Non-Equilibrium 2-point functions in AdS/CFT: Formalism and an example

Ville Keränen
University of Oxford

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Motivation

- The purpose of today's talk is to study far from equilibrium processes in AdS/CFT
- Learn more about the duality in extreme environments
  - Learn more about black holes
  - Learn more about the dictionary between bulk and boundary
  - Test whether the duality gives sensible results
- Tool for strongly coupled dynamics in QFT
  - Search for universality at strong coupling
  - Experimental systems to keep in mind: Cold atom systems, condensed matter systems, Heavy ion collisions etc.
Non-Equilibrium example

- Take a QFT and prepare it in the vacuum state
- Excite the system at $t=0$ homogeneously in space $\rightarrow$ injects a finite energy density into the system
- E.g. a time dependent coupling

$$H = H_0 + \lambda(t) \int dx \mathcal{O}(x).$$

Sometimes called a “global quench”

- Want to preserve spatial translational and rotational symmetries for simplicity
- Non-trivial dynamics is mainly in non-local observables
Non-Equilibrium example

- AdS version of the previous setup

\[ H = H_0 + \lambda(t) \int d^4x \mathcal{O}(x). \]

\[ S = S_0 - \int d^4x J(t) \mathcal{O}(x). \]

- Sources are dual to boundary values of fields
- Start from the vacuum (= AdS) → Suddenly perturb a boundary value of a field → The perturbation starts falling deeper to the bulk and forms a black brane
- A simple analytic model for this process is provided by the Vaidya spacetime, which corresponds to a null shock wave starting from the boundary and forming an AdS-Swarchild black brane
AdS version of the story

- Dictionary:
  - Thermalization $\leftrightarrow$ Black hole formation
  - Thermalization time scale? $\leftrightarrow$ When does the black hole form?
    (In gravity there is no preferred time coordinate, so there is no one correct answer)
  - More precise question: When/How does a specific observable thermalize? Choose to look at correlation functions of local operators.

$$H = H_0 + \lambda(t) \int d\mathbf{x} \mathcal{O}(\mathbf{x}).$$
Outline

1. On correlation functions
   • Heuristic picture of correlators
   • Formalism in non-equilibrium QFT

2. AdS/CFT dictionary out of equilibrium
   • Review of different dictionaries
   • Sketch of a proof of equivalence of the two “best” dictionaries

3. Explicit example of 2-point functions in a collapsing spacetime
   • Method
   • Results
Correlation functions

- Example 1: the Harmonic Oscillator

\[ H = \frac{1}{2}p^2 + \frac{1}{2}x^2 \]

- Classical ground state \( x=0 \)
- Quantum ground state

\[ \psi(x) = Ce^{-x^2/2} \]

- Prepare the same ground state and measure the position of the particle: On average find \( \langle x \rangle = 0 \), but due to quantum fluctuations the single measurements give non-zero values and the distribution of measured values has a width

\[ \Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \int dx x^2 |\psi(x)|^2 = \frac{1}{2} \]
Correlation functions

- Example 2: Free scalar QFT

\[ H = \frac{1}{2} \int dx \left( \Pi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right) \]

- Quantum ground state is again a Gaussian (as we are dealing with a set of coupled harmonic oscillators)

\[ \Psi [\phi] = \mathcal{N} e^{-\frac{1}{4} \int dx dy \phi(x) K(x,y) \phi(y)} \]

\[ K(x,y) = 2 \int \frac{dk}{(2\pi)^D} \sqrt{k^2 + m^2} e^{ik \cdot (x-y)} \]

- By measuring the field at two spatially separated points \( x \) and \( y \), and recording the measured values can construct

\[ \langle \phi(x) \phi(y) \rangle = \int [d\phi] |\Psi [\phi]|^2 \phi(x) \phi(y) = K^{-1}(x,y) \approx e^{-m|x-y|} \]

- This tells us two things, the measured values at spatially separated points are correlated (due to entanglement) and the wavefunction has a width due to quantum mechanics
Correlation functions

