Entanglement Entropy for Probe Branes

The Leading Backreaction

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November 6, 2013

Work with Andreas Karch [arXiv:1307.5325]

Oxford String Theory Group Seminar

Outline

- Entanglement Entropy - Review
  - Introduction
  - Holographic Entanglement Entropy - Ryu-Takayanagi Prescription
- Probe Brane Physics - Review
- Holographic Entanglement Entropy for Probe Branes
  - Naive Ansatz for Toy-Models
  - "Not-So-Naive-After-All": Coincidence with Some Top-Down Models
  - Concrete Examples
- Conclusion

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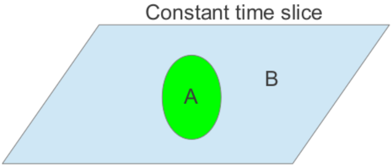
  Given a constant time slice of QFT, split the Cauchy surface into two disjoint regions, $D = A \cup B$, with the common boundary $\Sigma = \partial A = \partial B$, *the entangling surface.*



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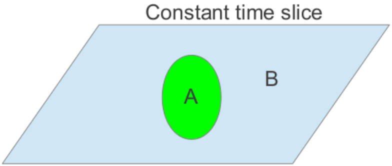

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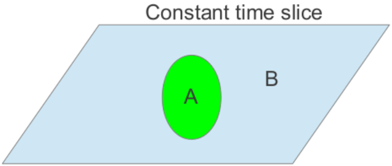

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  - Integrating out degrees of freedom in region B
    $\rightarrow \rho_A \equiv Tr_B[\rho_{A \cup B}]$
  - Calculate associated von Neumann entropy
    $\rightarrow S_{EE} \equiv -Tr_A[\rho_A \log(\rho_A)]$

Introduction of Entanglement Entropy: "Replica"

- "Replica trick" for calculating Entanglement entropy

$$S_{n-Renyi} \equiv \frac{1}{1-n} \log Tr_A[{\rho_A}^n] \xrightarrow{n \to 1} S_{EE}$$

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  - Trace out region B:

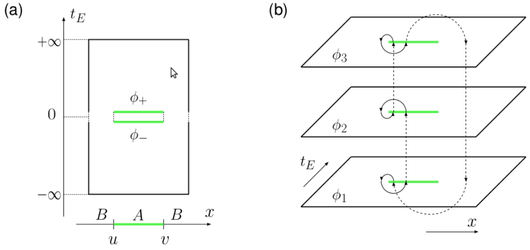
$$\rho_A = Tr_B|\Omega\rangle\langle\Omega|$$

$$\langle\psi_{A,2}|\rho_A|\psi_{A,1}\rangle = \int \mathbb{D}[\phi]_{\Big|^{\phi(x\in A,t=0^+)=\psi_{A,2}(x)}_{\phi(x\in A,t=0^-)=\psi_{A,1}(x)}} \cdot e^{-S[\phi]}$$

$$``S_{n-Renyi} \equiv \frac{1}{1-n} \log Tr_A[\rho_A{}^n]''$$

- Raise ρ_A to the n-th power, and trace in the end:

$$Tr_A[\rho_A{}^n] = \int \mathbb{D}[\phi_{A,1} \cdots \phi_{A,n}] \langle\psi_{A,1}|\rho_A|\psi_{A,2}\rangle\langle\psi_{A,2}|\rho_A|\psi_{A,3}\rangle \cdots \langle\psi_{A,n}|\rho_A|\psi_{A,1}\rangle$$



[1]

[1]Ryu and Takayanagi 2006[1], Calabrese and Cardy 2004[2]

Introduction of Entanglement Entropy: Ryu-Takayanagi Prescription

- Ryu and Takayanagi 2006[1]: " Just the horizon entropy of a surface that minimal extended from Σ"

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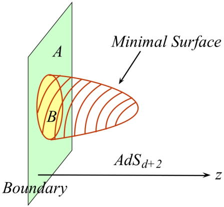

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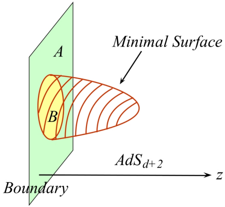

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No rigorous proof, but reproduced known results, *ex.* Calabrese&Cardy[2].

