Entanglement entropy in a holographic model of the Kondo effect

Mario Flory

Max-Planck-Institut für Physik

University of Oxford

05.05.2015

Overview

- Part I: The holographic Kondo model
  - The Kondo effect
  - Bulk model: Top down and bottom up
- Part II: Including backreaction
  - Israel junction conditions
  - General results for $AdS_3/BCFT_2$
  - Exact solutions for toymodels
  - Including Chern-Simons fields
- Part III: Entanglement entropy for Kondo model
  - Numerical results
  - Qualitative discussion

Main points:

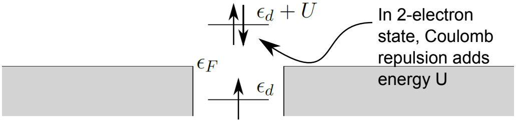
The holographic bottom up model is both tractable, and a good approximation to Kondo-like physics.

Our specific results may be numerical, but we have a thorough understanding of the qualitative features of our model due to energy conditions and geometrical considerations.

The results can be compared to field theory calculations, with interesting similarities.

Part I: Field theory side

- Spin-spin interaction of electrons with a localised magnetic impurity. This may be another type of atom (i.e. Fe in Au), or, in the *Anderson model*, an electron (*slave fermion*) bound in a quantum dot:



- Impact on resistivity at low temperatures.
- At low temperature, electrons form a bound state around impurity, the *Kondo cloud*.
- Can be mapped to a $1+1$ dimensional system [Affleck et. al. 1991]:

$$H = \frac{v}{2\pi}\psi_L^\dagger i\partial_x \psi_L + \frac{v}{2}\lambda_K \delta(x)\vec{S}\psi_L^\dagger \vec{\tau}\psi_L$$

v: Fermi velocity, λ_K: Kondo coupling.

The top-down Kondo model

Holographic top-down model [Erdmenger, Hoyos, O'Bannon, Wu: 1310.3271]:
Brane setup:

	0	1	2	3	4	5	6	7	8	9
N D3	x	x	x	x						
N_7 D7	x	x			x	x	x	x	x	x
N_5 D5	x				x	x	x	x	x	

- $D3/D7$ strings: chiral fermions in 1+1 d $\rightarrow$ electrons ψ_L.
- $D3/D5$ strings: *slave fermions* in 0+1 d $\rightarrow$ impurity spin $\vec{S} = \chi^\dagger \vec{T} \chi$.
- $D5/D7$ strings: tachyonic scalar $\rightarrow$ Formation of Kondo cloud:

$$\langle \mathcal{O} \rangle \equiv \left\langle \psi_L^\dagger \chi \right\rangle \neq 0$$

The bottom-up Kondo model

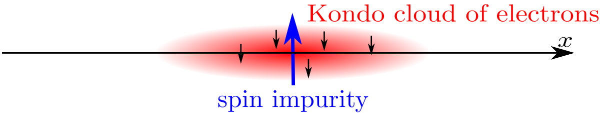
Idea: Construct top-down model, and strip it from everything that seems non-essential. [Erdmenger et. al.: 1310.3271]

→ Holographic bottom-up model:

- Dual gravity model has $2+1$ (bulk-) dimensions.
- Localised spin impurity is represented by co-dimension one hypersurface ("brane") extending from boundary into the bulk.
- Finite T is implemented by BTZ black hole background.
- Kondo cloud is described by condensation of scalar field Φ.

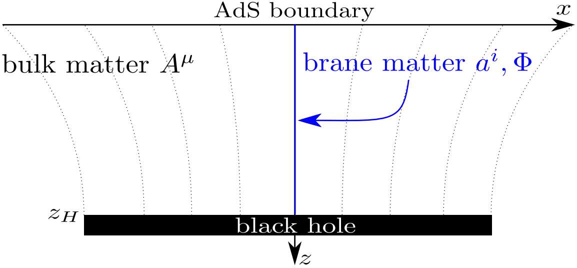
The bottom-up Kondo model

Field theory picture:



⇕

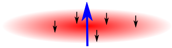
Gravity picture:



$$S = S_{CS}[A] - \int d^3x \delta(x) \sqrt{-g} \left(\frac{1}{4} f^{mn} f_{mn} + \gamma^{mn} (\mathcal{D}_m \Phi)^\dagger \mathcal{D}_n \Phi + V(\Phi^\dagger \Phi) \right)$$

What can we learn from this model?

How can we obtain information about the Kondo cloud from our model?

