"Black holes, Stokes flows and DC transport at strong coupling"

Talk at the "Oxford Holography Group"

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March 2015

Based on work with J. Gauntlett : 1506.01360
J. Gauntlett and E. Banks: 1507.00234
J. Gauntlett, T. Griffin and L. Melgar: 1511.00713

Outline

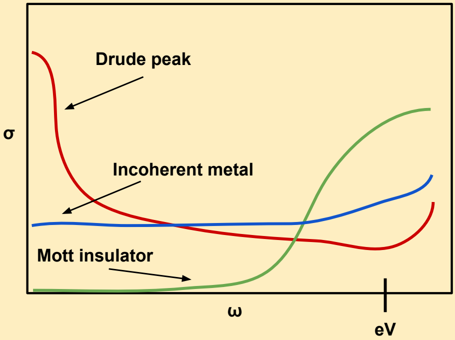
1. Motivation/Setup
2. Holography
3. Summary / Outlook

The Result

Things a black hole horizon knows about:

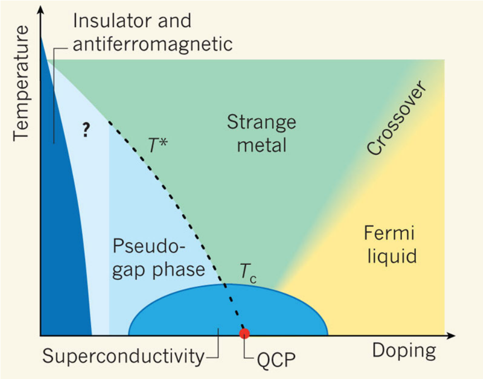
- Temperature
- Entropy $s = \frac{A}{4G}$
- Shear Viscosity (sometimes) $\eta = \frac{s}{4\pi}$
  [Son, Starinets, Policastro]
- We will add DC conductivities $\begin{pmatrix} \sigma & T\,\alpha \\ T\,\bar{\alpha} & \bar{\kappa} \end{pmatrix}$

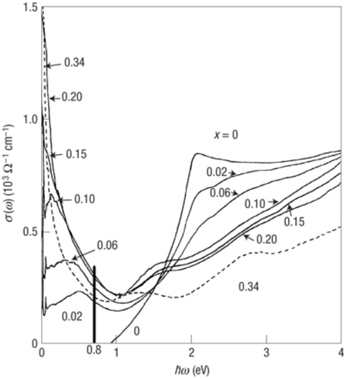
Charge transport in real materials



- Materials with charged d.o.f. can be
  - Coherent metals with a well defined Drude peak
  - Insulators
  - Incoherent conductors of electricity
- Interactions expected to become important in the incoherent phase → Possible description in AdS/CFT?

The Cuprates





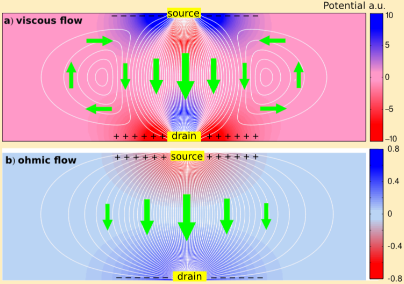
The Cuprates are real life example of :

- Incoherent transport
- Anomalous scaling of conductivity and Hall angle with T [Blake, AD]

$$\sigma_{DC}^{B=0} \propto T^{-1}, \quad \theta_H \propto T^{-2}$$

Electrons as a soup

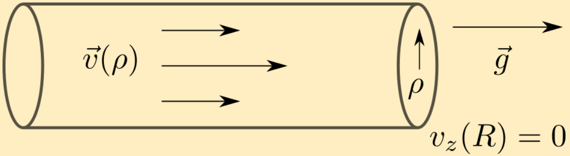
Recent evidence for high viscosity in strongly interacting electrons.
[1508.00836],[1509.04165], [1509.05691]



- Hydrodynamics accurate in the high T, momentum (quasi-) conserving regime
  [Hartnoll, Kovtun, Muller, Sachdev]
- Incoherent transport is away from this limit

