Anomalies of the Entanglement Entropy in Chiral Theories

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Quantum Entanglement

A quantum state in Hilbert space contains a great deal of information.

Much of this information does not obviously have a simple classical analog: it is stored in patterns of entanglement.

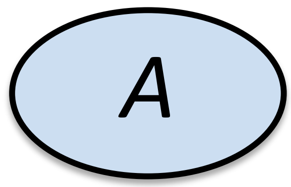
$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle \uparrow\rangle \right)$$

How can we organize this information?

Entanglement entropy

In this talk we will study entanglement entropy.

Consider a general QFT_d in some state, and a spatial region *A* in it.



Construct the reduced density matrix ρ_A by tracing out everything not in *A*.

The entanglement entropy is:

$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A$$

Anomalies and entanglement

Now sometimes a classical symmetry does not survive quantization: anomaly, e.g.

Axial anomaly: $$\partial_\mu j_5^\mu = c_A F \wedge F$$

Weyl anomaly: $$T_\mu^\mu = \frac{c}{12} R$$

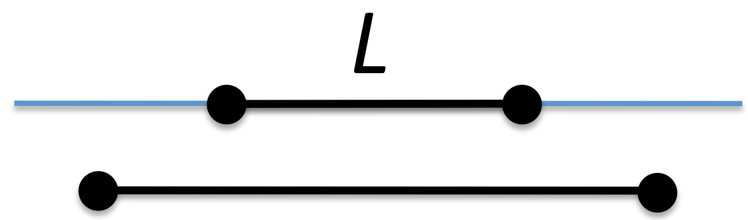
In this talk we will discuss the connection between anomalies and entanglement entropy.

Why should there be such a connection?

Entanglement entropy in 2d CFT

For example, consider the entanglement entropy of an interval in the vacuum of a 2d CFT:

$$S(L) = ?$$



Naive: in a CFT, all lengths are the same, so S(L) should not depend on *L*.

Not true (Holzhey, Larsen, Wilczek; Cardy, Calabrese):

$$S(L) = \frac{c}{3} \log\left(\frac{L}{a}\right)$$

This famous formula is an example of the interplay of the Weyl anomaly and entanglement.

In this talk we will extend similar ideas to other kind of anomalies.

1. Introduction
2. Gravitational anomalies and entanglement entropy in 2d: field theory
3. Mixed anomalies and entanglement entropy in 4d: field theory
4. Gravitational anomalies and entanglement entropy in 2d: holography
5. Future directions

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Gravitational Anomalies

We discussed examples of classical symmetries (U(1) axial, Weyl-invariance) that do not survive quantization.

When diffeomorphism invariance is such a symmetry, we have a gravitational anomaly, e.g. in 1+1d:

$$\nabla_\mu T^{\mu\nu} = c_g \, g^{\mu\nu} \epsilon^{\rho\sigma} \partial_\rho \partial_\beta \Gamma^\beta_{\mu\sigma}$$

↑ Anomaly coefficient

This is equivalent to (i.e. can be traded for) a Lorentz anomaly:

$$\tilde{T}^{\mu\nu} - \tilde{T}^{\nu\mu} = c_g \epsilon^{\mu\nu} R$$

Some examples:

A left-moving Weyl fermion in 1+1d:

$$S = \int dz d\bar{z}\ \psi \partial \psi \qquad c_g = \frac{1}{96\pi}$$

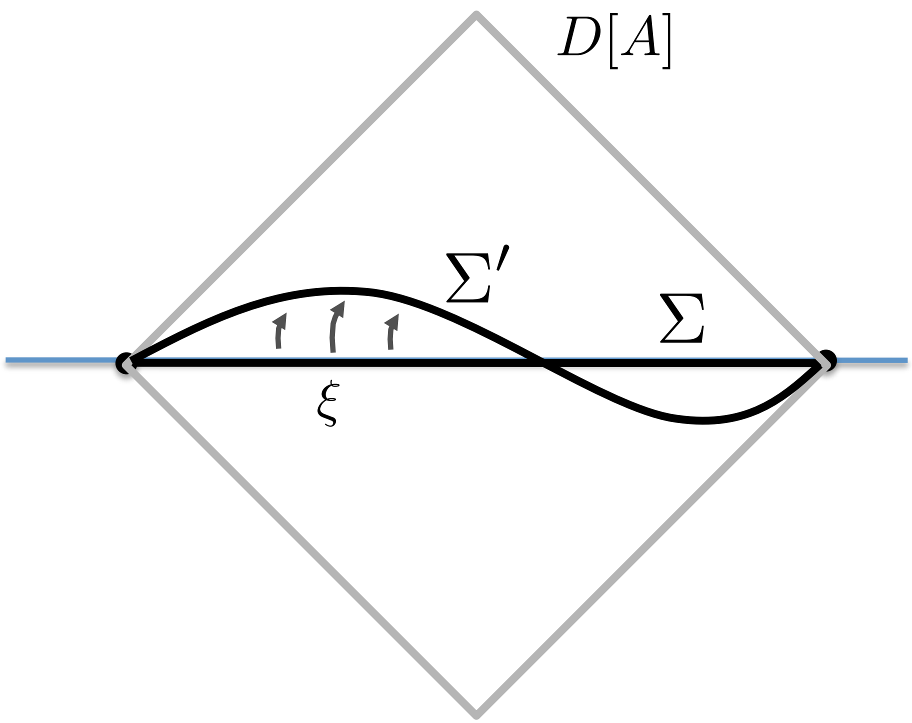
In fact any 2d CFT with an unequal number of right-moving and left-moving degrees of freedom has such an anomaly:

$$c_g = \frac{c_L - c_R}{96\pi}$$

Note that gravity here is not dynamical: thus energy non-conservation may be weird but is perfectly allowed.