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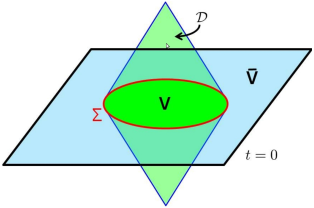
CFT and Special Spherical Entangling Surfaces Casini Huerta and Myers 2011 [4]



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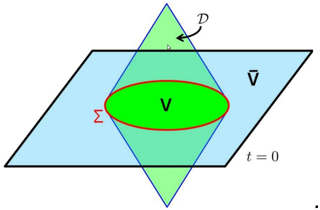

- $\Sigma \equiv S^{d-2} \to \mathcal{D}$, the whole causal development thereof
- Sphere is special!! Conformal Mapping: $\mathcal{D} \to \mathcal{H} = R_t \times H^{d-1}$

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- "KMS": $ds^2 = -dt^2 + dr^2 + r^2\, d\Omega_{d-2}^2$

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- The same von Neumann entropy $\Rightarrow S_{EE} = S_{Th}$

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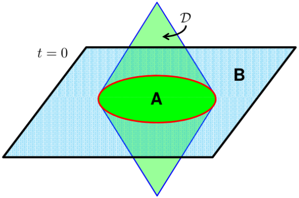

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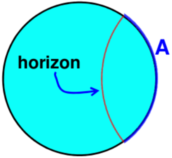

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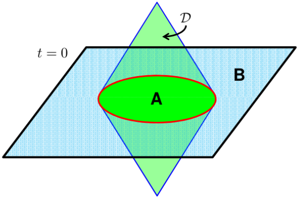

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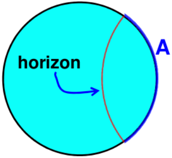

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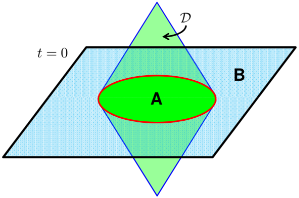

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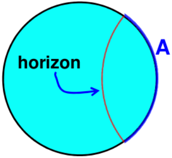

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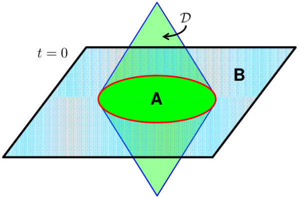

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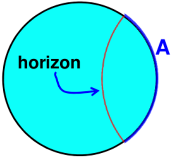

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The Ryu-Takayanagi prescription gives the same result

Further Progress

- Jensen and O'Bannon 2013[5]: Extending for any dCFT/BCFT, using the most general metric respecting the reduced conformal symmetry $\rightarrow$ conformal probe brane systems.

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- Lewkowycz and Maldacena 2013[6] provide an alternative argument for the Ryu-Takayanagi prescription.

- The original system of N_c D3 brane dynamics/ $N = 4$ $U(N_c)$ SYM theory only has adjoint field content.

Probe Branes Physics

- The original system of N_c D3 brane dynamics/ $N = 4$ $U(N_c)$ SYM theory only has adjoint field content.
- Karch and Katz 2002 [7]: the lack of fundamental matter content thereof can be fixed by inserting N_f probe branes, and a chosen separation of scale, to decouple the gauge field on the probe brane worldvolume, and introduce the fundamental matter content.

$$S_{bulk} = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g}\,\left(R + \mathcal{L}_{bulk}\right)$$

$$S_{probe} = T_0 \int d^{n+1}z\, \sqrt{-g_I}\mathcal{L}_{probe}\,(\mathcal{L}_{probe} = 1 + \ldots)$$

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- Separation of Scale: $\boxed{\frac{L^{d-1}}{16\pi G_N} \gg T_0 L^{n+1} \gg 1}$
  - L, curvature radius of background geometry, due to the choice of $\mathcal{L}_{bulk}$.
  - $G_N \sim O(N_c^2)$, Newton's constant
  - $T_0 \sim O(N_c)$, Probe brane tension
  - $\boxed{t_0 \equiv 16\pi G_N T_0 L^{n-d+2}}$

Probe Branes Physics: N_f vs. N_c

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However, the CFT dual is not known explicitly.

Probe Branes Physics: N_f vs. N_c

The top-down bulk sector is more complicated:

- $AdS_5 \times S^5$ metric with N_c RR flux
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Various top-down probe sectors are possible, *ex.* D3/D7:

- $AdS_5 \times S^3$-submanifold
- extra $N = 2$ hypermultiplet

For D-brane probes, S_{probe} has the form of the Dirac-Born-Infeld(DBI) action, as well as the Wess-Zumino term:

- $T_0 \int d^{n+1}z \sqrt{-det(g_I + 2\pi\alpha' F)} \cdots$

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Nonetheless, We will show the result for bottom-ups are the same as for top-downs!

Holographic Entanglement Entropy for the Probe Brane System

A priori, with "the Ryu-Takayanagi prescription" $+$ "$S_{bulk} + S_{probe}$" $= \cdots$

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A posteriori, these are very difficult.