Electrons as a soup

- Macroscopic effects of viscosity
  [1509.05691]



$$\cancelto{0}{\partial_t v_i} + \cancelto{0}{v^j \nabla_j v_i} + 2\,\eta\,\nabla^j \nabla_{(j}\,v_{i)} - \nabla_i p = -g, \quad \nabla_i v^i = 0$$

$$v_z = (g/4\eta)\,(R^2 - \rho^2)$$

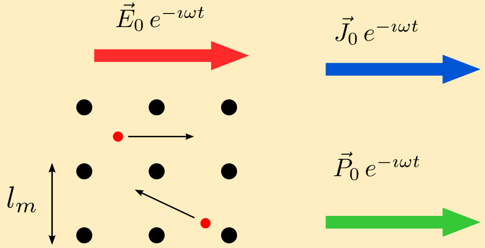
$$\Rightarrow \sigma_{DC} \approx R^2/\eta$$

btw This is a Stoke's flow

Drude Model

Put the lattice back!

- Lattice scattering (Drude physics)



- Average momentum obeys

$$\langle \dot{p} \rangle = qE - \frac{1}{\tau} \langle p \rangle \Rightarrow \sigma = \frac{nq^2}{m} \frac{\tau}{1 - \imath\omega\tau} \Rightarrow \sigma_{DC} \approx \tau \approx l_m$$

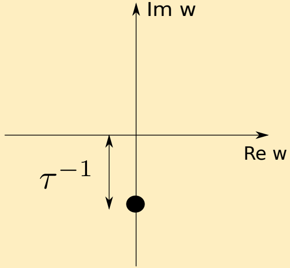
- Microscopically $\sigma = G_{JJ}(\omega)/(\imath\omega)$

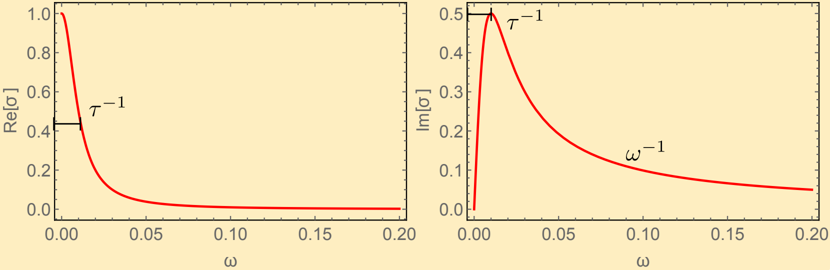
Viscosity vs Lattice Scattering

Don't need quasi-particles to have Drude physics.

Coherent metals arise when momentum relaxation is slow with dominant pole on imaginary axis.

[Hartnoll, Hofman]





Fourier/Ohm law

- We have electric currents J^i and a thermal current $Q^i = -T^i{}_t - \mu\, J^i$
- Transport coefficients are packaged in Ohm/Fourier law

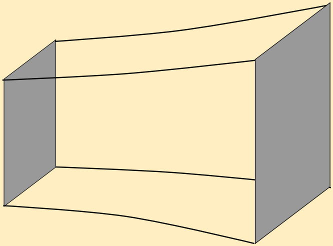
$$\begin{pmatrix} J \\ Q \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \bar{\alpha} T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E \\ -(\nabla T)/T \end{pmatrix}$$

- With ∇T a temperature gradient

Setup

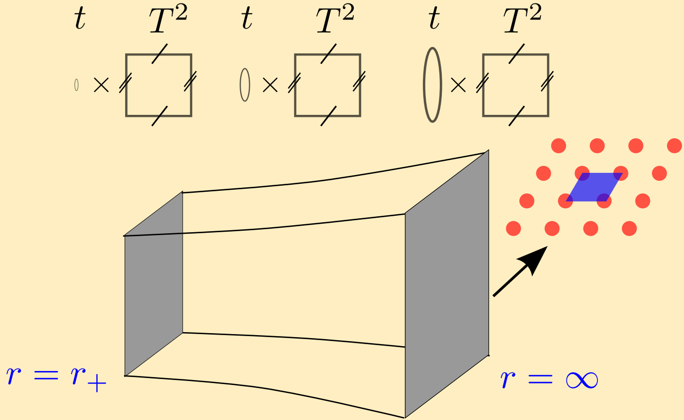
In $D=4$ Einstein-Maxwell with AdS asymptotics:

$$\mathcal{L}_{EM} = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 12$$
$$ds_4^2 = -U(r)\,dt^2 + U(r)^{-1}\,dr^2 + r^2\left(dx_1^2 + dx_2^2\right)$$
$$A = a(r)\,dt$$



