Holographic Lattices

Jerome Gauntlett

with

Aristomenis Donos

Christiana Pantelidou

Holographic Lattices

CFT with a deformation by an operator that breaks translation invariance

Why?

- Translation invariance $\Rightarrow$ momentum is conserved, hence no dissipation and hence DC response are infinite. To model more realistic metallic behaviour or insulating behaviour we can use a lattice

- The lattice deformation can lead to novel ground states at T=0. Can also model metal-insulator transitions

- Formal developments: thermo-electric DC conductivities in terms of black hole horizon data [Donos,Gauntlett]

Analogous to $\eta = \frac{s}{4\pi}$ [Policastro,Kovtun,Son,Starinets]

Plan

- Drude physics

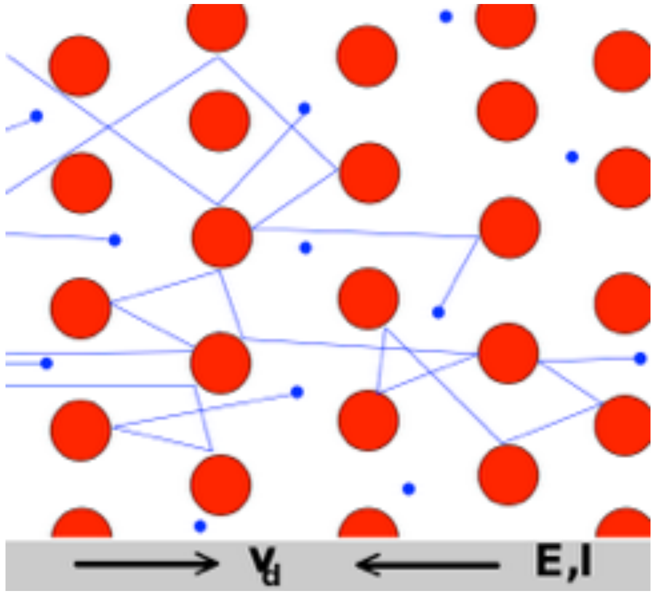
- Lattice with global U(1) symmetry and $\mu(x)$. In Einstein-Maxwell theory. Coherent metals.

- Q-lattices, using scalars and global symmetry. Can give coherent metals, incoherent metals and insulators and transitions between them.

- Helical lattices in D=5 pure gravity. Universal deformation. Coherent metals. Comments on calculating Greens functions

Drude Model of transport in a metal

Quasi-particle interactions ignored



$$m\frac{d}{dt}v = qE - \frac{m}{\tau}v \qquad \Rightarrow v = \frac{q\tau E}{m}$$

$$J = nqv$$

$$J = \sigma_{DC}E \qquad \sigma_{DC} = \frac{nq^2\tau}{m}$$

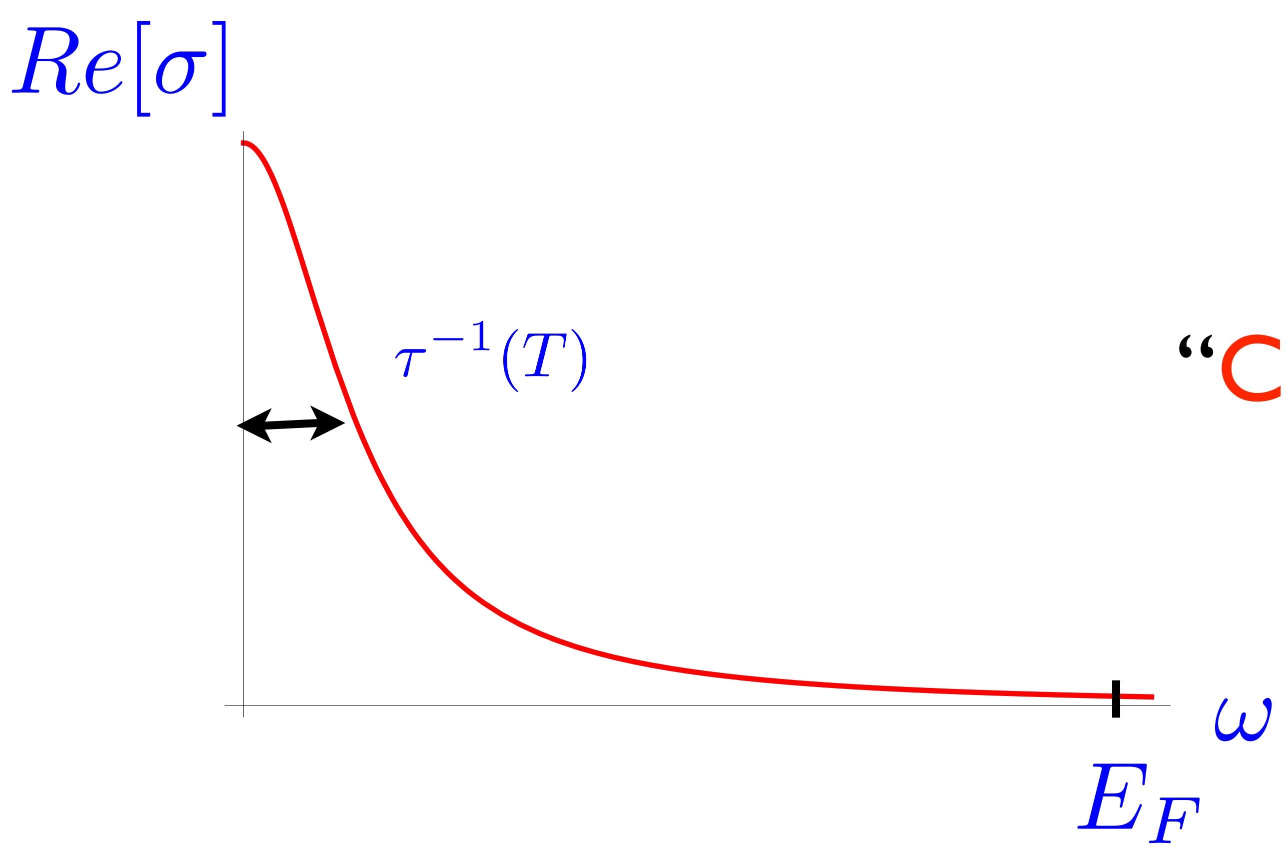
$$E = E(\omega)e^{-i\omega t}$$

$$J(\omega) = \sigma(\omega)E(\omega)$$

$$J = J(\omega)e^{-i\omega t}$$

$$\sigma(\omega) = \frac{\sigma_{DC}}{1 - i\omega\tau}$$

$$\sigma_{DC} = \frac{q^2\tau}{m}$$



"Coherent" or "good" metal

When $\tau \to \infty$ $\sigma(\omega) \sim \delta(\omega) + \frac{i}{\omega}$

- Drude physics doesn't require quasi-particles

  Coherent metals arise when momentum is nearly conserved [Hartnoll,Hofman]