Entanglement and causal domains

Normally, we think of the entanglement as being associated with a spatial region A.



In a diff-invariant theory, it is actually a property of the causal domain D[A] of A; does not care if we use Σ or Σ'.

However, in a theory with a gravitational anomaly, this is no longer true; it turns out to depend on the coordinate system used to regulate the theory.

We will call this phenomenon an entanglement anomaly.

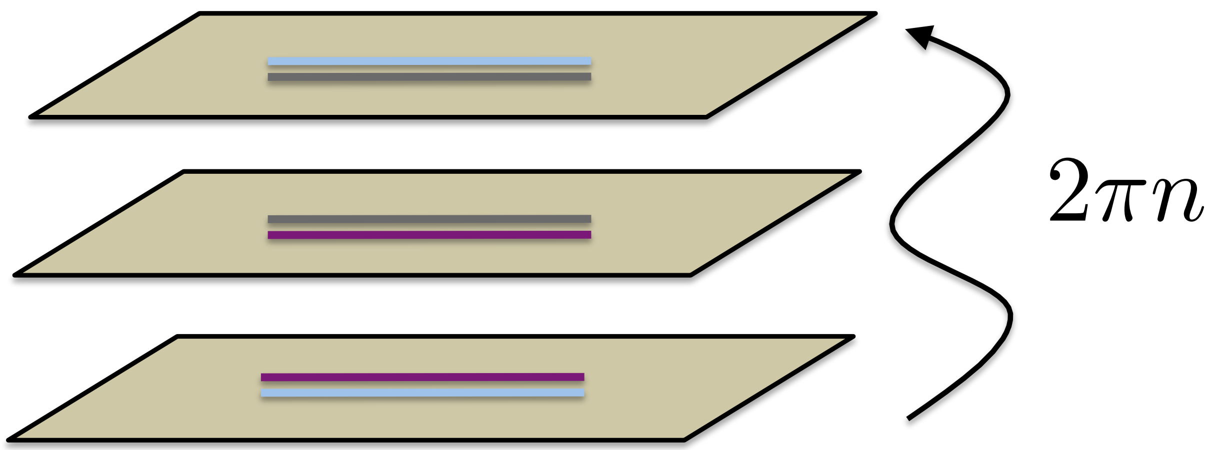
Computing the entanglement anomaly

Let's derive an explicit formula for the transformation of the EE under a diffeomorphism.

Renyi entropy: compute from the partition function on a funky manifold.

$$S_n = -\frac{1}{n-1}\log \mathrm{Tr}(\rho^n) \qquad \mathrm{Tr}(\rho^n) \sim Z[g^{(n)}]$$

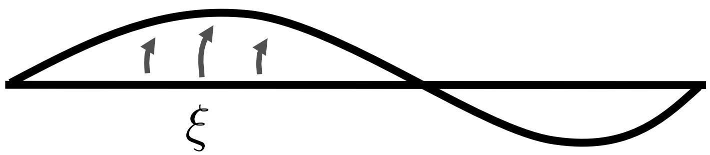
$$g^{(n)} =$$



How does this change under a diffeomorphism?

Computing the entanglement anomaly I

Under a small diff, partition function transforms:

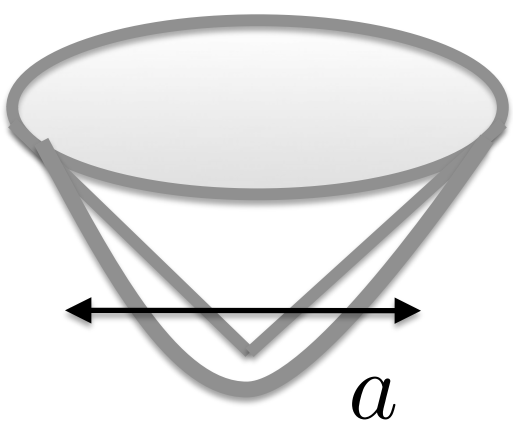


$$\delta_\xi \log Z \sim \int_{\mathcal{M}_2} \nabla_\mu T^{\mu\nu} \xi_\nu + \text{bdy}$$

In a theory with an anomaly, this can be explicitly calculated from the anomaly equation.

$$\nabla_\mu T^{\mu\nu} \sim c_g \partial^2 \Gamma$$

Contribution comes from endpoints: regulate conical surplus over a region *a*, evaluate Christoffel connection.