Background black hole has temperature T , energy E, pressure P, entropy s and charge q.

Setup

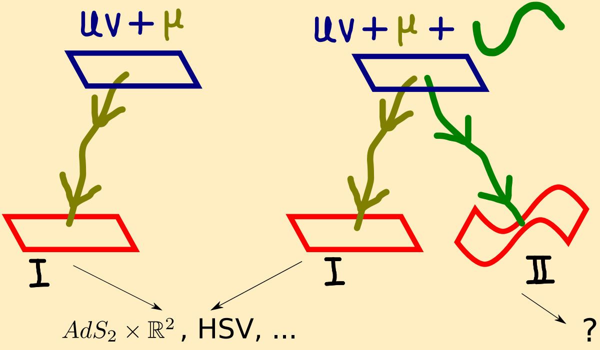


- Introduce periodic lattice (deformation) on the boundary
- Focus on simple black hole topologies
- More general statements
  [AD, Gauntlett, Griffin, Melgar]

Setup

- Deform by chemical potential μ_0 and magnetic field B
- Hold at finite temperature T
- Introduce periodic sources that can relax momentum:
  - Local chemical potential $\nabla\mu$
  - Local temperature ∇T
  - Magnetic impurities
  - Local stress + rotation
- Probe with external electric field $\nabla\delta\mu = E$ and thermal gradient $-\nabla\delta T/T = \zeta$ to extract conductivities

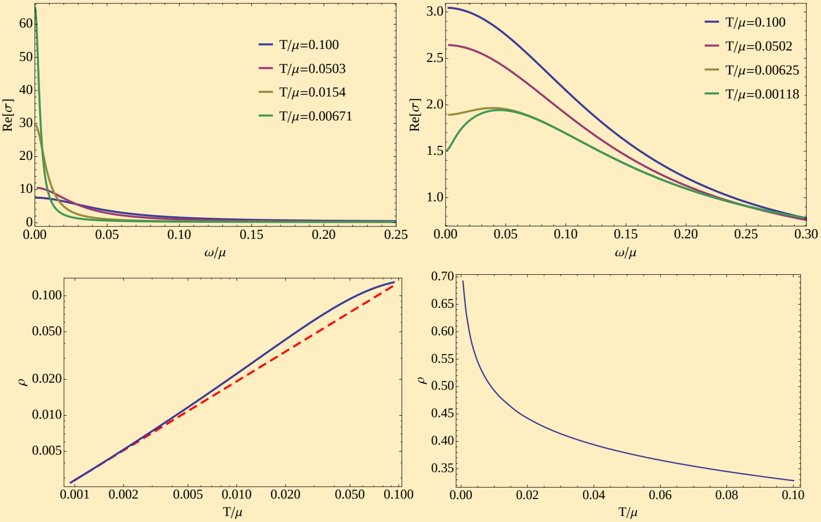
RG/Holographic picture



- I Charge dominated RG flows, translations restored in IR $\rightarrow$ Coherent transport
- II Lattice dominated RG flows, translations broken in IR $\rightarrow$ Incoherent transport

[AD, Hartnoll] [AD, Gauntlett]

Conductivity from Q-lattices [AD, Gauntlett]



- Can model Metal - Insulator transitions
- Similar story for inhomogeneous lattices
  [Rangamani, Rozali, Smyth]