- Kondo cloud is formed by anti-aligned spins 
- $\Rightarrow$ expect imprint on *entanglement entropy*:

$$\mathcal{S}_{EE}(A) = -\mathrm{Tr}_A[\rho_A \log(\rho_A)], \quad \rho_A \equiv \mathrm{Tr}_B[\rho_{A\cup B}].$$

  E.g. entanglement of state $|\Psi\rangle = \frac{1}{\mathcal{N}}\left(|\underbrace{\uparrow\downarrow}_{A}\underbrace{\downarrow\dots}_{B}\rangle - |\underbrace{\downarrow\uparrow}_{A}\underbrace{\uparrow\dots}_{B}\rangle\right)$ does not vanish.

- $\mathcal{S}_{EE}$ is determined by spacelike geodesics [Ryu, Takayanagi, 2006]:

$$\mathcal{S}_{EE}(A) = \frac{\mathrm{Area}\,(\mathcal{E}_A)}{4G_N}, \quad \mathcal{E}_A\text{: co-dim two extremal surface.}$$

  $\Rightarrow$ to calculate it, we need *backreaction* on the geometry.

- What is the backreaction of an infinitely thin hypersurface carrying energy-momentum? *Israel junction conditions!*

Part II: Including backreaction

In electromagnetism: To describe field around an infinitely thin charged surface Σ, integrate Maxwells equations in a box around Σ:

$$\Rightarrow \vec{E}_{||} \text{ continuous, } \; \vec{E}_{\perp} \text{ discontinuous on } \Sigma$$

In gravity: To describe backreaction of an infinitely thin massive surface, integrate Einsteins equations in a box

$\Rightarrow$ *Israel junction conditions* [Israel, 1966]:

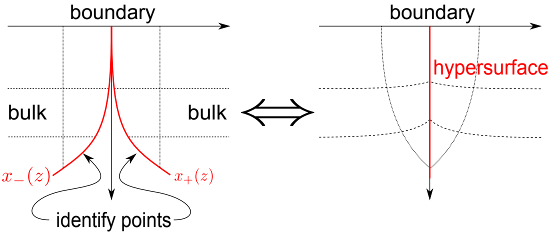
$$(K^+_{ij} - \gamma_{ij} K^+) - (K^-_{ij} - \gamma_{ij} K^-) = -\kappa \, S_{ij}$$

S_{ij}: energy momentum tensor on the brane, γ_{ij}: induced metric,
$K^{\pm}$: extrinsic curvatures depending on embedding.

Israel junction conditions

With mirror symmetry ($K^+ = -K^-$): $$K^+_{ij} - \gamma_{ij} K^+ = -\frac{\kappa}{2} S_{ij} \qquad (*)$$

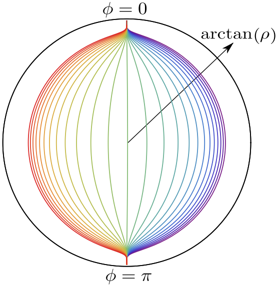
$\Rightarrow$ Embedding (location of the brane) will not be $x \equiv 0$ anymore, but a dynamical function $x(z)$ with $(*)$ its own equations of motion.

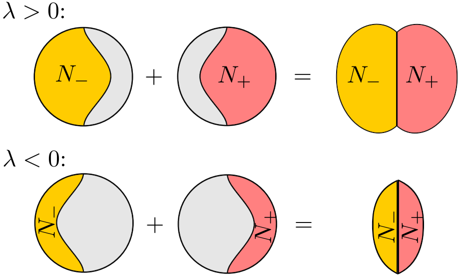


With $(*)$ we arrive at a *general* setting for the study of AdS/boundary CFT correspondence proposed by Takayanagi et. al.: [Takayanagi 2011, Fujita et. al. 2011, Nozaki et. al. 2012].

Israel junction conditions: example

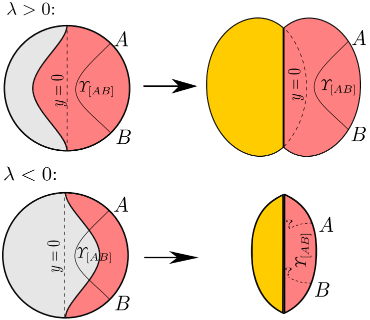
To explain this construction, look at the example of branes with constant tension $\left(S_{ij} = \frac{\lambda}{2}\gamma_{ij}\right)$ embedded in *global* AdS_3 [Azeyanagi et. al. 2007]:





- On the left: $x_+(z)$ for various λ ($\lambda >> 0$, $\lambda \approx 0$, $\lambda << 0$).
- On the right: full construction involving $x_\pm(z)$.