A Simple Double Integral

The philosophy/utility of probe brane limit $\Rightarrow$ ignoring $\mathcal{O}(t_0^2)$

- the leading order backreaction on metric $\rightarrow$ the leading order change of minimal area?
- $T^{\mu\nu}_{probe} = \frac{2}{\sqrt{-g_I}} \frac{\delta\left(\sqrt{-g_I}\mathcal{L}_{probe}\right)}{\delta g_{\mu\nu}}\Big|_{x^\mu \rightarrow x^\mu_P(z^i)}$
- $(\delta g)_{\mu\nu} = (8\pi G_N T_0) \int d^{n+1}z \sqrt{g_I}\, G_{\mu\nu\rho\sigma}\, T^{\rho\sigma}_{probe}$

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$G_{\mu\nu\rho\sigma}$: the Green's function for linearized Einstein gravity:

- static background, Euclidean signature.
- regular inside the bulk, vanishing on the boundary.

A Simple Double Integral

The embedding $x_M^\mu = X_M^\mu(w^a)$ is solved from the following "action":

- $S_{min} \equiv \frac{1}{4G_N} \int d^{d-1}w \, \sqrt{\gamma}$
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$$\Rightarrow \boxed{\Delta S_A = (\pi T_0) \int (d^{d-1}w \sqrt{\gamma}) \, (d^{n+1}z \sqrt{g_I}) \left(T^{\mu\nu}_{min} G_{\mu\nu\rho\sigma} T^{\rho\sigma}_{probe} \right) \Bigg|_{on-shell}} \quad .$$

Complications Due to Other Bulk Background

The $\mathcal{O}(t_0)$ part of δg is included. How about $\mathcal{O}(t_0)$ part of other bulk fields, $\delta\Phi_{bulk}$? Do they contribute any $\mathcal{O}(t_0)$ part of δg?

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- IIB action $= \frac{1}{G}(\Phi_{bulk})^2$
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Secondary backreaction is subleading?...
Misleadingly, for we assume there is no background Φ_{bulk}turned on!

Real Complications Due to Other Bulk Background

With background Φ_{bulk}turned on:

- $\Phi_{bulk} \sim 1,\ \delta\Phi_{bulk} \sim 1,\ G \sim \frac{1}{N^2}$
- IIB action $= \frac{1}{G}(\Phi_{bulk} + \delta\Phi_{bulk})^2$
- stress tensor $= \ldots + N^2(\Phi_{bulk} \times \delta\Phi_{bulk}) \sim O(N)$
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Examples include:

- D3/D5 with no D5 worldvolume gauge field F turned on: $C_4 \wedge F$.
- D3/D7 with no D7 worldvolume gauge field $F \wedge F$ turned on: $C_4 \wedge F \wedge F$.

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Codimension-0 Toy Model: Spacetime filling branes

Toy Model Setup:

- $S = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{L^2} \right)$
  - $AdS_{d+1}(L)$-metric: $ds^2 = \frac{L^2}{(x^0)^2} \delta_{\mu\nu} dx^\mu dx^\nu$
- $S_{probe} = -T_0 \int d^{n+1}z \sqrt{g_I}$

Codimension-0 Toy Model: Spacetime filling branes

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Codimension-0/Spacetime filling branes, $n = d$:
The fully backreacted metric is given as $AdS_{d+1}(L) \to AdS_{d+1}(l)$, with

- $l = L \left(1 + \frac{t_0}{2d(d-1)} \right)$
- $(\delta g)_{\mu\nu} = \frac{t_0 L^2}{d(d-1)} \frac{\delta_{\mu\nu}}{(x^0)^2}$

Codimension-1 Toy Model: Randall-Sundrum-type branes

Codimension-1/RS branes, $n = d - 1$:
- Using Israel junction equation

$$(K_{ij} - g_{ij}K)\big|_{r-\epsilon}^{r+\epsilon} = 8\pi G_N T_{ij}$$

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How about top-downs? Much more complicated.

- D3/D5: $AdS_4 \times S^2$ in $AdS_5 \times S^5$
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What about the change in the internal space metric?

Assertion: irrelevant if focused on the $\mathcal{O}(t_0)$ of EE, due to:

- The background is a product manifold, $\mathcal{M} \times \mathcal{I}$
- The minimal surface is of a special codimensional-2 type.

From Toy Models to Probe-Brane Models: Internal Space

A simple example: the Green's function for a scalar field on the product manifold.

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Therefore $G(x_S, x_I, x'_S, x'_I)$ only couples the same eigenmode between two different sources!
Therefore, if one is a constant in the internal space, forget about the non-constant term of another!