- Example 3: Free scalar QFT with a classical source

\[ H = \frac{1}{2} \int dx \left( \Pi^2 + (\nabla \phi)^2 + m^2 \phi^2 + 2J\phi \right) \]

- Treat the current term as an interaction and use the Dirac interaction picture. Then states evolve as

\[ |\psi(t)\rangle = T(e^{-i \int_0^t dt'dx' J(x')\phi(x')}|\psi(0)\rangle \]

- What is the average value of the field at some point \( x \), after turning on a small source?

\[ \langle \phi(x) \rangle = \langle \psi(0)|T(e^{i \int_0^t dt'dx' J(x')\phi(x')}\phi(x)T(e^{-i \int_0^t dt'dx' J(x')\phi(x')})|\psi(0)\rangle \]

\[ \approx \langle \psi(0)|\phi(x)|\psi(0)\rangle - i \int dt'dx' J(x')\theta(t-t')\langle \psi(0)|[\phi(x), \phi(x')]|\psi(0)\rangle \]
Correlation functions

- For a localized source \( J(x) = J_0 \delta(x) \)

\[
\delta \langle \phi(x) \rangle = -i J_0 G_R(x, 0)
\]

- The analogous question in QED would be: turn on a current in the light bulb at \( x'=0 \), what is the amount of light you will see at \( x \)?

- The retarded correlator quantifies response
Correlation functions

- Lessons:
  
  - Different correlation functions answer to different physical questions:
    
    One point functions = Average results for observables
    Spacelike separated correlator = Quantum fluctuations in the state
    Retarded correlator = Response of the system
Correlation functions: Formalism

- In the following will work in the Heisenberg picture

\[ |\psi(t)\rangle = |\psi(t_i)\rangle \quad \quad A(t) = U^\dagger(t, t_i) A(t_i) U(t, t_i) \]

- Consider the two point function

\[
\langle \psi | A(t_2) B(t_1) | \psi \rangle = \langle \psi | U^\dagger(t_2, t_i) A(t_i) U(t_2, t_i) U^\dagger(t_1, t_i) B(t_i) U(t_1, t_i) | \psi \rangle \\
= \langle \psi | U^\dagger(t_2, t_i) A(t_i) U(t_2, t_1) B(t_i) U(t_1, t_i) | \psi \rangle
\]

- Important to notice the time-evolution backwards in time (in particle physics often consider amplitudes from initial to final states so there is only forwards time-evolution)
Correlation functions: Formalism

- Apply this to QFT and go to the path integral formalism

\[
\langle \psi | A(t_2) B(t_1) | \psi \rangle = \langle \psi | U^\dagger (t_2, t_i) A(t_i) U(t_2, t_1) B(t_i) U(t_1, t_i) | \psi \rangle
\]

\[
= \int [d\varphi_+, d\varphi_-] e^{iS[\varphi_+] - iS[\varphi_-]} \langle \psi | \varphi_-(t_i) \rangle \langle \varphi_+(t_i) | \psi \rangle A(\varphi(t_2)) B(\varphi(t_1))
\]

- For the moment, the initial state wavefunction

\[
\langle \varphi_+ | \psi \rangle = \Psi [\varphi_+]
\]

is arbitrary. It is our initial data.

- Can also define a generating functional, from which correlation functions are obtained by differentiation

\[
Z [J_+, J_-] = \int [d\varphi_+, d\varphi_-] e^{iS[\varphi_+] - iS[\varphi_-] + i \int (J_+ \varphi_+ + J_- \varphi_-) \langle \psi | \varphi_-(t_i) \rangle \langle \varphi_+(t_i) | \psi \rangle}
\]
Correlation functions: Formalism

- Example of an initial state wavefunction: the ground state

- Consider the following quantity

\[ \langle \varphi | e^{-\tau H} | \varphi_0 \rangle = \sum_n e^{-\tau E_n} \langle \varphi | n \rangle \langle n | \varphi_0 \rangle \]