  Pole on negative imaginary axis near origin $\omega = -\frac{i}{\tau}$

- Similar comments apply to thermal conductivity $Q = -\bar{\kappa}\nabla T$

- There are also "incoherent" metals without Drude peaks

  Not dominated by single time scale τ

  Of particular interest to realise these in holography

- Insulators with $\sigma_{DC} = \bar{\kappa}_{DC} = 0$ at T=0

Holographic CFTs at finite charge density

Focus on d=3 CFT and consider D=4 Einstein-Maxwell theory:

$$S = \int d^4x \sqrt{-g} \Big[R + 6 - \frac{1}{4} F^2 + \dots \Big]$$

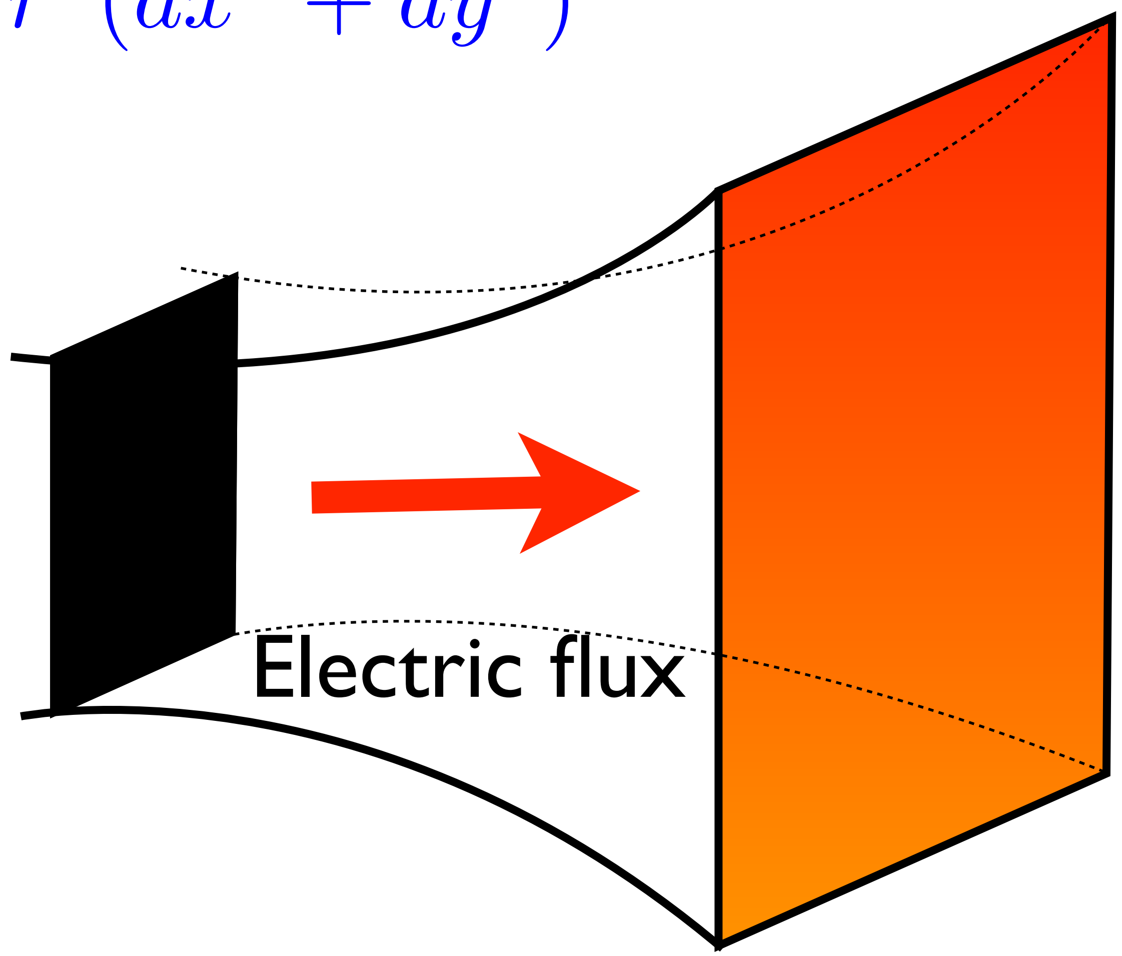
Admits AdS_4 vacuum $\leftrightarrow$ d=3 CFT with global U(1)

Electrically charged AdS-RN black hole (brane)

Describes holographic matter at finite charge density that is translationally invariant

$$ds^2 = -U dt^2 + \frac{dr^2}{U} + r^2(dx^2 + dy^2)$$

$$A_t = \mu(1 - \frac{r_+}{r})$$



d=3 CFT

$\mu \quad T$

T=0 limit: $AdS_2 \times \mathbb{R}^2 \longleftarrow AdS_4$

IR UV

By perturbing the black hole and using holographic tools we can calculate the electric conductivity and find a delta function at $\omega = 0$ [Hartnoll]

Construct lattice black holes dual to CFT with $\mu(x)$

$$A_t(x,r) \sim \mu(x) + \mathcal{O}(\frac{1}{r}) \qquad r \to \infty$$

$$g_{\mu\nu}(x,r)$$

Need to solve PDEs in two variables

e.g. Monochromatic lattice:

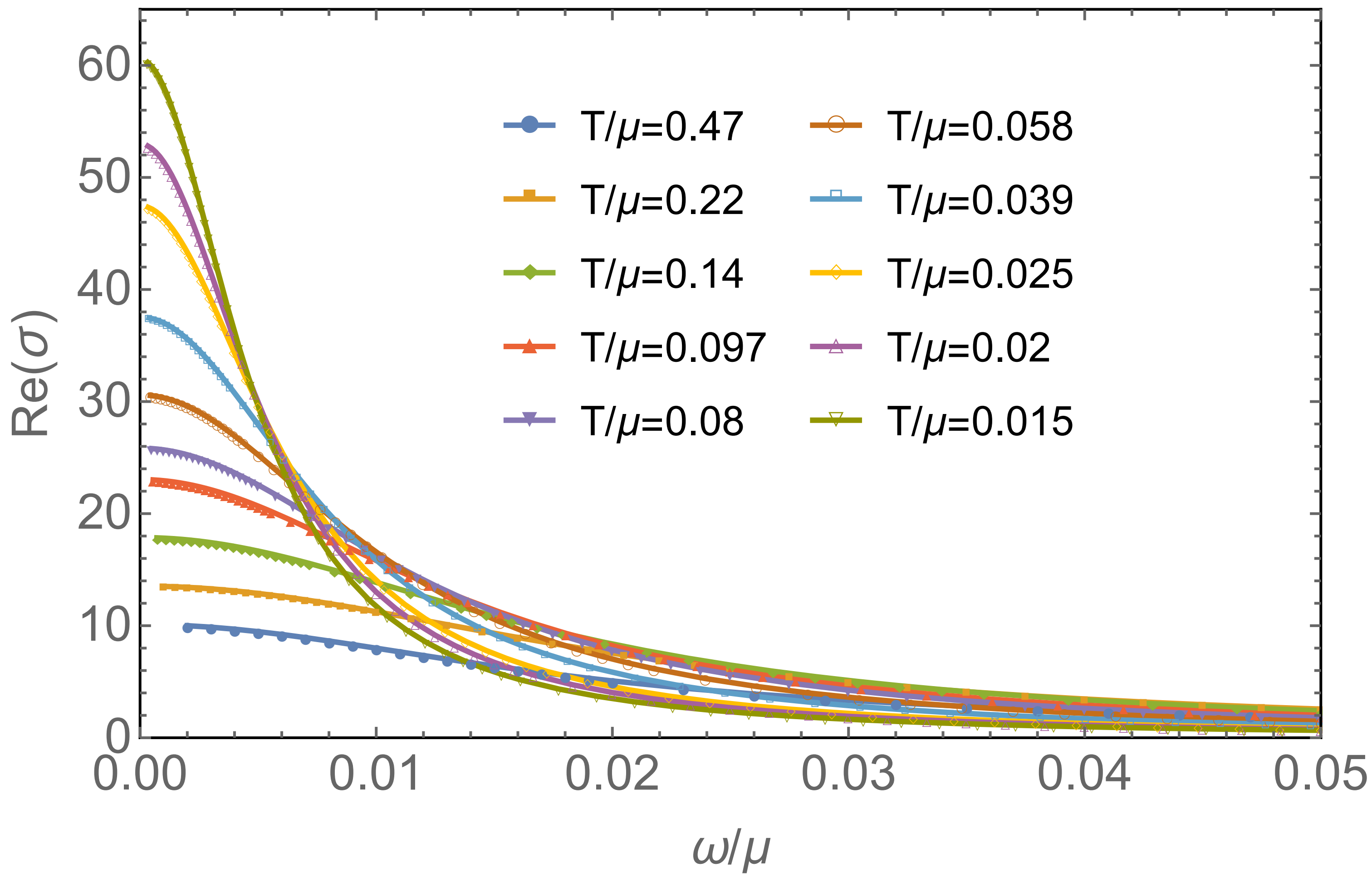
[Horowitz, Santos, Tong]