Israel junction conditions: example



- If brane bends the "wrong" way, curves may have to be refracted or reflected.
- Meaning for entanglement entropy, Wilson loops etc? Is $\lambda < 0$ physical?
- We will later see that such problems do not occur when WEC is satisfied.

Israel junction conditions: questions

$$K^+_{ij} - \gamma_{ij} K^+ = -\frac{\kappa}{2} S_{ij}$$

curvature = energy momentum

Geometric equations of a similar form as Einstein equations, *extrinsic* curvature tensors (K^+_{ij}) instead of *intrinsic* ones ($R_{\mu\nu}$).

General questions:

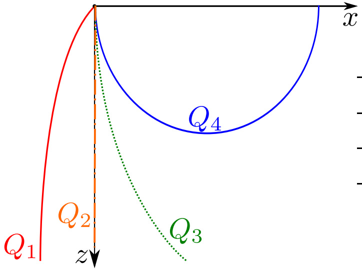
- Impact of energy conditions on possible geometries?
- Find exact solutions for simple toy models of S_{ij}?
- Investigate Kondo model?

Answers in [Erdmenger, M.F., Newrzella: 1410.7811].

Possible geometries in $2+1$ dimensions

Which impact do *energy conditions* have on the possible geometries?

Four examples:



	NEC	WEC	SEC	comment
Q_1	yes	yes	no	
Q_2	yes	yes	yes	$S_{ij}=0$
Q_3	yes	no	yes	
Q_4	yes	yes	yes	$\cup$ shaped

Null energy condition (NEC): $S_{ij}m^i m^j \geq 0 \;\; \forall \; m^i m_i = 0$

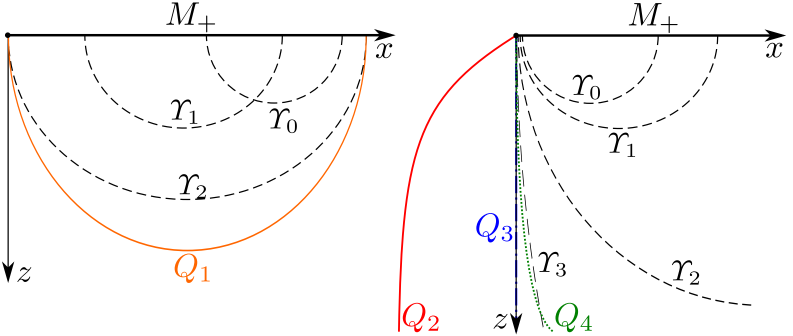
Weak energy condition (WEC): $S_{ij}m^i m^j \geq 0 \;\; \forall \; m^i m_i < 0$

Strong energy condition (SEC): $(S_{ij} - S\gamma_{ij})m^i m^j \geq 0 \;\; \forall \; m^i m_i < 0$

Possible geometries in $2+1$ dimensions

Barrier Theorem [Engelhardt, Wall: 1312.3699]

Let Q be a hypersurface splitting the spacetime N in two parts $N_\pm$ with boundaries $M_\pm$ such that $K^+_{ij} v^i v^j \leq 0$ for any vector field v^i on Q. Then any spacelike extremal surface Υ which is anchored in M_+ remains in N_+.

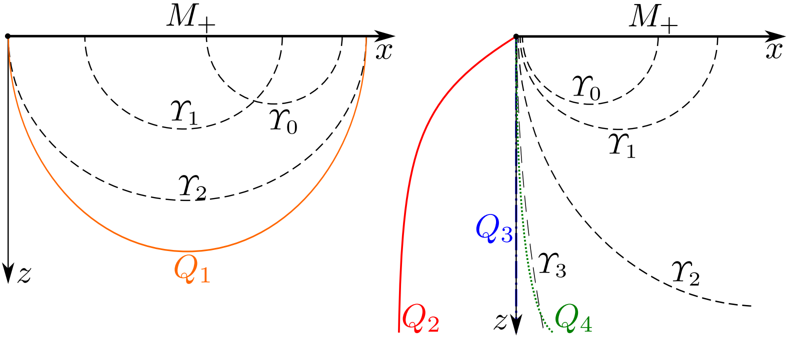


$K^+_{ij} v^i v^j \leq 0$ for Q_1, Q_3. We call Q_1, Q_2, Q_3 *extremal surface barriers.*