- In the large tau limit this is dominated by the ground state, and thus

\[ \langle \varphi | 0 \rangle \propto \lim_{\tau \to \infty} \langle \varphi | e^{-\tau H} | \varphi_0 \rangle = \lim_{\tau \to \infty} \int [d\varphi_E] |_{\varphi_E(\tau=0)=\varphi} e^{-\int_{-\tau}^{0} d\tau dx \mathcal{L}_E(\varphi_E)} \]

- By adding non-trivial sources to the Euclidean action, one can prepare more general states. In the following specialize to states that can be prepared in this way
Correlation functions: Formalism

- Collecting all the pieces we obtain the generating functional

$$Z[J_+, J_-] = \int [d\varphi_C] e^{iS_C[\varphi_C] + i \int_C \phi_C J_C}$$

- Non-equilibrium correlators can be calculated from a generating functional that is obtained by gluing together Euclidean and Lorentzian spacetimes and performing a path integral over the fields in all of the parts.
The AdS/CFT dictionary

- Recall the standard AdS/CFT dictionary

\[ Z_{CFT} [J] = \int [d\Phi] e^{-S_E[\Phi]} \bigg|_{\Phi=z^{\Delta} - J + ...} \]

- Where all the bulk fields are denoted as

\[ \Phi = (g, A_\mu, \phi, ...) \]

- At weak coupling (large-N in CFT), can perform a saddle point approximation

\[ Z_{CFT} [J] \approx e^{-S_E[\Phi_{cl}]} \quad \frac{\delta S_E}{\delta \Phi} \bigg|_{\Phi=\Phi_{cl}} = 0 \]

- Leads to a well posed problem as the boundary sources are enough to determine the unique classical solution, since the equations of motion are elliptic
The AdS/CFT dictionary

- For some Lorentzian situations (ground state or thermal state) one can take the Euclidean correlator and analytically continue it to Lorentzian time.

- One way to generalize the dictionary to non-equilibrium situations is to build a holographic version of the complex time contour path integral. An obvious candidate dictionary is

\[ Z_{\text{CFT}}[J_+, J_-] = \int [d\Phi_C] e^{iS_C[\Phi_C]} \bigg|_{\Phi_\pm = z^\Delta - J_\pm + \ldots} \]
The AdS/CFT dictionary

- Again at weak coupling in the bulk, we can perform a saddle point approximation

\[
Z_{CFT} \left[ J_+, J_- \right] \approx e^{iS[\Phi_+] - iS[\Phi_-] - S_E[\Phi_{E,1}] - S_E[\Phi_{E,2}]}
\]

\[
\frac{\delta}{\delta \Phi} \left( iS[\Phi_+] - iS[\Phi_-] - S_E[\Phi_{E,1}] - S_E[\Phi_{E,2}] \right) = 0
\]

- Variations on the Lorentzian parts leads to Lorentzian eoms. Variations at the Euclidean parts leads to Euclidean eoms. Variations at the joining surfaces lead to “matching conditions”.
The AdS/CFT dictionary

- In the following we will assume that the metric has been appropriately matched and consider a free scalar field in this metric background.

- The Euclidean on-shell action of a scalar field can be written in the following form, where $K$ is the inverse of the equal time two point function:

$$S_E = \frac{1}{4} \int_{\mathbb{R}_\mathbb{E}} d\mathbf{x}_1 d\mathbf{x}_2 \, \phi(\mathbf{x}_1) K(\mathbf{x}_1, \mathbf{x}_2) \phi(\mathbf{x}_2)$$

- Using this the equations of motion become:

$$\left(\Box - m^2\right) \phi_{\pm} = 0$$

$$D^t \phi_+(\mathbf{x}, t)|_{t=t_m} = D^t \phi_-(\mathbf{x}, t)|_{t=t_m}$$

$$D^t \phi_+(\mathbf{x}, t)|_{t=t_i} = -\frac{i}{2} \int d\mathbf{x}_1 \phi_+(\mathbf{x}_1, t_i) K(\mathbf{x}_1, \mathbf{x})$$

$$D^t \phi_-(\mathbf{x}, t)|_{t=t_i} = \frac{i}{2} \int d\mathbf{x}_1 \phi_-(\mathbf{x}_1, t_i) K(\mathbf{x}_1, \mathbf{x})$$