For the graviton propagator in a product manifold, the same thing happens:

$$G_{\mu\nu\rho\sigma} = \sum_{k=0}^{\infty} a_k h^k_{\mu\nu} h^k_{\rho\sigma} + \sum_{k=1}^{\infty} b_k V^k_{\mu\nu} V^k_{\rho\sigma} + \sum_{k=2}^{\infty} c_k W^k_{\mu\nu} W^k_{\rho\sigma} +$$

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Moreover, there is a term, such that both indices on the "spacetime" part, a scalar in the internal space:

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...A probe will leads to non-vanishing components of *all* modes: vectors, tensors on the internal space....

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internal metric is not sourced from the minimal surface

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From Toy Models to Probe-Brane Models: Internal Space

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- EE for D3/D7 ($AdS_5 \times S^5$) $\overset{!}{\leftrightarrow}$ spacetime filling probe in AdS_5
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- Double UV-divergence in the double integral!
  - The UV divergence in the z-integral: removed by a gauge choice.
  - The remaining UV divergence in the w-integral: the physical short distance effect. $\rightarrow$ A careful holographic regularization procedure is in order.

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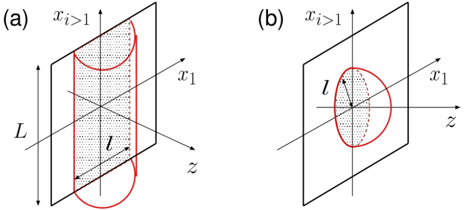
$$= \boxed{\left(A\left[g + \delta g\right]_\Sigma - A[g]_\Sigma\right) + \left(A[g]_{\Sigma'(g';g,\Sigma)} - A[g]_\Sigma\right)}$$

$$+ \; O\left(t_0^2\right)$$

Codimension-1 Randall-Sundrum type probe branes, or D3/D5 Probe Brane Systems

We investigate the following 2 cases in AdS_{d+1}:

- Spherical entangling surface bisected by the defect: the minimal surface is given by $R^2 = w_0^2 + w_1^2 + \vec{w}^2 = y^2 + \vec{w}^2$,
- Strip entangling surface bisected by the defect: the minimal surface is given by $\frac{dx^1}{dw^0} = \frac{\pm 1}{\sqrt{\left(\frac{L_U}{w^0}\right)^{2d-2} - 1}}$.



[3]

[3] Ryu and Takayanagi 2006[1]

Codimension-1 Randall-Sundrum type probe branes or D3/D5 Probe Brane Systems

Recall:

$$S_A = (\pi T_0) \int (d^{d-1} w \sqrt{\gamma})\,(d^{n+1} z \sqrt{g_I}) \left(T^{\mu\nu}_{min} G_{\mu\nu\rho\sigma} T^{\rho\sigma}_{probe} \right) \Big|_{on-shell}$$

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Spherical Entanglement Entropy for Codimension-1 Probe Branes

$$\sqrt{\gamma} = \left(\frac{\cosh r}{y}\right)^{d-2} (1-y^2)^{\frac{d-4}{2}}\, vol^S_{d-3}$$

$$\frac{1}{2}\gamma^{ab} x^{\mu}_{,a} x^{\nu}_{,b} \delta g_{\mu\nu} = -c \tanh r * Tr[\mathbb{1}_{(d-2)}] = -c\,(d-2) \tanh r$$

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$$I_1 = \frac{1}{4G_N} V^S_{d-3}\, 2 \int_{\epsilon}^{1} dy \int_{1}^{\frac{y}{\epsilon}} dc_r\, [-c(d-2)c_r^{d-3}] \frac{(1-y^2)^{\frac{d-4}{2}}}{y^{d-2}}$$

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$$\Rightarrow \tilde{I} \equiv -c(d-2)\int_1^{\frac{y}{\epsilon}} dc_r\, c_r^{d-3} + \int_{\frac{y}{\epsilon}}^{\frac{y}{\epsilon}(1+c\sqrt{1-\frac{\epsilon^2}{y^2}})} dc_r \, \frac{c_r^{d-2}}{\sqrt{c_r^2-1}}$$

Spherical Entanglement Entropy for Codimension-1 Probe Branes

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For the leading $c \sim t_0$ behavior,expand out the second term in a power series in c,

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The probe contribution to the EE then becomes

$$S_A = \frac{1}{2G_N} V^S_{d-3} c \int_\epsilon^1 dy\, \frac{(1-y^2)^{\frac{d-4}{2}}}{y^{d-2}} = \boxed{\frac{2\pi T_0}{d-1} V^H_{d-2}}$$

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V_{d-2}^H is exactly the structure to expect from a conformal field theory is $d-1$ dimensions: the EE for the defect degrees of freedom has the same functional form as that of a CFT living on the defect!