Possible geometries in $2+1$ dimensions

With the junction conditions, we can express the assumption made in the barrier theorem in terms of energy conditions:

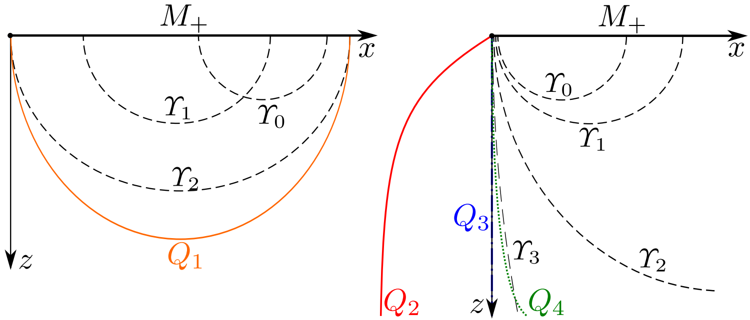
WEC and SEC satisfied on Q in 1+1 d $\Rightarrow K^+_{ij} v^i v^j \leq 0 \; \forall v^i$



$K^+_{ij} v^i v^j \leq 0$ for Q_1, Q_3. We call Q_1, Q_2, Q_3 *extremal surface barriers.*

Q_2 violates SEC, Q_4 violates WEC. For Q_3, $S_{ij} = 0$.

Possible geometries in $2 + 1$ dimensions



$K^+_{ij} v^i v^j \leq 0$ for Q_1, Q_3. We call Q_1, Q_2, Q_3 *extremal surface barriers.*
Q_2 violates SEC, Q_4 violates WEC. For Q_3, $S_{ij} = 0$.

Whether or not a brane Q bends back to the boundary or goes deep into the bulk depends on whether S_{ij} satisfies or violates WEC and SEC.

Q is an extremal surface barrier if the WEC is satisfied.

Exact analytical solutions

We first studied simple models for S_{ij} and obtained some exact analytical solutions to the junction conditions for:

- Perfect fluids: $S_{ij} = (\rho + p)u_i u_j + p\gamma_{ij}$ with $p = a \cdot \rho,\ a \in \mathbb{R}$.
- As the special case thereof with $a = 1$: The free massless scalar ϕ with

$$S_{ij} = \partial_i\phi\partial_j\phi - \frac{1}{2}\gamma_{ij}(\partial\phi)^2.$$

- The $U(1)$ Yang-Mills field a_i in the absence of sources:

$$S_{ij} = -\frac{1}{4}f^{mn}f_{mn}\gamma_{ij} + \gamma^{mn}f_{mi}f_{nj}$$

- Constant tension solutions $S_{ij} = \frac{\lambda}{2}\gamma_{ij}$.

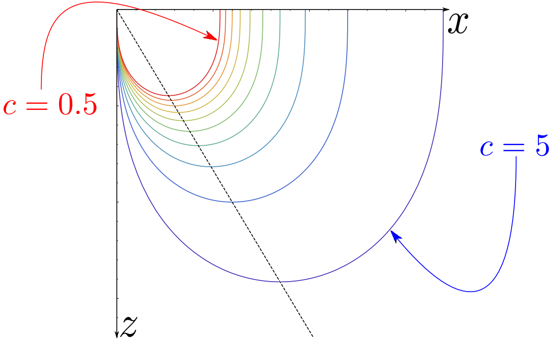
All of these were studied in AdS and BTZ backgrounds.

Exact analytical solutions

For the free massless scalar ϕ with $S_{ij} = \partial_i \phi \partial_j \phi - \frac{1}{2}\gamma_{ij}(\partial\phi)^2$, we obtain

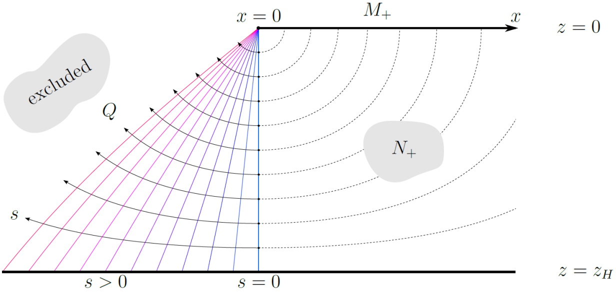
$$x_+(z) = \frac{cz^3}{3}\,{}_2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; c^2 z^4\right).$$

with ${}_2F_1(a, b; c; d)$ the hypergeometric function. WEC and SEC are satisfied, hence the brane bends back to the boundary.