[Diagram of AdS space with $E$, $L$, $E$, and $L$]
The AdS/CFT dictionary

- There is also another independent version of the AdS/CFT dictionary where one identifies bulk and boundary operators

\[ \hat{O}(x) = \lim_{z \to 0} z^{-\Delta} \hat{\Phi}(x, z) \]

- In addition, to calculate correlation functions, one has to make a map between bulk and boundary theory state

- For the states that can be prepared with a Euclidean path integral, this map is the same as before, the bulk wavefunction of the quantum field is

\[ \Psi[\Phi] = \mathcal{N} e^{-S_E[\Phi_E]} \bigg|_{\Phi_E(\tau=0)=\Phi} \]

- This is the “extrapolate” dictionary, and is simpler to use in practice
The AdS/CFT dictionary

- We have two versions of the AdS/CFT dictionary, that are supposed to make sense in non-equilibrium situations in a class of initial states.

- There are three options:
  - Both of them are wrong
  - One of them is correct and the other one wrong
  - Both of them are correct and lead to the same result

- We will argue that in the case of a free scalar, they lead to the same results. We take this as evidence for the third option.
The AdS/CFT dictionary

• We will prove the equivalence by constructing a solution to the equations of motion following from the gluing approach.

• The ansatz for the solution is motivated by the “extrapolate” dictionary.

• First we need to slightly reformulate the problem.

\[(\Box - m^2)\phi_{\pm} = 0\]
\[D^t \phi_+(x, t)|_{t=t_i} = -\frac{i}{2} \int d\mathbf{x}_1 \phi_+(x_1, t_i)K(x_1, x)\]
\[D^t \phi_(x, t)|_{t=t_i} = \frac{i}{2} \int d\mathbf{x}_1 \phi_-(x_1, t_i)K(x_1, x)\]
\[\phi_{\pm}(x_B, z) = z^{\Delta_{\pm}} J_{\pm}(x_B) + \ldots\]

• A standard approach to solving equations like this is to define bulk to boundary propagators (in the following take \( J_- = 0 \)).

\[\phi_+(x) = \int d\mathbf{x}_B K_{++}(x, x_B) J_+(x_B)\]
\[\phi_-(x) = \int d\mathbf{x}_B K_{--}(x, x_B) J_-(x_B)\]
The AdS/CFT dictionary

- The bulk to boundary propagators have to satisfy all the same equations of motion as the scalar field itself, except that the boundary condition near the AdS boundary is different

\[ K_{\alpha\beta}(x_1, z_1; x_2) = z_1^{d-1} \delta_{\alpha\beta} \delta^{d-1}(x_1 - x_2) + z_1^\Delta K_{\alpha\beta}^{(1)} + \ldots \]

- The gluing dictionary leads to the correlators

\[ \langle T\mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \lim_{z_1 \to 0} z_1^{-\Delta} K_{++}(x_1, z_1; x_2) \]

\[ \langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \lim_{z_1 \to 0} z_1^{-\Delta} K_{--}(x_1, z_1; x_2) \]

- On the other hand the corresponding correlators according to the “extrapolate” dictionary are

\[ \langle T\mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \lim_{z_1, z_2 \to 0} z_1^{-\Delta} z_2^{-\Delta} \langle T\phi(x_1, z_1)\phi(x_2, z_2) \rangle \]

\[ \langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \lim_{z_1, z_2 \to 0} z_1^{-\Delta} z_2^{-\Delta} \langle \phi(x_1, z_1)\phi(x_2, z_2) \rangle \]
The AdS/CFT dictionary