$$\mu(x) = \mu + A\cos kx$$

[Donos, Gauntlett]

After constructing black holes, one can perturb, again solving PDEs, to extract thermo-electric conductivities

Find Drude physics at finite T

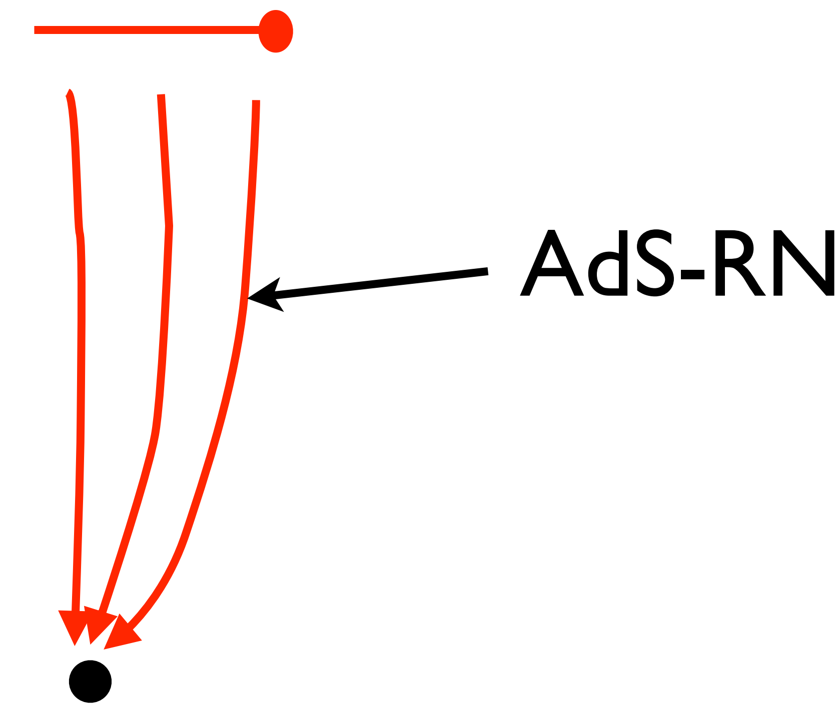


Coherent metal phases

Can be understood by analysing T=0 solutions:

UV data

$k/\mu \quad A/\mu$



IR fixed point $\quad AdS_2 \times \mathbb{R}^2$

At T=0 the black holes approach $AdS_2 \times \mathbb{R}^2$ in the IR

perturbed by irrelevant operator with $\Delta(k_{IR}) \geq 1$

Don't find exceptions to this behaviour even for dirty lattices e.g.

$$\mu(x) = 1 + A \sum_{n=1}^{10} \cos(n\, k\, x + \theta_n)\,,$$

Holographic Q-lattices

[Donos,Gauntlett]

- Illustrative D=4 model

$$\mathcal{L} = R - \frac{1}{2}|\partial\varphi|^2 + V(|\varphi|) - \frac{Z(|\varphi|)}{4}F^2$$

- Choose V, Z so that AdS-RN is a solution at $\varphi = 0$

- Now $\varphi \leftrightarrow \mathcal{O}$ in CFT. Want to build a holographic lattice by deforming with the operator $\mathcal{O}$

- The model has a gauge $U(1)$ and a global $U(1)$ symmetry

  Exploit the global bulk symmetry to break translations so that we only have to solve ODEs

Ansatz for fields

$$ds^2 = -U dt^2 + U^{-1} dr^2 + e^{2V_1} dx^2 + e^{2V_2} dy^2$$

$$A_t = a(r)$$

$$\varphi(r,x) = \phi(r) e^{ikx}$$

UV expansion:

$$U = r^2 + \ldots, \qquad e^{2V_1} = r^2 + \ldots \qquad e^{2V_2} = r^2 + \ldots$$