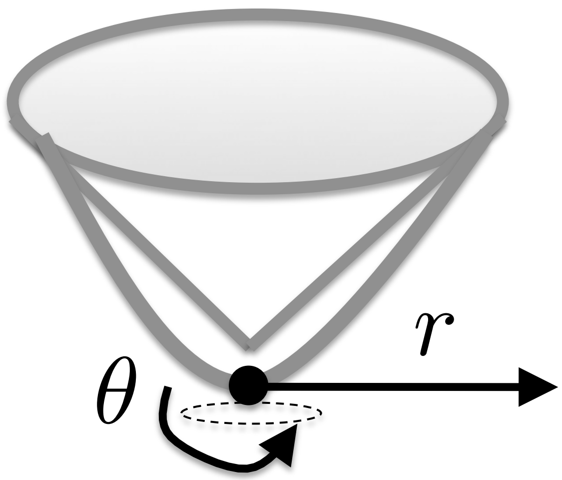


$$\delta S^{\text{bulk}} = \sum_{i \in \partial A} 4\pi c_g \epsilon^{\mu\nu} \nabla_\mu \xi_\nu (x_i)$$

Computing the entanglement anomaly II

We are not done: what about the boundary term?

$$\delta_\xi \log Z \sim \text{bulk} + \int_{\partial \mathcal{M}_2} T^{\mu\nu} n_\mu \xi_\nu$$



$$ds^2 = f\left(\frac{r}{a}\right)^2 dr^2 + r^2 d\theta^2$$

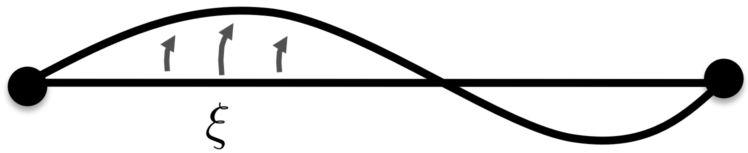
The theory is not covariant, and so it is not smart enough to know that *r = 0* is not a boundary. Can compute its contribution.

Physics: whenever you smooth out a cone, you leave behind a coordinate singularity that the theory knows about.

$$\delta S^{\text{bdy}} = \sum_{i \in \partial A} 4\pi c_g \epsilon^{\mu\nu} \nabla_\mu \xi_\nu(x_i)$$

Entanglement anomaly in 2d

Final universal answer:



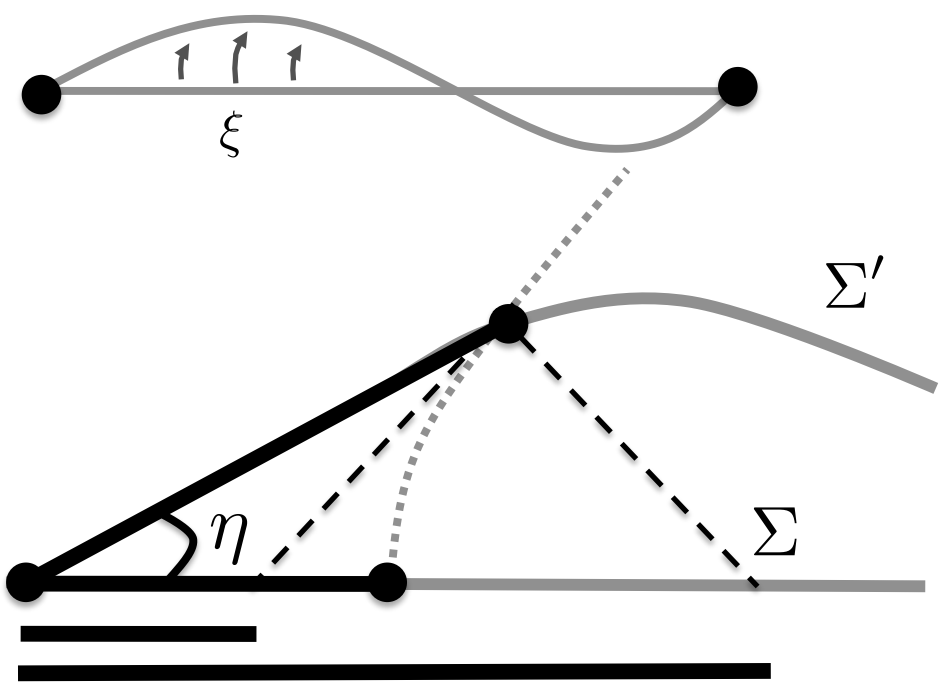
$$\delta S = 8\pi c_g \sum_{i \in \partial A} \epsilon^{\mu\nu} \nabla_\mu \xi_\nu (x_i)$$

(see also: Nishioka, Yarom; Hughes, Leigh, Parrikar, Ramamurthy)

Transformation of the EE is completely fixed by the anomaly: measures the "curl" of the diff at the entangling surface.

Geometric explanation

There is a simple geometric argument for this. Consider a 2d CFT with $c_L \neq c_R$. Impose a proper-distance cutoff.



But after a local boost the cutoff region includes less left-movers, and more right-movers! (Wall 2012)

$$\delta S = \sum_{i \in \partial A} \frac{(c_R - c_L)}{12} \eta_i$$

This is in perfect agreement with the expression before.

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Mixed anomalies in 4d

In 4d there are no purely gravitational anomalies. However, there are mixed gauge-gravitational anomalies, e.g. a right-moving Weyl fermion:

$$\nabla_\mu j^\mu = c_m R \wedge R \qquad \nabla_\mu T^{\mu\nu} = \text{“}0\text{”}$$

The current is not conserved in the presence of nontrivial metric sources. (The stress-energy is morally conserved: its non-conservation preserves diff-invariance.)