- Assuming that the dictionaries are equivalent leads to the identities

\[ K_{++}(x_1, z_1; x_2) = \lim_{z_2 \to 0} z_2^{-\Delta} \langle T \phi(x_1, z_1) \phi(x_2, z_2) \rangle \]

\[ K_{--}(x_1, z_1; x_2) = \lim_{z_2 \to 0} z_2^{-\Delta} \langle \phi(x_1, z_1) \phi(x_2, z_2) \rangle \]

- So proving the equivalence of the dictionaries is equivalent to showing that the above K's satisfy all the equations of motion arising from the gluing construction
The AdS/CFT dictionary

- It is clear that they satisfy the correct bulk equation of motion as

\[(\Box - m^2)\langle T\phi(x_1)\phi(x_2)\rangle = i\delta(x_1 - x_2)\]

\[(\Box - m^2)\langle \phi(x_1)\phi(x_2)\rangle = 0\]

- The initial and final conditions are the trickiest to show. There need to use the fact that the kernel K in the wavefunction is the inverse of the bulk to bulk correlator.

- The delta function boundary condition at AdS boundary follows from the delta function on the right hand side of the Klein-Gordon equation for the bulk correlator.

- This is the proof.
AdS-Vaidya correlator

- Consider the example in the beginning

- The Vaidya spacetime provides a simple analytic example of the above process

\[ ds^2 = \frac{1}{z^2} \left[ -(1 - \theta(v) z^2) dv^2 - 2dv\,dz + dx^2 \right] \]

- By itself this does not solve the vacuum Einstein's equations, but needs a source. In a realistic case, this would be a scalar field that is collapsing, and the theta function would be a smooth function.
AdS-Vaidya correlator

- We will want to work out the correlation functions in this spacetime.
- Energy-momentum tensor one point functions become time independent immediately
- Consider a scalar field
  \[ S = -\frac{1}{2} \int d^d x \sqrt{-g} \left( \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) \]
- The simplest case is when \( m^2 = -3/4 \) (there is a hidden Weyl symmetry in this case)
- We want the two point function of the scalar. Use the extrapolate dictionary, and work in the Heisenberg picture. State is the initial AdS vacuum.
We choose to calculate the time ordered 2-point correlator (all others can be obtained from this one)

\[ G_F(x_2, x_1) = \langle \psi | T\phi(x_2)\phi(x_1) | \psi \rangle \]

From the Heisenberg equation of motion, it follows that

\[(\Box_1 - m^2) G_F(x_2, x_1) = i \frac{\delta(x_2 - x_1)}{\sqrt{-g}} = (\Box_2 - m^2) G_F(x_2, x_1)\]

Thus, we are lead to solve a 6 dimensional PDE.

The initial data is given by the initial state (the AdS vacuum)

\[ G_{F}^{\text{AdS}}(v_2, x_2, z_2; v_1, x_1, z_1) = \frac{1}{4\sqrt{z_1 z_2}} \left( \frac{1}{\sqrt{-(v_2 - v_1)^2 - 2(v_2 - v_1)(z_2 - z_1) + (x_2 - x_1)^2 + i\epsilon}} - \frac{1}{\sqrt{-(v_2 - v_1)^2 - 2(v_2 - v_1)(z_2 - z_1) + 4z_1 z_2 + (x_2 - x_1)^2 + i\epsilon}} \right) \]
AdS-Vaidya correlator

- Since the eom is linear we can use the method of Green's functions

\[ G_F(x_3, x_1) = i \int_{t_2=\text{const}} dx_2 \, G_F(x_2, x_1) \overrightarrow{D}^{t_1} G_R(x_3, x_2) \]

- We will use the above formula 3 times as follows

\[ G_F(x_4, x_3) = -i \int dx_2 dx_1 dx_0 \left( G_F(x_1, x_0) \overrightarrow{D}^{t_1} G_R(x_4, x_1) \right) \overrightarrow{D}^{t_0} \left( G_R(x_2, x_0) \overrightarrow{D}^{t_2} G_R(x_3, x_2) \right) \]
AdS-Vaidya correlator