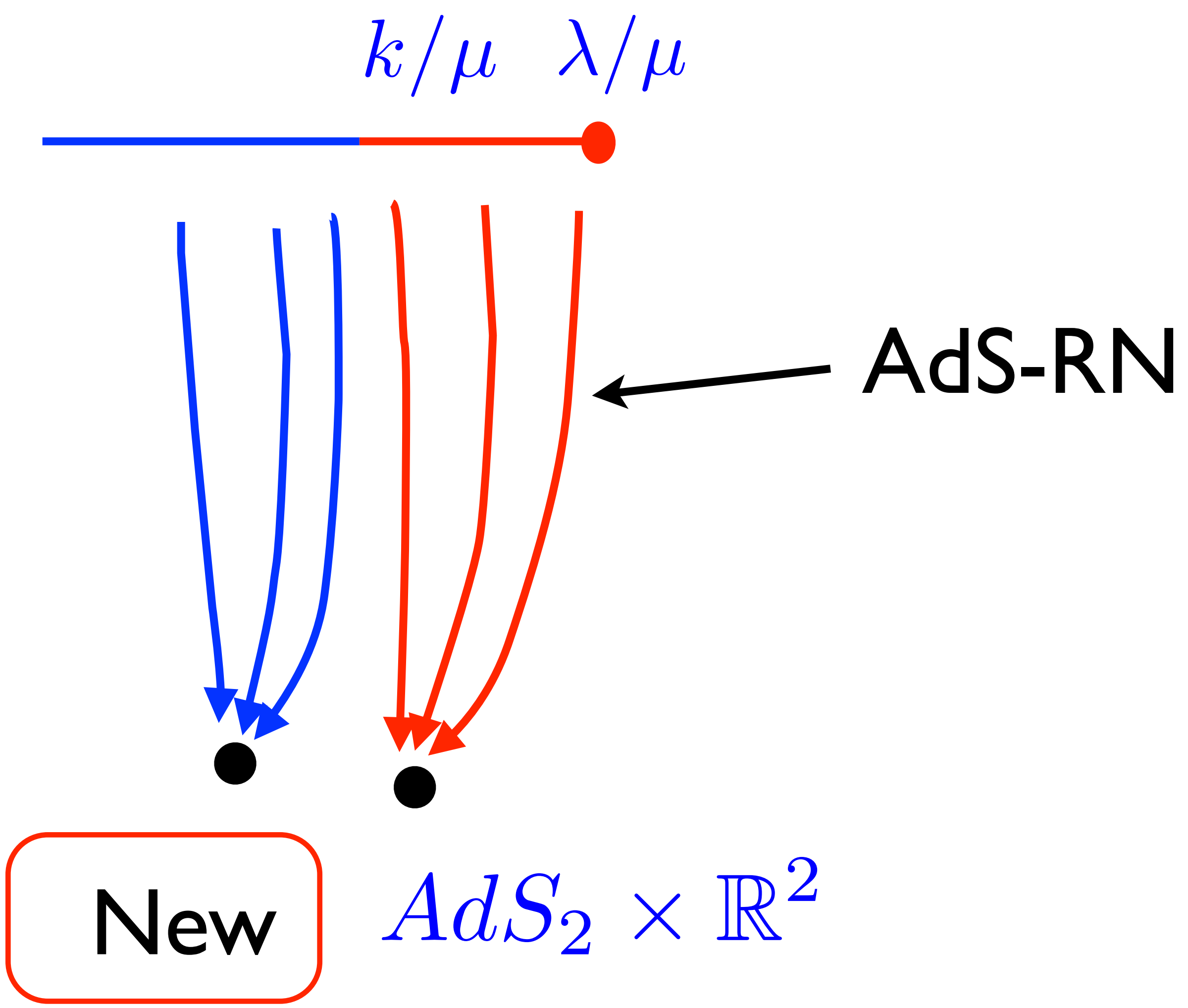
$$a = \mu + \frac{q}{r} \ldots, \qquad \phi = \frac{\lambda}{r^{3-\Delta}} + \ldots$$

Homogeneous and anisotropic and periodic holographic lattices

UV data: T/μ $\quad \lambda/\mu^{3-\Delta}$ $\quad k/\mu$

For small deformations from AdS-RN we find Drude peaks corresponding to coherent metals.

This can be understood by examining T=0 behaviour of solutions:



For larger deformations, for specific models, we find a transition to new behaviour. The new ground states can be both insulators and also incoherent metals!

See also: [Gouteraux][Andrade, Withers]

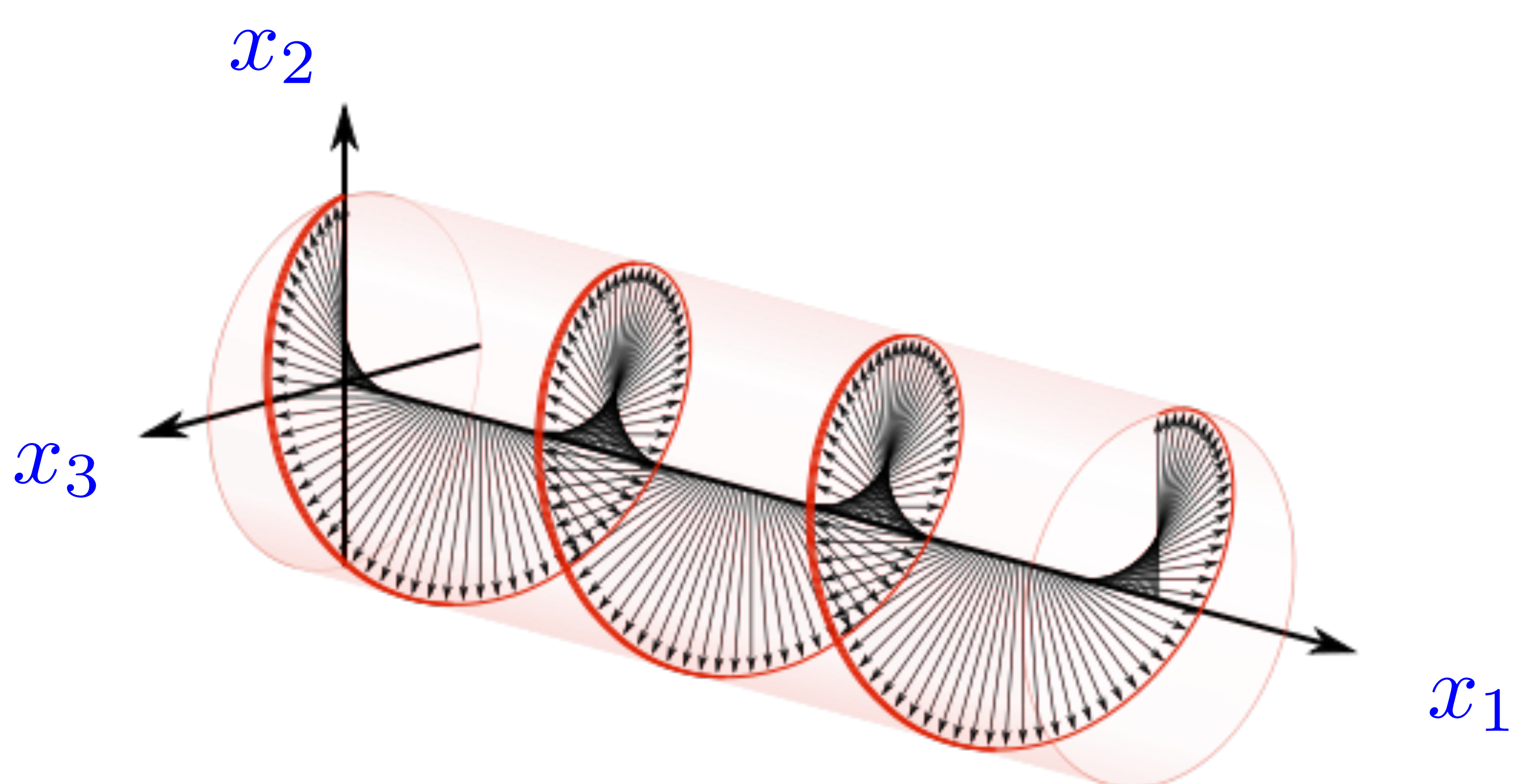
D=4 CFTs with a Helical Twist [Donos,Gauntlett,Pantelidou]

Study a universal helical deformation that applies to all d=4 CFTS

First recall the Bianchi VII_0 Lie algebra

$$[L_1, L_2] = -kL_3 \qquad [L_1, L_3] = kL_2 \qquad [L_2, L_3] = 0$$

$$L^1 = \partial_{x_1} + k(x_3\partial_{x_2} - x_2\partial_{x_3}) \qquad L_2 = \partial_{x_2} \qquad L_3 = \partial_{x_3}$$