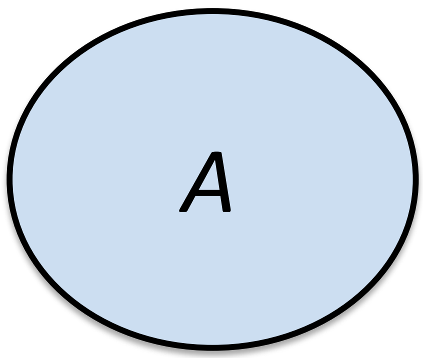
By adding local counter-terms we can shift the anomaly around: e.g. there is an equivalent formulation of the theory where:

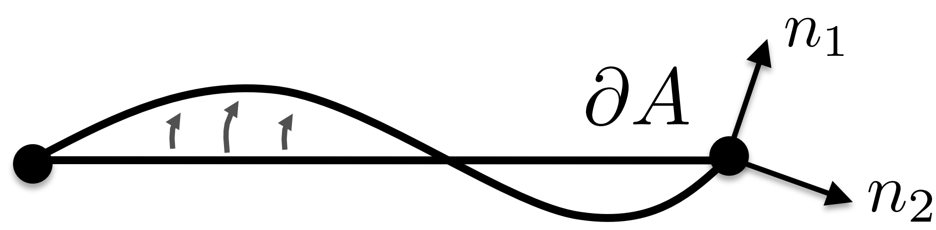
$$\nabla_\mu j^\mu = 0 \qquad \nabla_\mu T^{\mu\nu} = c_m \partial_\lambda F \wedge d\Gamma^\lambda_\nu$$

We will work with this theory in this formulation, because we want to turn on a background field for *j*.

Entanglement anomalies in 4d

We may compute the entanglement anomaly in the same way as before: entangling surface is now a closed 2-surface.





Bi-normal: $\epsilon^{\mu\nu}_{\partial A} = n^{\mu}_{[1} n^{\nu}_{2]}$

$$\delta S = 8\pi c_m \int_{\partial A} (d\Sigma^{\alpha\beta} F_{\alpha\beta}) \epsilon^{\mu\nu}_{\partial A} \nabla_\mu \xi_\nu$$

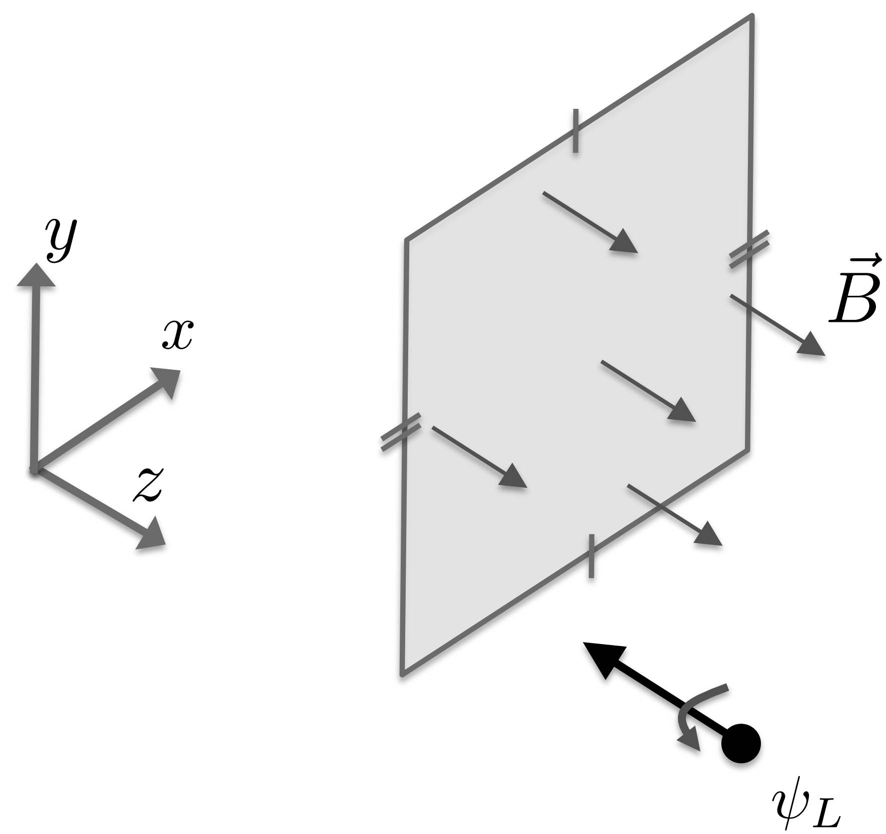
(see also: Azeyanagi, Loganayagam, Ng)

It measures the "curl" of the diff around the entangling surface, if there is a magnetic field through it.

Free Weyl fermions

There is a simple way to understand this from free fermions.