Currents At Equilibrium

- For homogeneous systems we have T_0, μ_0, $\vec{B}_0$, ...
- First consider hydrodynamic limit
- Weakly break translations $\mu_0 \to \mu_0 + \delta\mu(x)$, $\vec{B}_0 \to \vec{B}_0 + \delta\vec{B}(x)$, $T \to T + \delta\mu(x)$
- In hydrodynamic limit, magnetization becomes local

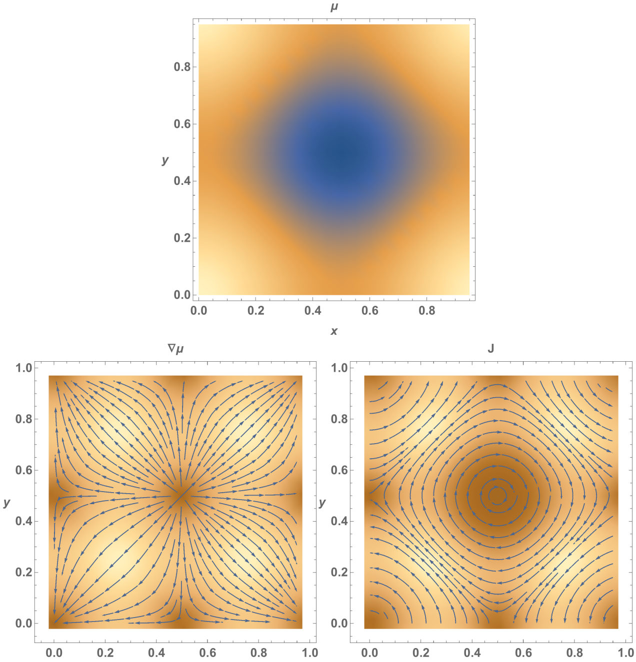
$$\delta\vec{M} = \partial_\mu \vec{M}_0 \, \delta\mu(x) + \partial_T \vec{M}_0 \, \delta T(x) + \cdots$$

$\Rightarrow$ Presence of local magnetization currents

$$\vec{J} = \vec{\nabla} \times \delta\vec{M}$$

Similar for heat currents

Currents At Equilibrium



Currents At Equilibrium

In the non-hydrodynamic limit $k = \partial_t$ is a symmetry

$$\mathcal{L}_k * J = 0 \Rightarrow i_k(d * J) + d(i_k * J) = 0$$
$$d(i_k * J) = 0$$

Assuming $\mathcal{R}_t \times M_{D-1}$ topology

$$i_k * J = d *_D M + \omega$$

with ω harmonic. Currents relax

$$\int_{C_{D-2}} i_k * J = 0 \Rightarrow \omega = 0 \Rightarrow J^i = \partial_j(\sqrt{g_{D-1}} M^{ij})$$

Similarly for the heat current

$$Q^i = T^i{}_\mu k^\mu - k^\mu A_\mu J^i = \partial_j(\sqrt{g_{D-1}} M_T^{ij})$$

DC conductivities from BH horizons

- Bulk theory is Einstein-Maxwell
- Consider E/M charged, static black branes

$$ds^2 = -UG\,(dt+\chi)^2 + \frac{F}{U}\,dr^2 + ds^2(\Sigma_d)$$
$$A = a_t\,(dt+\chi) + a_i\,dx^i$$
$$ds^2(\Sigma_d) = g_{ij}(r,x)dx^i dx^j$$

- Asymptotically, $r \to \infty$

$$U \to r^2, \qquad F \to 1$$
$$a_t(r,x) \to \mu(x), \qquad a_i(r,x) \to a_i(x)$$
$$G \to \bar{G}(x), \qquad g_{ij}(r,x) \to r^2 \bar{g}_{ij}(x), \quad \chi_i(r,x) \to \bar{\chi}_i(x)$$