Spherical Entanglement Entropy for Codimension-1 Probe Branes

we can actually obtain the complete solution to all order of t_0!

$$ds^2 = dr^2 + \cosh^2\left(\frac{|r| - c}{L}\right) ds^2_{AdS_d}$$

$$t_0 = 4(d-1)\tanh\left(\frac{c}{L}\right) \to c = \tanh^{-1}(4\pi G_N T_0/(d-1))$$

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$$S_A = \frac{1}{2G_N} V^S_{d-3} \int_0^{-coshc} d\bar{c}_r \, \frac{\bar{c}_r^{d-2}}{\sqrt{\bar{c}_r^2 - 1}} \int_\epsilon^1 dy \, \frac{(1-y^2)^{\frac{d-4}{2}}}{y^{d-2}} = \frac{V^H_{d-2}}{2G_N} F(c),$$

$$F(c) = \int_0^{\cosh c} dc_r \frac{c_r^{d-2}}{\sqrt{1-c_r^2}} \xrightarrow{c\to 0} c + \mathcal{O}(c^2)$$

$$S_A \to \frac{2\pi T_0}{d-1} V^H_{d-2} + \mathcal{O}(T_0^2)$$

Indeed our previous result is reproduced.

Strip Entanglement Entropy for Codimension-1 Probe Branes, A Glimpse

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For the Strip Entangling Surface, bisected by the defect, with $U^{d-1} = \{(w^0, \vec{w})\} \hookrightarrow AdS_{d+1}$:

$$\frac{dx^1}{dw^0} = \frac{\pm 1}{\sqrt{\left(\frac{L_U}{w^0}\right)^{2d-2} - 1}}$$

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$$\left(\gamma^{ab} x^{\mu}_{,a} x^{\nu}_{,b} \delta g_{\mu\nu}\right)_{\vec{x}\text{-subspace}} \to \frac{-2\tilde{c}|x^1|}{\sqrt{(x^0)^2 + (x^1)^2}} \mathrm{Tr}[\mathbb{1}_{d-2}] = \frac{-2\tilde{c}|x^1|}{\sqrt{(x^0)^2 + (x^1)^2}}(d-2)$$

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$$\int_\epsilon^1 \frac{dw^0}{(w^0)^{d-2}\sqrt{(w^0)^2-(w^0)^{2d}}} \frac{(-2\tilde{c}|x^1|)\left(d-2+\frac{(w^0)^2-(w^0)^{2d}}{(x^1)^2+(w^0)^2}\left(1+\frac{x^1(w^0)^{d-1}}{\sqrt{(w^0)^2-(w^0)^{2d}}}\right)^2\right)}{\sqrt{(x^1)^2+(w^0)^2}}$$

$$\frac{S_{sub}}{\mathrm{Vol}^{d-2}_{Span\{\vec{w}\}}} = \left(\frac{2\tilde{c}x^1}{\sqrt{(x^1)^2+(w^0)^2}}\frac{1}{(w^0)^{d-2}}\right)_{w^0\to\epsilon}$$

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$$\Rightarrow \tilde{s}_A = \frac{(d-1)L_U^{d-2}}{\pi T_0 L^d} \frac{S_A}{\mathrm{Vol}^{d-2}_{Span\{\vec{w}\}}} = \frac{(d-1)L_U^{d-2}}{4G_N \pi T_0 L^d}\left(2.0\ \tilde{c}\ \frac{L^{d-1}}{L_U^{d-2}}\right) = \boxed{2.0} \quad , \text{ for } d \in \{4, 5, 6, 7\}$$

Toy-2 Spacetime filling probe branes or D3/D7 Probe Brane Systems

- Recall the exact solution is just AdS_{d+1} with a new curvature radius
  $l = L\left(1 + \frac{t_0}{2d(d-1)}\right) \rightarrow (\delta g)_{\mu\nu} = \frac{t_0 L^2}{d(d-1)} \frac{\delta_{\mu\nu}}{(x^0)^2}$.

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\begin{aligned}
S_A &= \frac{L^{d-1}}{4G_N} V^S_{d-2} \int_{R/a}^{1} dy \, \frac{(1-y^2)^{(d-3)/2}}{y^{d-1}} \\
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This also agrees with our result, and the result from Jensen and O'Bannon.

Conclusion

- We propose an formula for probe brane systems, and we calculate the simplest examples of the toy models:
  - spherical entangling surfaces,
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Thank you very much for your attention!

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