Put a free left-moving Weyl fermion on $R^{1,1}$ x T^2 and turn on a magnetic field $F_{xy} = B$ on the T^2



Classic Landau problem: energy of lowest modes is

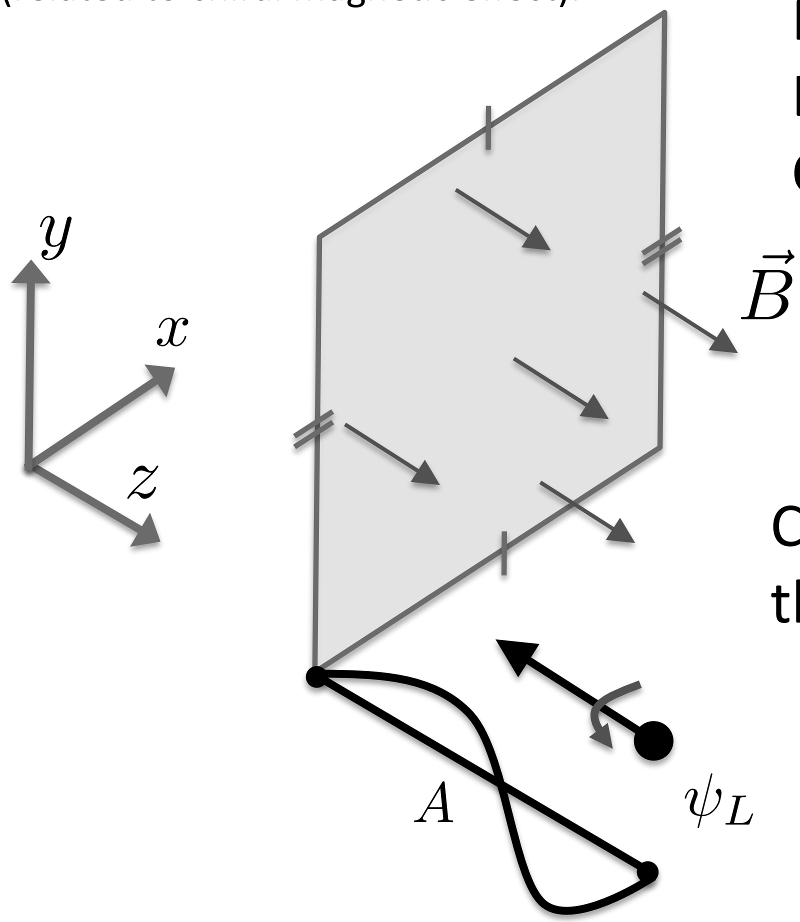
$$E = \frac{\omega_c}{2} - \vec{\sigma} \cdot \vec{B}$$

Spin wants to align with B field: if so, terms cancel, lowest mode has zero energy.

This follows from an index theorem.

Free Weyl fermions

Now, Weyl: definite helicity means that velocity is correlated with spin! Thus, zero modes propagate chirally along the magnetic field (related to chiral magnetic effect).



Number of modes given by the Landau degeneracy. Effective chiral 2d CFT with

$$c_L - c_R = \frac{q}{2\pi}\int_{T^2} F$$

Can now apply intuition from 2d to find the entanglement anomaly of an interval:

$$\delta S = \sum_i \left(\frac{q}{24\pi}\int_{T^2} F\right)\eta_i$$

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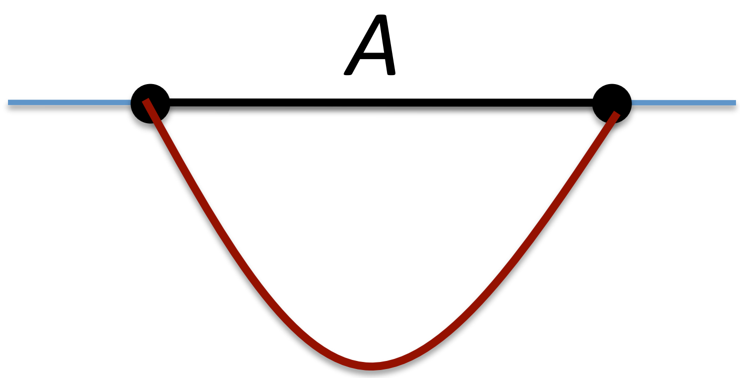
Holographic entanglement entropy

Consider now a 2d CFT with a "normal" gravity dual.

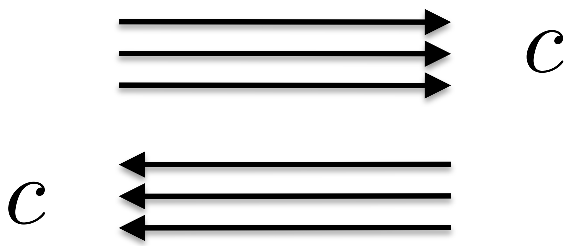


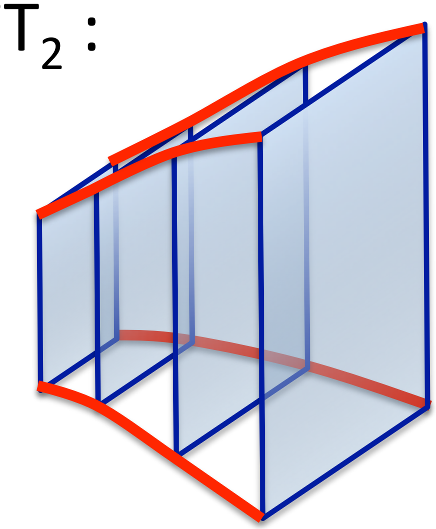
Ryu and Takayanagi: the entanglement entropy is equal to the area of the bulk minimal surface ending on A:

$$S_{EE} = \frac{1}{4G_N} L_{min}$$

Gravity Duals of CFT$_2$

We now want to study theories with gravitational anomalies.
Recall the dual of an ordinary CFT$_2$:





"Normal" CFT$_2$

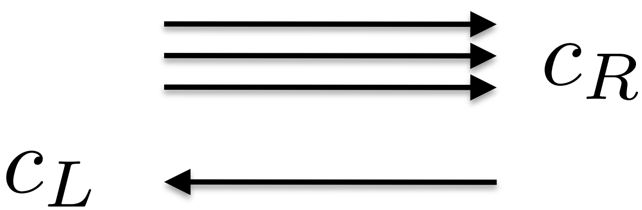
$$S = \frac{1}{16\pi G_N}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right)$$

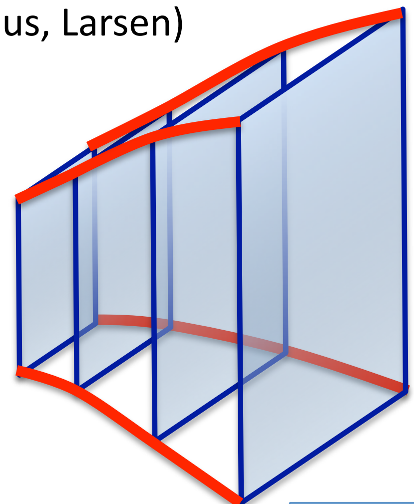
Relation between central charge and AdS radius (Brown, Henneaux):

$$c = \frac{3\ell}{2G_N}$$

Topologically massive gravity

The dual of a theory with an anomaly is topologically massive gravity (Deser, Jackiw, Templeton; Kraus, Larsen)





$$S = \frac{1}{16\pi G_N}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2} + \frac{1}{\mu}\mathrm{CS}(\Gamma)\right)$$

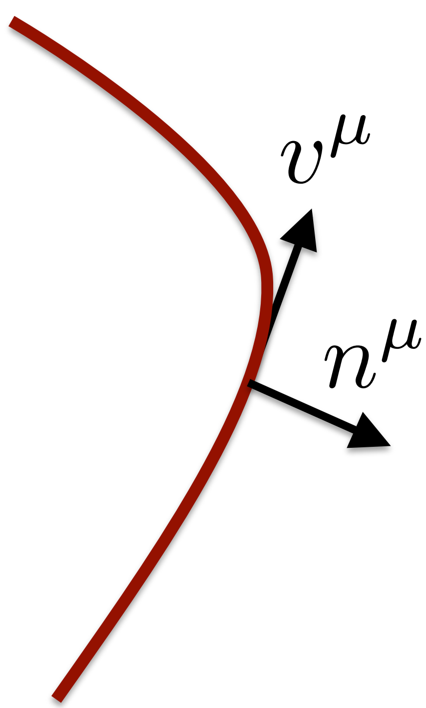
Bulk Christoffel symbol.

$$\mathrm{CS}(\Gamma) \equiv \Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma$$

An action for spinning particles in (2+1)d

How does this term change the RT formula?

Pick a normal vector *n* on the worldline.



$$S = m \int d\tau + s \int d\tau \left(\epsilon_{\mu\nu\rho} v^\mu n^\nu \frac{Dn^\rho}{d\tau} \right)$$

The introduction of *n* has made the worldline into a ribbon.

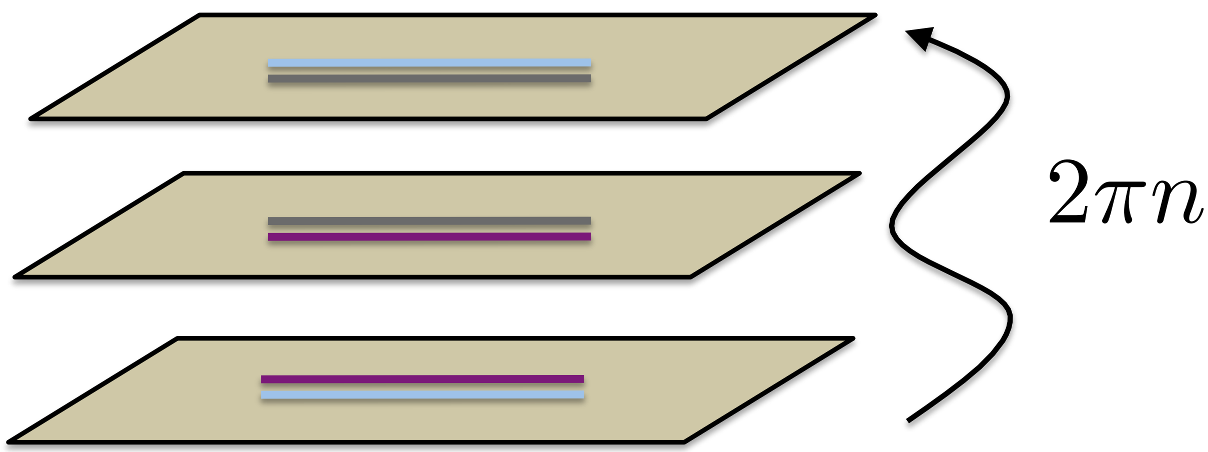
This torsion term measures the twist in the ribbon.

Entanglement entropy in AdS_3 / CFT_2

Sketch of how to justify this, following Lewkowycz + Maldacena.