- Local μ, B, T, mag impurities, surface forces

DC conductivities from BH horizons

For the perturbation write

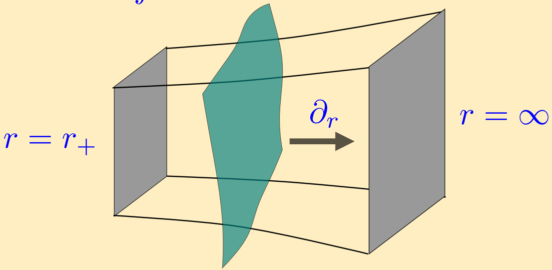
$$\delta(ds^2) = \delta g_{\mu\nu}(r,x)dx^\mu dx^\nu - 2tGU\zeta_i dt dx^i\,,$$
$$\delta A = \delta a_\mu(r,x)dx^\mu - tE_i dx^i + ta_t\zeta_i dx^i$$

- $E(x^i)$ and $\zeta(x^i)$ are closed forms
- ζ is boundary temperature gradient
- E is boundary electric field
- Count functions:
  - $g_{\mu\nu} \to \frac{1}{2}(d+2)(d+3) - (d+2)$ functions
  - $A_\mu \to (d+2) - 1$ functions

Radial Hamiltonian

- Imagine radial foliation by hypersurfaces e.g. normal to ∂_r
- Radial evolution Hamiltonian is sum of constraints

$$H_{\partial_r} = \int N\,\mathcal{H} + N_\mu\,\mathcal{H}^\mu + D\,\mathcal{G} + \mathrm{b.t.}$$

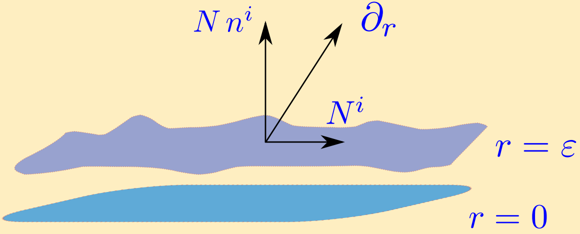


- At infinity they yield Ward identities

$$\nabla_\mu \langle T^{\mu\nu} \rangle = F^{\mu\nu}\ \langle J_\nu \rangle\,, \qquad \nabla_\mu \langle J^\mu \rangle = 0, \qquad \langle T^\mu{}_\mu \rangle = \mathrm{anom}$$

- Meaningful but not closed system without hydro

DC conductivities from BH horizons

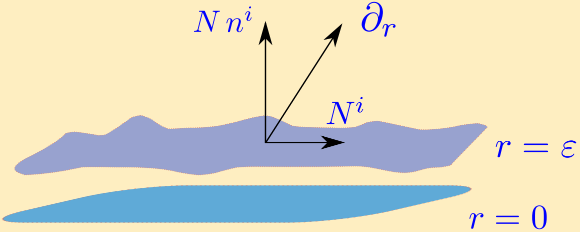


- Projection of metric $h_{\mu\nu}$ and gauge field b_μ on $r = \varepsilon$ surface
- Conjugate momentum densities $\pi^{\mu\nu}$ and π^μ with respect to ∂_r
- "Evolution" equations

$$\dot{h}_{\mu\nu} = \frac{\delta H_{\partial_r}}{\delta \pi^{\mu\nu}}, \quad \dot{\pi}^{\mu\nu} = -\frac{\delta H_{\partial_r}}{\delta h_{\mu\nu}}$$

$$\dot{b}_\mu = \frac{\delta H_{\partial_r}}{\delta \pi^\mu}, \quad \dot{\pi}^\mu = -\frac{\delta H_{\partial_r}}{\delta b_\mu}$$

DC conductivities from BH horizons



- And constraints

$$\mathcal{H}_\nu = D_\mu t^\mu{}_\nu - \frac{1}{2} f_{\nu\rho} j^\rho = 0$$
$$\mathcal{G} = D_\mu j^\mu = 0$$

- With $t^{\mu\nu} = (-h)^{-1/2}\,\pi^{\mu\nu}$ and $j^\mu = (-h)^{-1/2}\,\pi^\mu$.
- Continuity equations on the surface

DC conductivities from BH horizons

Examine constraints close to the horizon

- Impose infalling conditions
- Define

$$v_i \equiv -\delta g_{it}^{(0)}, \qquad w \equiv \delta a_t^{(0)}\,,$$

$$p \equiv -4\pi T \frac{\delta g_{rt}^{(0)}}{G^{(0)}} - \delta g_{it}^{(0)} g_{(0)}^{ij} \nabla_j \ln G^{(0)}$$