Useful to introduce the left-invariant one-forms

$$\omega_1 = dx_1$$
$$\omega_2 = \cos(kx_1)\, dx_2 - \sin(kx_1)\, dx_3,$$
$$\omega_3 = \cos(kx_1)\, dx_2 + \sin(kx_1)\, dx_3$$

We want to explicitly break the $ISO(3)$ spatial symmetry of the CFT down to Bianchi VII_0

Achieve by introducing suitable sources for the stress tensor

Equivalently, consider CFT not on $\mathbb{R}^{1,3}$ but on

$$ds^2 = -dt^2 + \omega_1^2 + e^{2\alpha_0}\, \omega_2^2 + e^{-2\alpha_0}\, \omega_3^2$$

with k, α_0 parametrising the deformation

Study in holography by considering

$$S = \int d^5x \sqrt{-g}(R + 12)$$

This is a consistent truncation of all $AdS_5 \times M$ solutions in string/M-theory. Hence analysis applies to entire class of CFTs

Ansatz

$$ds^2 = -g\, f^2\, dt^2 + g^{-1} dr^2 + h^2\, \omega_1^2 + r^2\, \left(e^{2\alpha}\, \omega_2^2 + e^{-2\alpha}\, \omega_3^2\right)$$

Equations of motion

$$f' = \ldots, \quad g' = \ldots, \quad h'' = \ldots, \quad \alpha'' = \ldots$$

AdS-Schwarzschild: $\quad f = 1, \;\; g = r^2 - \dfrac{r_+^4}{r^2}, \;\; h = r, \;\; \alpha = 0$

Expand functions at UV boundary

$$f = 1 + \frac{k^2}{12r^2}(1 - \cosh 4\alpha_0) - \frac{c_h}{r^4} + \frac{k^4}{96r^4}(3 + 4\cosh 4\alpha_0 - 7\cosh 8\alpha_0) + \log r() + \dots,$$

$$g = r^2\left(1 - \frac{k^2}{6r^2}(1 - \cosh 4\alpha_0) - \frac{M}{r^4} + \log r() + \dots\right),$$

$$h = r\left(1 - \frac{k^2}{4r^2}(1 - \cosh 4\alpha_0) + \frac{c_h}{r^4} + \log r() + \dots\right),$$

$$\alpha = \alpha_0 - \frac{k^2}{4r^2}\sinh 4\alpha_0 + \frac{c_\alpha}{r^4} + \log r() + \dots.$$

Source parameters: α_0, k

Vev parameters: c_h, c_α, M

Together these give $T^{\mu\nu}$ of helically deformed CFT

Log terms arise because of conformal anomaly

$$T^\mu{}_\mu = \frac{k^4}{3}\left(\cosh(8\alpha_0) - \cosh(4\alpha_0)\right)$$

Boundary conditions in the IR - smooth black hole horizon

$$g = g_+(r - r_+) + \cdots \,, \qquad f = f_+ + \cdots \,,$$
$$h = h_+ + \cdots \,, \qquad \alpha = \alpha_+ + \cdots$$

with $\quad g_+ = 4r_+$

Parameter count: expect two parameter family of black holes labelled by k/T , α_0 (for fixed dynamical scale)

Thermodynamics from $T^{\mu\nu}$ [Donos,Gauntlett]

Free energy density w

Boundary metric γ $\leftrightarrow$ $ds^2 = -dt^2 + \omega_1^2 + e^{2\alpha_0}\,\omega_2^2 + e^{-2\alpha_0}\,\omega_3^2$

Killing vector ∂_t

$$w = -Ts - \frac{k}{2\pi}\int_0^{2\pi/k} dx_1 \sqrt{-\gamma}\,(T^{tt}\gamma_{tt})$$

Killing vectors ∂_{x_2} and ∂_{x_3}

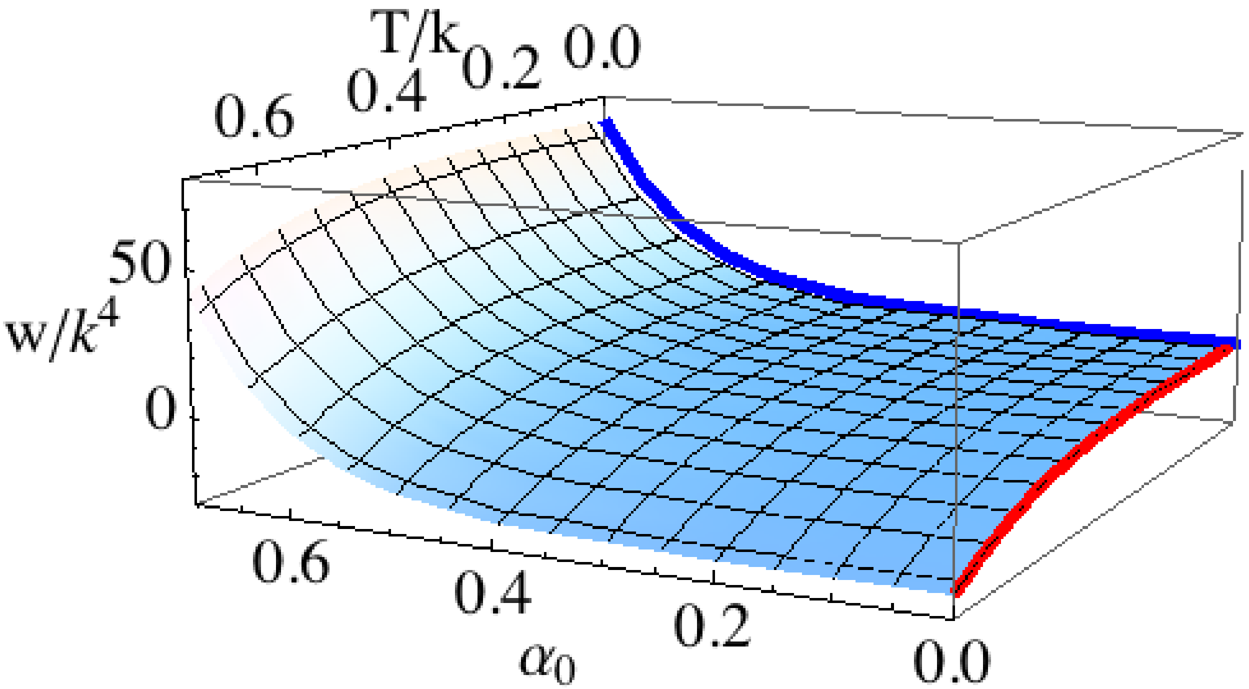
$$w = -\frac{k}{2\pi}\int_0^{2\pi/k} dx_1 \sqrt{-\gamma}\left(T^{x^2x^2}\gamma_{x^2x^2} + T^{x^2x^3}\gamma_{x^2x^3}\right) ,$$