Again, we compute Renyi entropies, continue *n* to 1:

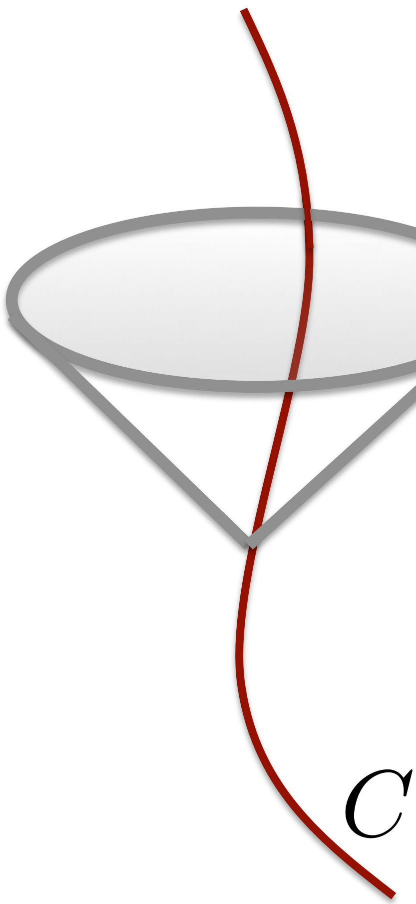
$$S_n = -\frac{1}{n-1}\log \mathrm{Tr}(\rho^n)$$



"Fill in" with 3d manifold, look at its action as a function of *n* (Faulkner, see also Hartman).

Conical geometries in ordinary gravity

For n close to 1, can extend conical defect into the bulk: its action computes the entanglement entropy.



$$\frac{1}{16\pi G_N}\int d^3x\sqrt{g}R$$

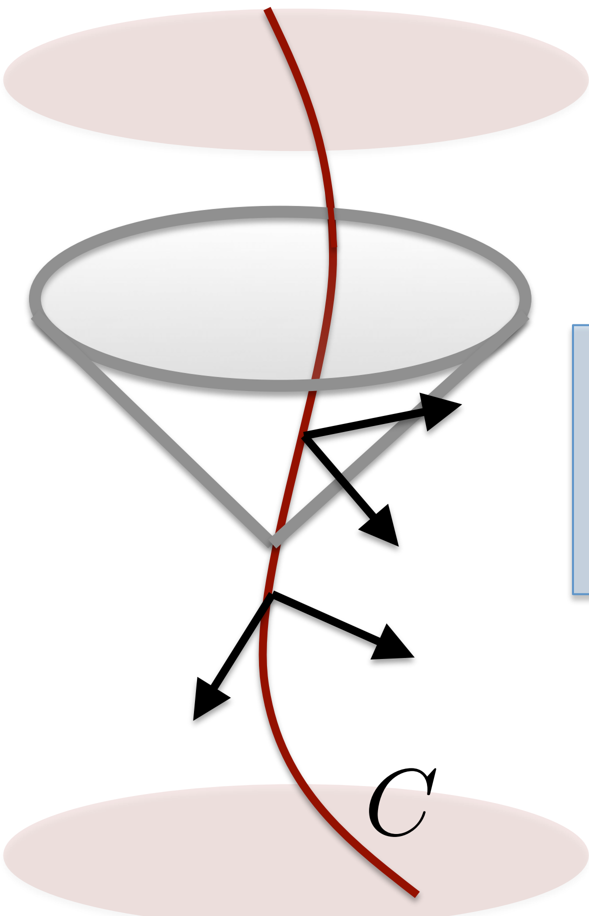
$$\downarrow$$

$$\frac{\epsilon}{4G_N}\int_C d\tau$$

Full bulk information required to find action can be found from the worldline of the defect.

Conical geometries in topologically massive gravity

In TMG, the Chern-Simons term is not quite coordinate invariant: answer depends also on the bulk choice of coordinates.



$$\frac{1}{32\pi G_N \mu} \int d^3x \, \mathrm{CS}(\Gamma)$$

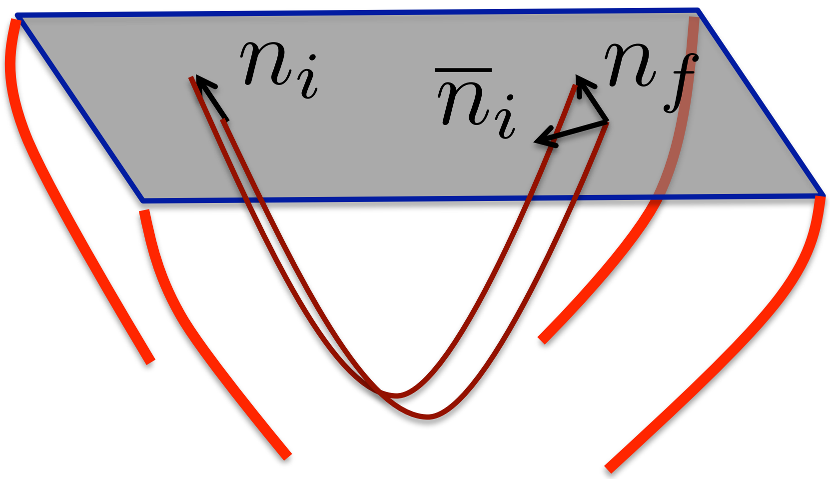
$$\downarrow$$

$$\frac{\epsilon}{4G_N \mu} \int_C d\tau \left(\epsilon_{\mu\nu\rho} v^\mu n^\nu \frac{Dn^\rho}{d\tau} \right)$$

It is the information of the choice of coordinates that is encoded in n; anomaly has brought to life a bulk degree of freedom that used to be pure gauge.