- This is useful because the retarded correlator happens to be independent of the initial state (a proof in the next slide)
- Thus, we can use the thermal retarded correlator in the black hole part, which is analytically known

\[ G_{BTZ}^{F}(v_2, x_2, z_2; v_1, x_1, z_1) \]

\[
= \sqrt{\frac{z_1 z_2}{32\pi^2}} \left( \frac{1}{\sqrt{\cosh(x_2 - x_1) - z_1 z_2 - (1 - z_1 z_2) \cosh(v_2 - v_1) - (z_2 - z_1) \sinh(v_2 - v_1) + i\epsilon}} \right)
\]

\[
- \sqrt{\cosh(x_2 - x_1) + z_1 z_2 - (1 - z_1 z_2) \cosh(v_2 - v_1) - (z_2 - z_1) \sinh(v_2 - v_1) + i\epsilon}} \right)
\]
AdS-Vaidya correlator

- The retarded correlator is independent of the state because
  1) It satisfies a second order differential equation
  2) The initial data is all determined by the equal time commutation relations

\[ G_R(x_2, x_1) = \theta(t_2 - t_1) \langle [\phi(x_2), \phi(x_1)] \rangle \]

- At equal times we thus have

\[ G_R = 0 \]
AdS-Vaidya correlator

- The task is to compute the 6 dimensional integral

\[ G_F(x_4, x_3) = -i \int d^2 x_1 d^4 x_0 \left( G_F(x_1, x_0) \mathcal{D}^{t_1} G_R(x_4, x_1) \right) \mathcal{D}^{t_0} \left( G_R(x_2, x_0) \mathcal{D}^{t_2} G_R(x_3, x_2) \right) \]

- Technical details:
  - The integrand has singularities at lightlike separated points
  - It is better to Fourier transform to k-space, which gets rid of 3 integrals and softens the lightcone divergences to logarithmic
AdS-Vaidya correlator

- Blue curve = AdS vacuum correlator
- Green curve = BTZ thermal correlator

- The real part is close to the vacuum, while the imaginary part is thermal right away

\[ v_2 = 0.051, \quad v_1 = 0.001 \text{ and for } z_1 = 0.10 \text{ and } z_2 = 0.20, \]

\[ G_R(x_2, x_1) = 2i\theta(t_2 - t_1) \Im G_F(x_2, x_1) \]
AdS-Vaidya correlator

- Fast (exponential?) approach to thermality. Smallest momentum has the slowest approach.

\[
\langle T \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \lim_{z_1, z_2 \to 0} z_1^{-\Delta} z_2^{-\Delta} \langle T \phi(x_1, z_1) \phi(x_2, z_2) \rangle
\]
AdS-Vaidya correlator

- At the qualitative level the results are explained by a simple geodesic estimate

\[ G(x_2; x_1) \propto e^{-mL[x_{cl}]} \]

\[ G(x, t_2; 0, t_1) = \begin{cases} 
G_{th}(x, t_2; 0, t_1), & |x| < t_1 + t_2 \\
\frac{1}{|x|^{2\Delta} (\cosh(t_1/2) \cosh(t_2/2))^{2\Delta}}, & |x| > t_1 + t_2
\end{cases} \]
Conclusions and open questions

- There are two versions of the AdS/CFT dictionary that are suitable for non-equilibrium settings (for a class of initial states that can be prepared with a Euclidean path integral).
- For a free scalar, the dictionaries agree.
- For 2-point functions of bulk gauge fields and gravity the previous proof probably goes through. One has to work out the appropriate gauge fixings etc.
- For higher point functions could possibly do a perturbative proof of equivalence.
- Also a non-perturbative proof using path integrals is possible, and has been done in Euclidean time for interacting bulk quantum scalar fields.
Conclusions and open questions

- What about states that cannot be prepared with Euclidean path integrals?
- What is the CFT dual of the bulk wavefunction?