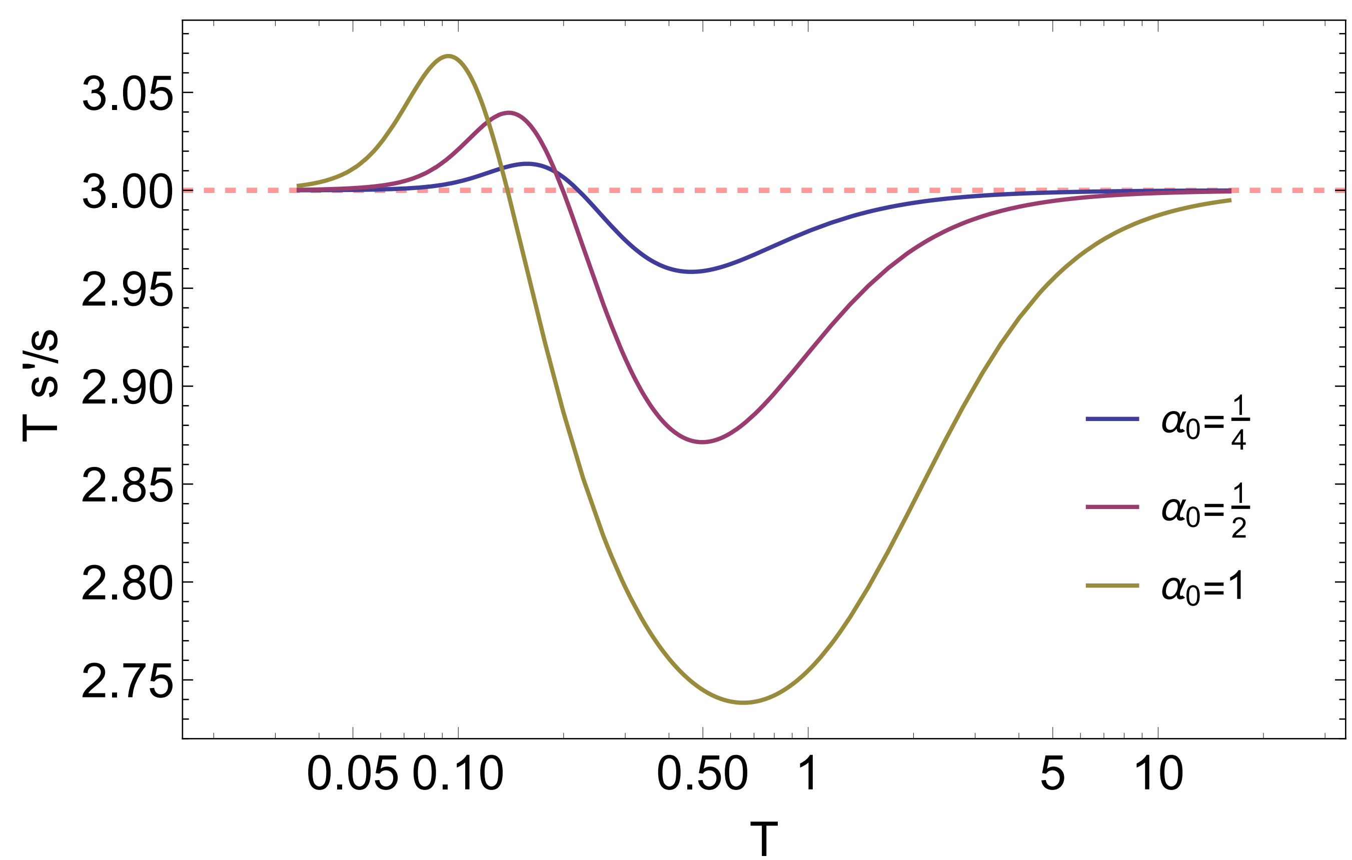
$$w = -\frac{k}{2\pi}\int_0^{2\pi/k} dx_1 \sqrt{-\gamma}\left(T^{x^3x^2}\gamma_{x^3x^2} + T^{x^3x^3}\gamma_{x^3x^3}\right)$$

First law

$$\delta w = -s\delta T - \frac{2\pi}{k}\int_0^{2\pi/k} dx_1\sqrt{-\gamma}\left(\tfrac{1}{2}T^{\mu\nu}\delta\gamma_{\mu\nu}\right) + \frac{\delta k}{k}\left(w + \frac{2\pi}{k}\int_0^{2\pi/k} dx_1\sqrt{-\gamma}\left(T^{x_1x_2}\gamma_{x_1x_2} + T^{x_1x_3}\gamma_{x_1x_3}\right)\right)$$

Results of numerics





At T=0 the solution might be approaching AdS5?

T=0 interpolating solutions

Consider small perturbation of α about AdS5 which one solve in terms of Bessel functions