DC conductivities from BH horizons

Constraints on the horizon give

$$\mathcal{H}^t \Rightarrow \quad \nabla_i v^i = 0$$

$$\mathcal{G} \Rightarrow \quad \nabla^2 w + \nabla_i (F^{(0)i}{}_k v^k) + v^i \nabla_i a_t^{(0)} = -\nabla_i E^i$$

$$\begin{aligned} \mathcal{H}^j \Rightarrow \quad & 2\,\nabla^i \nabla_{(i} v_{j)} + a_t^{(0)} \nabla_j w - \nabla_j\, p \\ & + 4\pi T\, d\chi^{(0)}_{ji}\, v^i + F^{(0)}_{ji}\, (\nabla^i w + a_t^{(0)} v^i + F^{i(0)}{}_k\, v^k) \\ & = -4\pi T\, \zeta_j - a_t^{(0)} E_j - F^{(0)}_{ji} E^i \end{aligned}$$

- Solve for a Stokes flow on the curved black hole horizon
- Closed system of equations in d dimensions
- Nowhere made hydro assumptions!
- Related (?) work
  [Damour][Thorne, Price][Eling, Oz][Bredberg, Keeler, Lysov, Strominger]

DC conductivities from BH horizons

Electric Current

Define

$$J^i = \sqrt{-g}F^{ir}$$

- At $r \to \infty$ gives field theory current densities J^i_∞
- Anywhere in the bulk

$$\partial_r J^i = \partial_j \left(\sqrt{-g}F^{ji}\right) + \sqrt{-g}\, F^{ij}\, \zeta_j$$
$$\partial_i J^i = J^i \zeta_i$$

DC conductivities from BH horizons

Heat Current

Let $k = \partial_t$ and define

$$G^{\mu\nu} = -2\,\nabla^{[\mu}k^{\nu]} - k^{[\mu}F^{\nu]\sigma}A_\sigma - \frac{1}{2}\,(\phi - \theta)\,F^{\mu\nu}$$

and

$$Q^i = \sqrt{-g}G^{ir}$$

- At $r \to \infty$ gives field theory heat current densities

$$Q^i_\infty = -\left\langle \delta T^i{}_t \right\rangle - \mu\,\left\langle \delta J^i \right\rangle$$

- Anywhere in the bulk

$$\partial_r Q^i = \partial_j\left(\sqrt{-g}G^{ji}\right) + 2\sqrt{-g}G^{ij}\zeta_j + \sqrt{-g}ZF^{ij}E_j$$
$$\partial_i Q^i = 2Q^i\zeta_j + J^i E_i$$

DC Conductivities from BH horizons

For the background ($E_i = \zeta_i = 0$) we have

$$J_\infty^{(B)i} = \partial_j M^{(B)ij}, \quad Q_\infty^{(B)i} = \partial_j M_T^{(B)ij}$$

with the magnetizations

$$M^{ij}(x) = -\int_{r_+}^{\infty} dr\sqrt{-g}\,F^{ij}, \quad M_T^{ij}(x) = -\int_{r_+}^{\infty} dr\sqrt{-g}\,G^{ij}$$

satisfying

$$\partial_i J_\infty^{(B)i} = 0, \quad \partial_i Q_\infty^{(B)i} = 0$$

and giving and no fluxes!

DC Conductivities from BH horizons

Back to perturbations we write...

$$J^i_\infty = J^i_{(0)} + \partial_j M^{ij} - M^{(B)ij}\zeta_j$$
$$Q^i_\infty = Q^i_{(0)} + \partial_j M^{ij}_T - M^{(B)ij}E_j - 2\, M^{(B)ij}_T\zeta_j$$

The "transport components" of the currents are then [Cooper, Halperin, Ruzin]

$$\mathcal{J}^i_\infty = J^i_{(0)}, \quad \mathcal{Q}^i_\infty = Q^i_{(0)}$$

Important point is

$$\partial_i \mathcal{J}^i_\infty = 0, \quad \partial_i \mathcal{Q}^i_\infty = 0$$

$\Rightarrow$ Meaningful to examine fluxes through $d-1$ cycles!