Geodesic normal flow construction

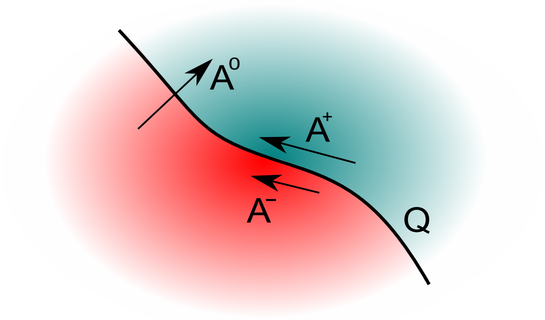
For constant curvature backgrounds, we can analytically construct constant tension solutions:



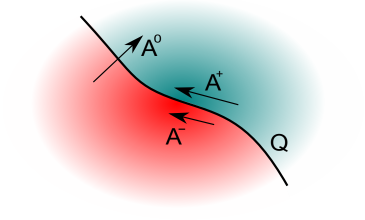
- Start in the trivial embedding with $S_{ij} = 0$.
- Construct the geodesics normal to this hypersurface.
- Shift every point of the initial hypersurface along the normal geodesics by a distance s related to the value of the tension λ.

Junction conditions for Chern-Simons field

- Our Kondo model contains both the metric field and a *Chern-Simons field* in the bulk. Assume CS field to be $U(1)$ in simplest case.
- Similarly to the metric, we get junctions conditions for the CS field along the hypersurface Q (located at $\eta \equiv 0$) if it carries a current in its worldvolume.
- Split up field: $A \sim \theta(\eta)A^+ + \theta(-\eta)A^- + \delta(\eta)A^0$.



Junction conditions for Chern-Simons field



With $D_m \equiv (A_m^{+||} + A_m^{-||})/2$ (projected mean value),

$C_m \equiv A_m^{+||} - A_m^{-||}$ (projected discontinuity),

and $A^0_\mu = A^0 n_\mu$ (component localised on Q is normal)

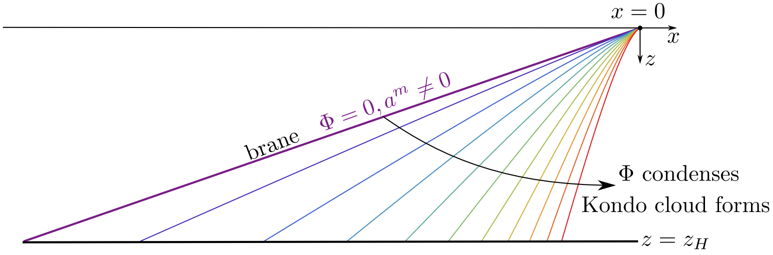
we find: $\epsilon^{mn}\left(C_n - \partial_n A^0\right) = 2\pi J^m\left[\gamma, \Phi, a, D\right]$

- What about non-abelian case, gravitational Chern-Simons terms?

Part III: Entanglement entropy for Kondo model

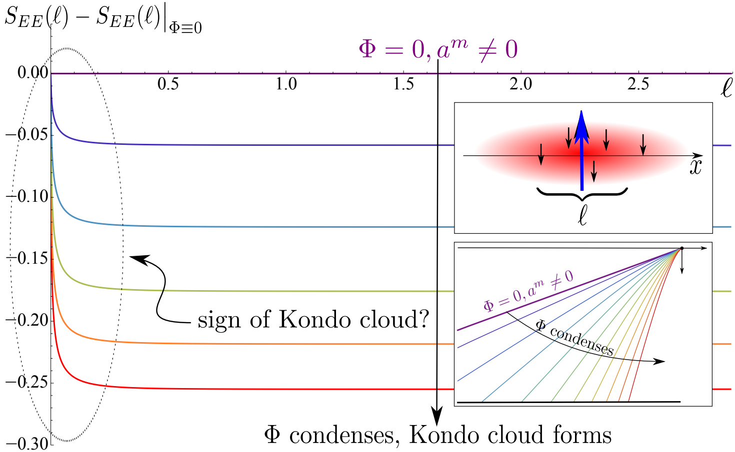
$$S_{brane}[a^m, \Phi] = -\int dV_{brane} \left(\frac{1}{4} f^{mn} f_{mn} + \gamma^{mn} (\mathcal{D}_m \Phi)^\dagger \mathcal{D}_n \Phi + V(\Phi^\dagger \Phi)\right)$$

- Due to Yang-Mills field a^m, SEC is violated everywhere in the bulk.
- Hence brane starts at boundary and falls into black hole, does *not* turn around and bend back to boundary.
- Preliminary numerical results:



Numerical results

Preliminary results on entanglement entropy: Difference of $S_{EE}(\ell)$ relative to solution with $\Phi \equiv 0$.