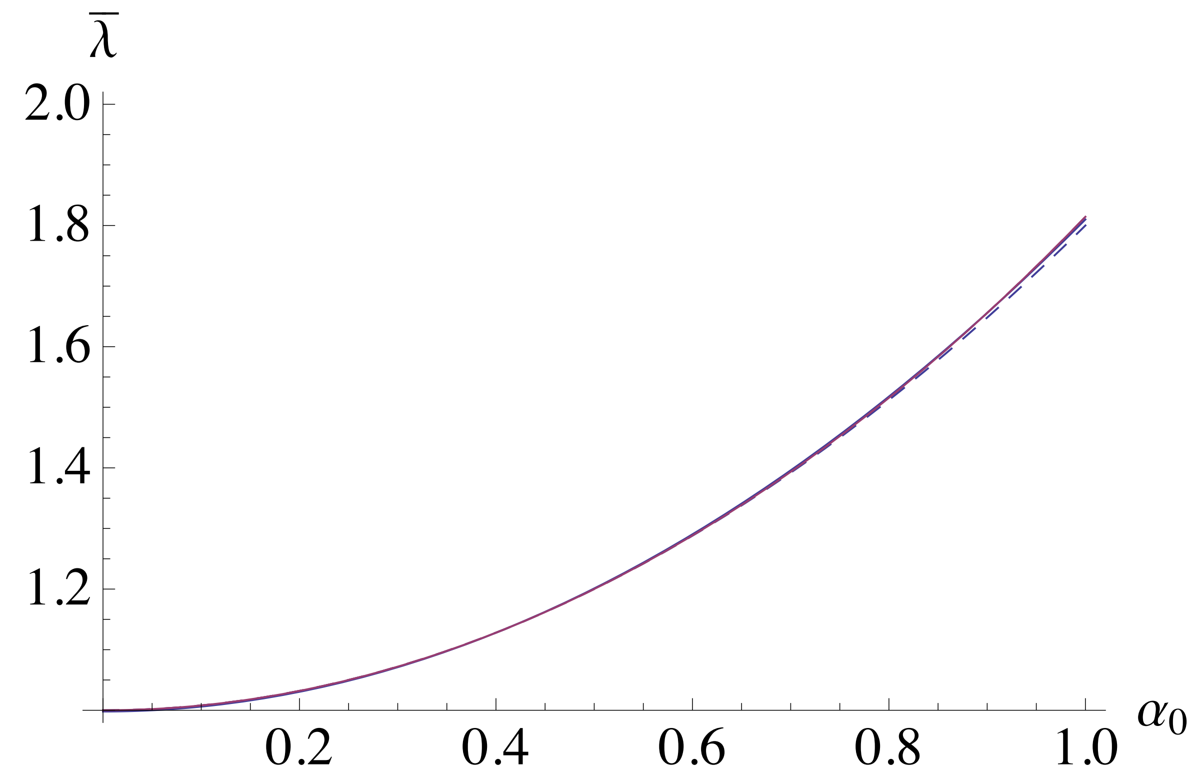
Suggests the IR expansion as $r \to 0$

$$
\begin{aligned}
g &= r^2 + \frac{k^3 \bar{\alpha}_+^2}{r} e^{-4k/\bar{h}_+ r}\left(1 + \frac{5\bar{h}_+}{8k} r + \mathcal{O}(r^2)\right) + \cdots\,, \\
f &= \bar{f}_+ - \frac{k^3 \bar{\alpha}_+^2 \bar{f}_+}{2r^3} e^{-4k/\bar{h}_+ r}\left(1 + \frac{5\bar{h}_+}{8k} r + \mathcal{O}(r^2)\right) + \cdots\,, \\
h &= \bar{h}_+ r - \frac{k^3 \bar{\alpha}_+^2 \bar{h}_+}{2r^2} e^{-4k/\bar{h}_+ r}\left(1 + \frac{21\bar{h}_+}{8k} r + \mathcal{O}(r^2)\right) + \cdots\,, \\
\alpha &= \frac{\bar{\alpha}_+ 2k^2}{\sqrt{\pi \bar{h}_+} r^2} K_2\left(\frac{2k}{\bar{h}_+ r}\right) + \cdots\,,
\end{aligned}
$$

Note that there can be a renormalisation of length scales

Length scale renormalisation

$$\bar{\lambda} \equiv \sqrt{\frac{g_{x_1x_1}(r \to 0)}{g_{x_1x_1}(r \to \infty)}}$$



Note similar T=0 ground states have been seen before

Chemical potential lattice $\mu(x)$ with no zero-mode [Chesler,Lucas,Sachdev]

s-wave superconductors [Horowitz,Roberts]

p-wave superconductors [Basu,He,Mukherjee,Rozali,Shieh] [Donos,Gauntlett,Pantelidou]

Greens functions for thermal conductivity at finite T

Perturb black hole

$$\delta(ds^2) = 2\delta g_{tx_1}(t,r)dtdx_1 + 2\,\delta g_{23}(t,r)\omega_2\omega_3$$

with

$$\delta g_{tx_1}(t,r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} h_{tx_1}(\omega, r)$$
$$\delta g_{23}(t,r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} h_{23}(\omega, r)$$

We obtain linear ODEs:

$$h''_{2\,3} = \ldots \qquad\qquad h''_{t\,x} = \ldots$$

and a constraint equation involving $h'_{t\,x}$ and h'_{23}

IR boundary conditions

Ingoing at black hole horizon [Son,Starinets]

UV boundary conditions

$$h_{tx_1} = r^2 s_1 + \frac{i\omega k}{2} \sinh 2\alpha_0 s_2 + \frac{v_1}{r^2} + \cdots$$

$$h_{2\,3} = r^2 s_2 + (\frac{1}{2} i\omega k \sinh 2\alpha_0 s_1 + \frac{\omega^2}{4} s_2 - k^2 \cosh^2 2\alpha_0 s_2) + \frac{v_2}{r^2} + \ldots$$

with the constraint

$$64 i\omega v_1 + 128 k \sinh 2\alpha_0 v_2 + i\omega(-128 c_h + 16 k^4 \sinh 2\alpha_0{}^4) s_1$$
$$- 4k \left(64 c_\alpha \cosh 2\alpha_0 + 4k^2 \sinh 2\alpha_0{}^3 (2k^2 - \omega^2 + 2k^2 \cosh 4\alpha_0)\right) s_2 = 0$$

Greens function $J_i = G_{ij} s_j$

with

$$J_1 = \langle T^{t\,x_1} \rangle = \lim_{r\to\infty} \frac{1}{\sqrt{-g_\infty}} \frac{\delta S^{(2)}}{\delta h_{t\,x_1}(r)}\,,$$

$$J_2 = \langle T^{\omega_2\,\omega_3} \rangle = \lim_{r\to\infty} \frac{1}{\sqrt{-g_\infty}} \frac{\delta S^{(2)}}{\delta h_{2\,3}(r)}\,.$$

Calculate the variation of the action and discard a horizon contribution

$$\delta S^{(2)}_\infty = \int d^2x \int_{\omega\geq 0} \frac{d\omega}{2\pi} \Big(\delta\bar{s}_i(\omega) J_i(\omega) + \delta s_i(\omega) \bar{J}_i(\omega) \Big)$$

with

$$J_1 = s_1(\dots) + s_2(\dots) - 4v_1$$

$$J_2 = s_1(\dots) + s_2(\dots) + 4v_2$$

dots are fixed by ω and black hole background: $\alpha_0, k, M, c_\alpha, c_h$

To obtain $G_{ij} = \frac{\partial J_i}{\partial s_j}$ we need to work out $\frac{\partial v_i}{\partial s_j}$

This is subtle due to residual gauge invariance

Take the background black hole with $x_1 \to x_1 + e^{-i\omega t}\epsilon_0$

which induces

$$
\begin{aligned}
s_1 &\to s_1 - i\epsilon_0\omega, & v_1 &\to v_1 - 2i\epsilon_0\omega(c_h + \frac{k^4}{8}\sinh^4 2\alpha_0) \\
s_2 &\to s_2 - 2\epsilon_0 k \sinh 2\alpha_0, & v_2 &\to v_2 - \epsilon_0 k \cosh 2\alpha_0 (4c_\alpha + k^4 \cosh 2\alpha_0 \sinh^3 2\alpha_0)
\end{aligned}
$$

with some work one can find $\frac{\partial v_i}{\partial s_j}$ consistent with these

Alternative procedure: work in a gauge with $s_1 = 0$

and calculate $G_{i2} = \left.\frac{J_i}{s_2}\right|_{s_1=0}$

Then work in a gauge with $s_2 = 0$ and $G_{i1} = \left.\frac{J_i}{s_1}\right|_{s_2=0}$

Comment: suppose we calculate on-shell action

$$S_{\infty}^{(2)} = \int_{\omega\geq 0} \frac{d\omega}{2\pi}\Big(s_2\bar{s}_2(\dots) + s_1\bar{s}_1(\dots) + s_1\bar{s}_2(\dots) + \bar{s}_1 s_2(\dots)$$