DC conductivities from BH horizons

- Solutions for v^i, w and p are uniquely fixed by sources E and ζ
- Then

$$J^i_{(0)} = \frac{s}{4\pi}\left(\partial^i w + E^i + F^{(0)i}{}_j v^j\right) + \rho\, v^i$$

$$Q^i_{(0)} = T s\, v^i, \quad s = 4\pi\sqrt{g_{(0)}}, \quad \rho = \sqrt{g_{(0)}}\, a_t^{(0)}$$

- To find field theory currents $\bar{\mathcal{J}}^i_\infty$ and $\bar{\mathcal{Q}}^i_\infty$ in e.g. $d = 2$

$$\bar{\mathcal{J}}^1_\infty = \int dx^2\, \mathcal{J}^1_\infty, \quad \bar{\mathcal{J}}^2_\infty = \int dx^1\, \mathcal{J}^2_\infty$$

- Conductivities determined by BH horizon data!

Hydro temptation

- Meaningful quantities are

$$Q = \mathrm{vol}_d^{-1} \int \sqrt{g_{(0)}}\, a_t^{(0)}, \quad S = \mathrm{vol}_d^{-1} \int 4\pi \sqrt{g_{(0)}}$$

- Very tempting to think of it as

$$\begin{aligned}
&\nabla_i v^i = 0 \\
&\nabla^2 \delta\mu + v^i \nabla_i \rho + \nabla_i (F^{(0)i}{}_k v^k) = -\nabla_i E^i \\
&2\eta\, \nabla^i \nabla_{(i} v_{j)} + d\chi^{(0)}_{ji}\, Q^i_{(0)} + F^{(0)}_{ji} J^i_{(0)} \\
&\qquad = T\, s\, (\zeta_j + T^{-1} \nabla_j \delta T) + \rho\, (E_j + \nabla_j \delta\mu)
\end{aligned}$$

- Tempting to see it as first order hydro
- Can be misleading...
- Lorentz + Coriolis force for electric and heat currents!

DC conductivities from BH horizons

Can show (strict) positivity of transport coefficients:

$$0 < \int d^d x \sqrt{h^{(0)}} \Big(2\nabla^{(i} v^{j)} \nabla_{(i} v_{j)} + |E_i + \nabla_i w + F^{(0)}_{ij} v^j|^2 \Big)$$

$$= \int d^d x (J^i_{(0)} E_i + Q^i_{(0)} \zeta_i)$$

$$= \begin{pmatrix} \bar{E}_i & \bar{\zeta}_i \end{pmatrix} \begin{pmatrix} \sigma^{ij} & \alpha^{ij}\, T \\ \bar{\alpha}^{ij}\, T & \bar{\kappa}^{ij}\, T \end{pmatrix} \begin{pmatrix} \bar{E}_i \\ \bar{\zeta}_i \end{pmatrix}$$

In the absence of Killing vectors

$$\mathcal{L}_v g^{(0)}_{ij} = 2\, \nabla_{(i} v_{j)} = 0, \quad \mathcal{L}_v a^{(0)}_t = 0$$

- The eigenvalues are positive definite... No insulators at finite T with regular BH horizons.
- In specific cases can come up with specific numbers for the bound.

