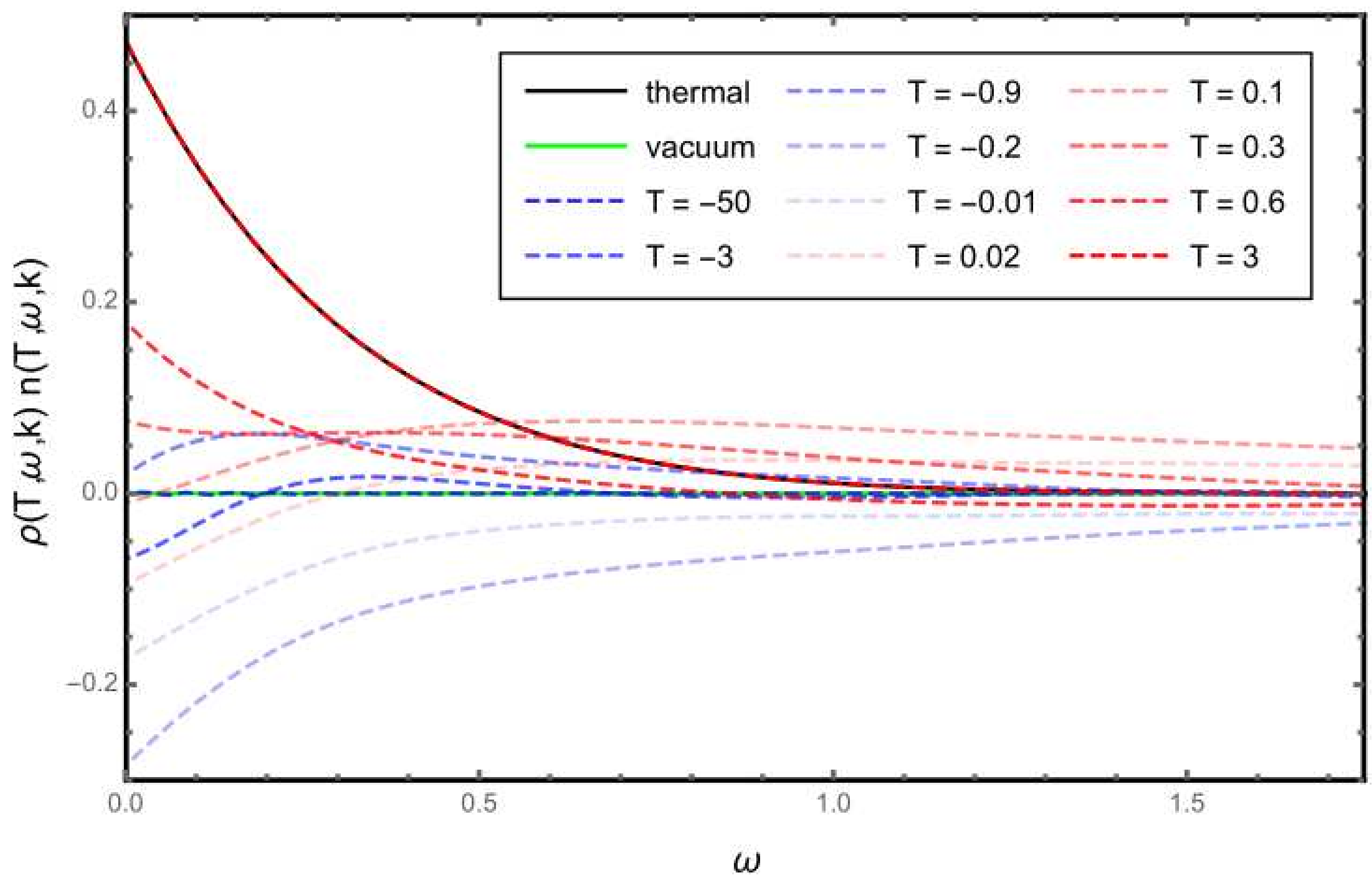






5. Occupation numbers

- Results



6. Conclusions and future directions

- The calculations of Wightman (or Feynman) correlators in holography are very rare
- There are no good simple methods for doing this
- We used a particular method based on the method of Green's functions. This works for Vaidya spacetime with a sharp theta function mass, but it does not generalize to any other non-equilibrium spacetimes
- Thus, new calculational methods are certainly welcome

6. Conclusions and future directions

- We obtained the formula

$$G_+(t_1, t_2; k) - G_+^{(thermal)}(t_1, t_2; k) \propto e^{-i\omega_*(t_1+t_2)} + c.c.$$

and I tried arguing this might be true more generally in AdS-Vaidya than the case we did the numerical calculation for

- Another example where one can easily (analytically) show that the above is true is in the dual of a "Cardy-Calabrese state" with a specific boundary state dual to an RP^2 geon

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arXiv.org > cond-mat > arXiv:cond-mat/0503393

Condensed Matter > Statistical Mechanics

Evolution of Entanglement Entropy in One-Dimensional Systems

Pasquale Calabrese, John Cardy

arXiv.org > hep-th > arXiv:1412.1084

High Energy Physics - Theory

Behind the geon horizon

Monica Guica, Simon F. Ross

6. Conclusions and future directions

- Let's assume that it is generally true in collapsing spacetimes and see what it would imply

  1) Thermalization time scale becomes easy to calculate: just calculate the quasinormal modes of the final static black hole

  2) The "top-down" pattern of holographic thermalization that has been around in the literature is incorrect for low scaling dimension operators → It is "bottom-up"

  3) CFT correlation functions in far from equilibrium states thermalize with the same rate as (infinitesimally) small perturbations around thermal equilibrium

Thanks for listening!

Possible new methods

- One possibility: “mode function approach”

$$\hat{\phi}(x) = \int d\omega dk \Big(f(\omega, k, t, z, x)\hat{a}_{\omega,k} + h.c. \Big)$$

  Start from some state where the mode functions and the expectation values of the creation operators are known (e.g. vacuum or thermal equilibrium)

- The function f is a solution to the Klein-Gordon equation that is straightforward to solve numerically in any background spacetime

- The initial mode function is some known smooth function (e.g. a Bessel in the case of AdS vacuum)

Possible new methods

- Everything works fine until one plugs the expressions into a correlation function. One gets expressions of the form

$$\langle \phi(x_1)\phi(x_2)\rangle \propto \int d\omega dk\, f^*(x_1,\omega,k) f(x_2,\omega,k)$$

  Now the problem is that these integrals do not converge very fast at large values of the momenta, while numerics usually breaks down at large momenta

- I don't know how to proceed
- Perhaps one could do something analytically at large momenta and then patch that to the numerical solution…