[Erdmenger, M.F., Hoyos, Newrzella, O'Bannon, Wu: work in progress]

Discussion: Entanglement entropy

While the precise numerics may still be improved, some of the *qualitative features* of these results follow directly from the energy conditions and simple geometric considerations.

- Entanglement entropy for given ℓ *decreases* as Kondo cloud forms: because Φ satisfies NEC, brane bends to the right.
- As $\ell \to \infty$, curves go to a *constant*, which decreases as Kondo cloud forms.
- The fall-off towards this constant value is for large ℓ *exponential*, due to simple geometric arguments.
- The holographic g-theorem for BCFTs is satisfied as the NEC is satisfied [Takayanagi 2011].

Discussion: Entanglement entropy

The fall-off towards this constant value is for large ℓ *exponential*, due to simple geometric arguments:

$$\Delta S_{EE}(\ell) \xrightarrow{\sim} c_0 + c_1(T)T\left(1 + 2e^{-4\pi\ell T} + ...\right)$$

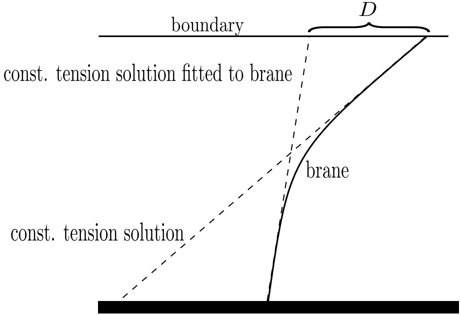
Qualitative agreement with results of field theory calculations [Affleck et. al. 2007, 2009; Eriksson, Johannesson 2011]:

$$\Delta S_{EE}(\ell) = \tilde{c}_0 + \frac{\pi^2 \xi_K T}{6v}\coth\left(\frac{2\pi\ell T}{v}\right) \to \tilde{c}_0 + \frac{\pi^2 \xi_K T}{6v}\left(1 + 2e^{-\frac{4\pi\ell T}{v}} + ...\right)$$

v: Fermi velocity, ξ_K: Kondo scale

Matching exponential behaviour in both expressions is *satisfactory*. Can we reproduce the coth behaviour?

Discussion: Large ℓ behaviour



- For large ℓ, the holographic entangling curves pass trough the brane close to the black hole event horizon.
- We can then approximate our dynamic brane by a constant tension brane, characterised only by its tension and the geometrical quantity D.
- Constant tension solutions are *analytically known*, and so is their impact on entanglement entropy.

Discussion: Large ℓ behaviour

Hence for $\ell \to \infty$ we find:

$$\Delta S_{EE}(\ell) = c_0 + S_{BH}(\ell + D) - S_{BH}(\ell)$$

with $S_{BH}(\ell) = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon} \sinh \left(\frac{2\pi\ell}{\beta} \right) \right)$.

Numerically, it turns out D becomes *small* as Kondo cloud forms!

$$\begin{aligned} \Delta S_{EE}(\ell) &\sim c_0 + D \partial_\ell S_{BH}(\ell) \\ &= c_0 + \frac{c}{3} \frac{2\pi D}{\beta} \coth \left(\frac{2\pi\ell}{\beta} \right) \end{aligned}$$

Can we read off Kondo scale ξ_K from $c \cdot D$? $\to$ Work in progress...

Summary and Outlook

- We studied a holographic model of the Kondo effect.
- Gravity dual involves thin brane carrying energy-momentum.
- Backreaction of the brane is described by Israel junction conditions.
- We obtained general results constraining possible geometries of the brane by energy conditions [Erdmenger et. al. 1410.7811].
- These results may also be applicable to holographic duals of BCFTs [Takayanagi, 2011] or the Hall effect [Melnikov et. al, 2012] involving thin branes.
- Specific Kondo model will be solved numerically, results on entanglement entropy can be compared to field theory literature.

Thank you for your attention