Evaluation of on-shell action

Now we need to evaluate this action on-shell, ending on AdS boundary.



We have boundary conditions on normal vector: torsion integral measures how much normal vector twists along the way. What does this mean in curved space?

$\overline{n}_i$: parallel transport of n_i along curve.

In Lorentzian signature, SO(1,1) transformation relates them:

$$\overline{n}_i = \Lambda(\eta) n_f$$

Torsion integral measures boost angle η.

Example: entanglement anomaly

This is exactly what we need to capture the coordinate-dependence from before; coordinate transformations shift the boundary conditions on n:

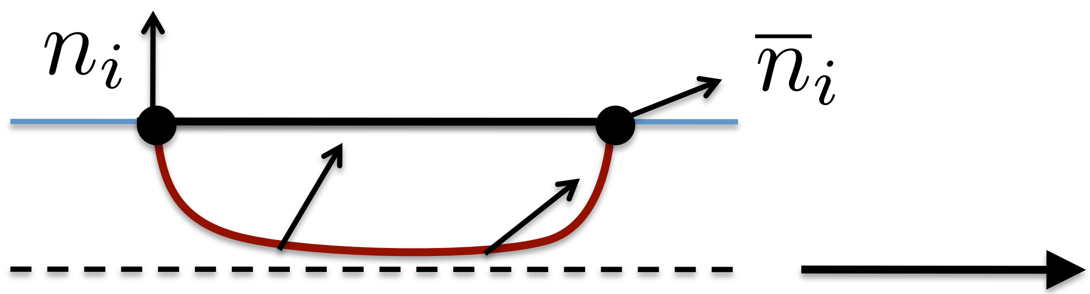
Now consider boosting one endpoint:

$$S_{boost} = S_{rest} + \boxed{\frac{c_L - c_R}{12}\eta}$$

The twist in the ribbon captures the effect of the coordinate transformation.

Example: Boosted BTZ black brane

As an application, consider the entanglement entropy of a boosted BTZ black brane (different left and right temperatures).



Normal vector dragged by moving horizon.

$$S = \frac{c_L + c_R}{6} S_{old} + \frac{c_L - c_R}{12} \log \left(\frac{\beta_L \sinh\left(\frac{\pi L}{\beta_L}\right)}{\beta_R \sinh\left(\frac{\pi L}{\beta_R}\right)} \right)$$

This result can be checked from CFT_2.

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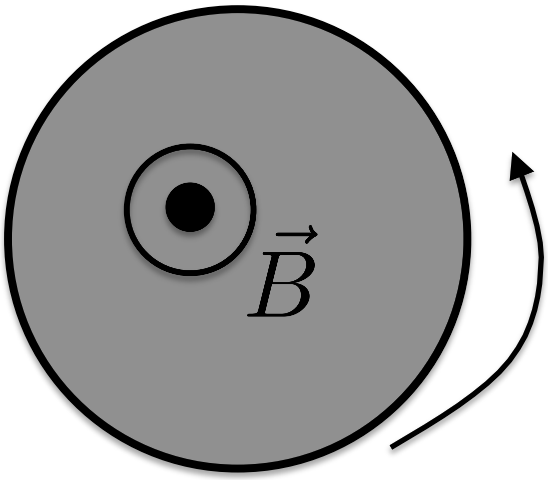
Future directions

In real life, theories with such anomalies arise on the boundaries of other systems (e.g. Hall physics). Full system is non-anomalous, but interesting interplay between boundary and bulk entanglement (see NI, Wall; Hughes, Leigh, Parrikar, Ramamurthy).

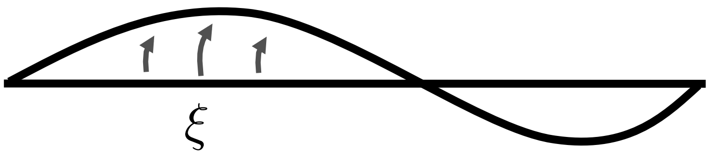


Can we extract this universal contribution from a microscopic wave-function computation?

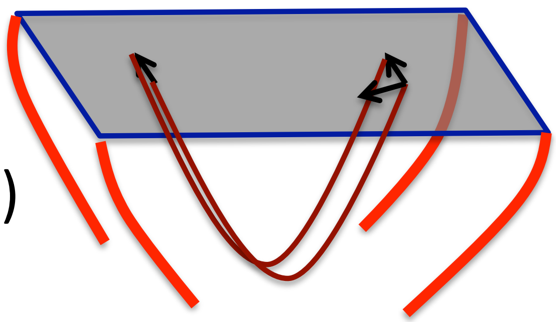
Can this be helpful in classifying interesting gapped phases of matter?

Summary

Anomalies manifest themselves in a universal manner in the entanglement structure of QFT.



In field theories with gravitational anomalies, this structure is geometrized (e.g. in 3d: a twistable ribbon in the bulk.)



It remains to be seen what more we can learn from the interplay of entanglement and anomalies.

The End