  [Grozdanov, Lucas, Sachdev, Schalm]

DC conductivities from BH horizons

The same equation shows uniqueness of solution

- Need to show that only solution to homogeneous problem is trivial
- Set sources E_i and ζ_i to zero
- Then only non-trivial solution for v^i is a Killing vector
- In translationally invariant case these zero modes generate boosted black branes
- Connected to infinity of conductivity

DC conductivities from BH horizons

Examples

- Can recover earlier results for e.g. Q-lattices and 1-dim lattices
- Perturbative, periodic lattices about AdS-RN black brane
  Let λ be the expansion parameter
  The black hole horizon is a small expansion about flat space

$$g_{(0)ij} = g\,\delta_{ij} + \lambda\,h^{(1)}_{ij} + \lambda^2\,h^{(2)}_{ij} + \cdots$$
$$a^{(0)}_t = a + \lambda\,a_{(1)} + \lambda^2\,a_{(2)} + \cdots$$
$$G^{(0)} = f_{(0)} + \lambda\,f_{(1)} + \cdots$$

Solve Navier-Stokes perturbatively in λ

DC conductivities from BH horizons

At leading order in λ we find

$$\alpha_{ij} = \bar{\alpha}_{ij} = \frac{L_{ij}^{-1}}{\lambda^2} 4\pi\rho + \ldots, \quad \bar{\kappa}_{ij} = \frac{L_{ij}^{-1}}{\lambda^2} 4\pi\, sT + \ldots$$

$$\sigma_{ij} = \frac{L_{ij}^{-1}}{\lambda^2} \frac{4\pi\rho^2}{s} + \ldots$$

Where $L_{ij} = \int_H l_{ij}\left(h_{kl}^{(1)}, a^{(1)}\right)$

Consistent with memory matrix formalism

[Barkeshli,Hartnol,Mahajan]

DC conductivities from BH horizons

Can easily include neutral scalars in the action

$$\mathcal{L} = \sqrt{-g}\,\left(R - V(\phi) - \frac{Z(\phi)}{4}\,F^2 - \frac{1}{2}(\partial\phi)^2\right)$$

$$J^i_{(0)} = Z^{(0)}\frac{s}{4\pi}\left(\partial^i w + E^i + F^{(0)i}{}_j v^j\right) + \rho\, v^i$$

$$Q^i_{(0)} = T s\, v^i, \quad s = 4\pi\,\sqrt{g_{(0)}}, \quad \rho = \sqrt{g_{(0)}}\, Z^{(0)}\, a_t^{(0)}$$

- Local change in expression for horizon electric “current density”
- Local change in expression for horizon “charge density”

DC conductivities from BH horizons

Can easily include neutral scalars in the action

$$\mathcal{L} = \sqrt{-g}\left(R - V(\phi) - \frac{Z(\phi)}{4}F^2 - \frac{1}{2}(\partial\phi)^2\right)$$

$$\nabla_i v^i = 0$$

$$\nabla^2 w + v^i \nabla_i \rho + \nabla_i (F^{(0)i}{}_k v^k) = -\nabla_i E^i$$

$$2\eta \nabla^i \nabla_{(i} v_{j)} + d\chi^{(0)}_{ji} Q^i_{(0)} + F^{(0)}_{ji} J^i_{(0)} - \nabla_j \phi^{(0)} \nabla_i \phi^{(0)} Q^i_{(0)}$$

$$= T\, s\,(\zeta_j + T^{-1}\nabla_j \delta T) + \rho\,(E_j + \nabla_j w)$$

- Extra "friction" term in Navier-Stokes equation

Onsager relations

We can easily find the time reversed background bh horizons by simply

$$\chi_i^0 \to -\chi_i^{(0)}, \quad F_{ij}^{(0)} \to -F_{ij}^{(0)}$$

- The transport coefficients of the new geometry are simply related to the original ones through Onsager relations

$$\begin{pmatrix} \tilde{\sigma} & \tilde{\alpha} \\ \tilde{\bar{\alpha}} & \tilde{\bar{\kappa}} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ \bar{\alpha} & \bar{\kappa} \end{pmatrix}^T$$

- If the background is symmetric under time reversal then these reduce to a relation among the transport coefficients
- Non-obvious after subtracting magnetisation currents in the UV theory. Proof relatively easy!

Summary / Outlook

- Holography is a tool to study transport in strongly coupled systems
- No assumption of quasiparticles
- Understand better the physics of new ground states
  [AD, Gauntlett][Withers]
- Fluid/gravity can be used to obtain (exact) DC thermoelectric conductivities
- Connection with fluid/gravity beyond DC?
- Other applications? Disorder?