$$+ 2(s_2\bar{v}_2 + \bar{s}_2 v_2 - s_1\bar{v}_1 - \bar{s}_1 v_1)\Big)$$

dots are fixed by ω and black hole background: $\alpha_0, k, M, c_\alpha, c_h$

Here we did NOT discard any terms arising from the black hole horizon

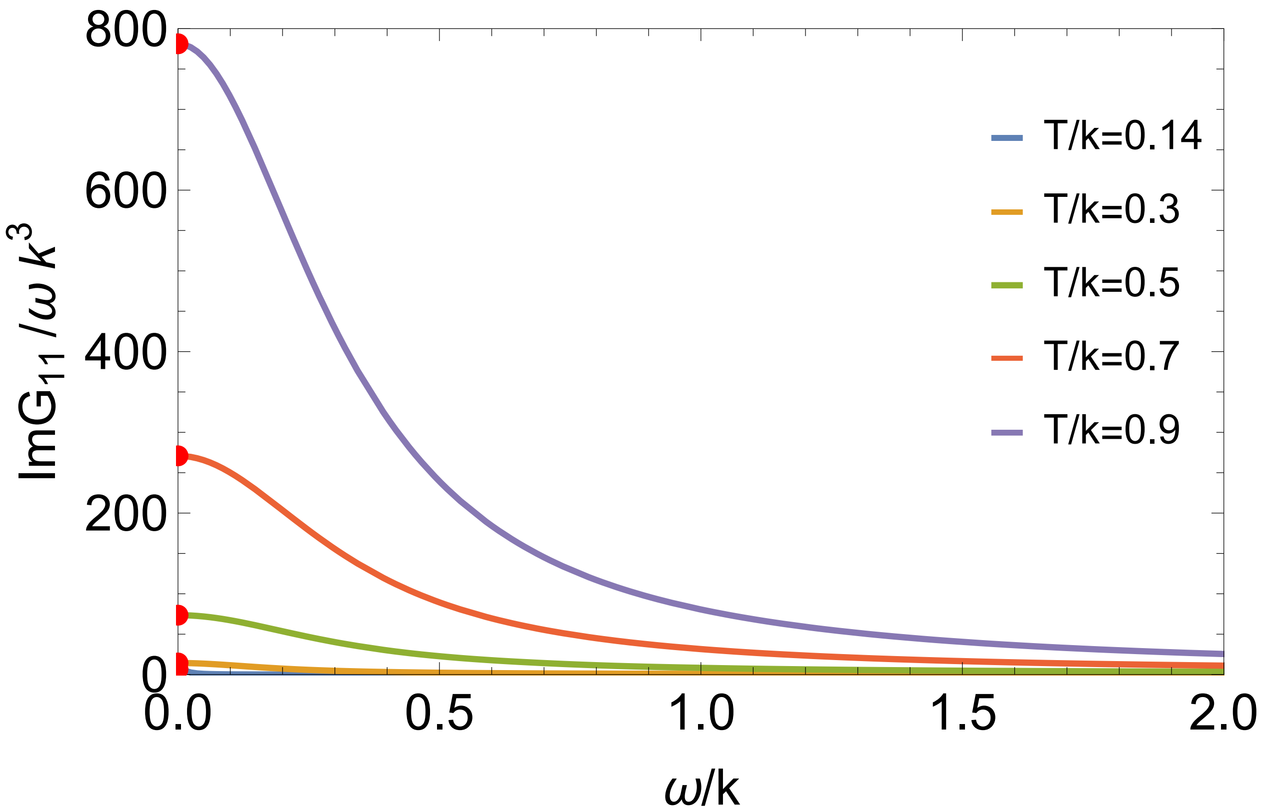
Using the same derivatives for $\frac{\partial v_i}{\partial s_j}$ as above we find

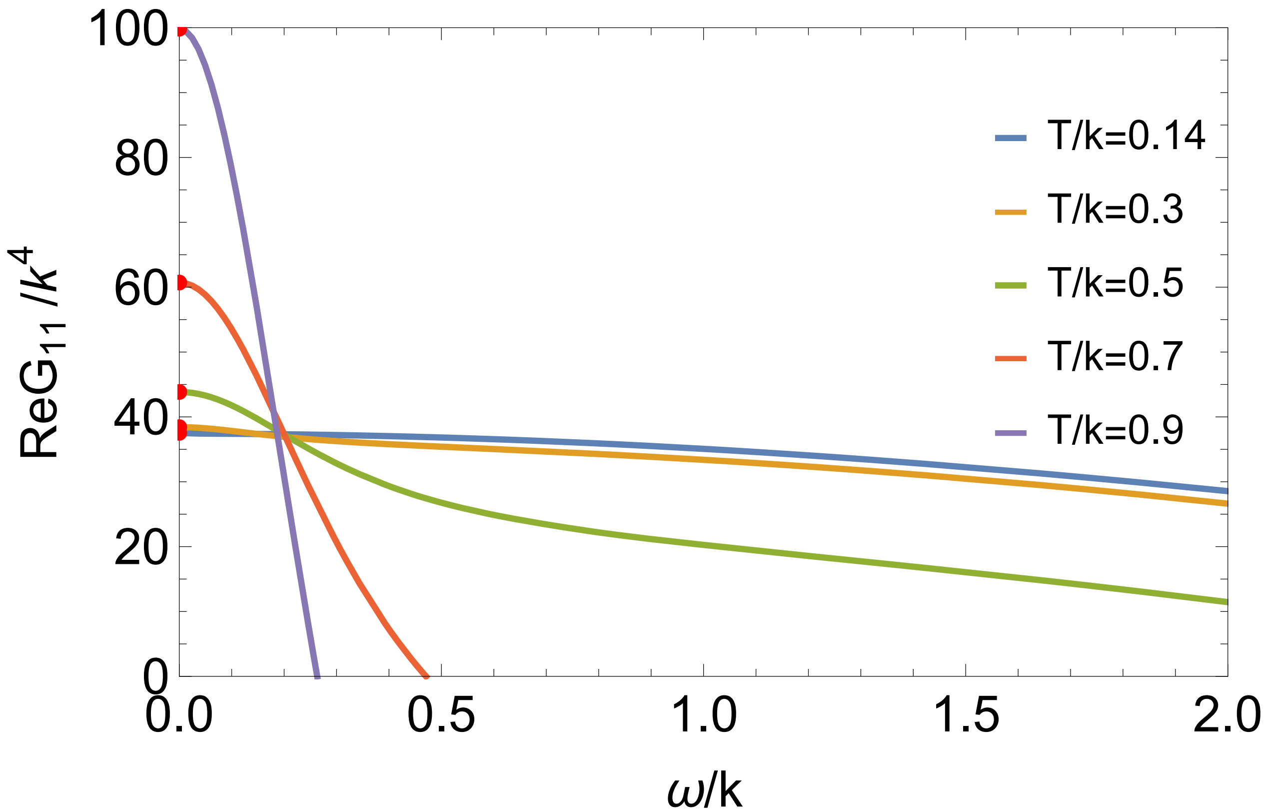
$$\frac{\partial^2 S^{(2)}}{\partial s_i \partial \bar{s}_j} = G_{ij} + G_{ij}^{\dagger}$$

Numerical results

Focus on $G_{11}(\omega) = \langle T^{tx_1} T^{tx_1} \rangle$

and recall that $T\kappa(\omega) \equiv \dfrac{G_{11}}{i\omega}$





Summary/Final Comments

- Holographic lattices are interesting

d=3,4 CFTs with global U(1) symmetry:

Einstein-Maxwell theory and $\mu(x)$ deformation (PDEs)

Q-lattice: Einstein-Maxwell plus scalar field with global symmetry in the bulk (ODEs)

d=4 CFTs with universal helical deformation (ODEs)

- All of these included a realisation of strongly coupled Drude physics at finite T, at least for small deformations

The Drude physics can be understood by the appearance of translationally invariant ground states in the far IR: $AdS_2 \times \mathbb{R}^2$ or AdS_5

- For larger deformations the Q-lattices realised incoherent metallic an insulating phases

The T=0 ground states break translation invariance

The phases have novel thermoelectric transport properties (not determined by memory matrix formalism)

- What is the landscape of such spatially modulated ground states?