

Generalized Entropy and higher derivative Gravity

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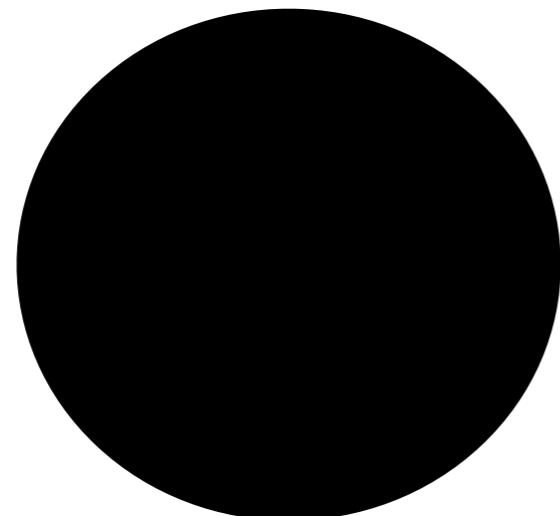
based on 1310.6659, see also 1310.5713 by Xi Dong

Oxford, 27 May 2014

Entropy in General Relativity

Bekenstein, Hawking

$$I = \frac{1}{16\pi G} \int \sqrt{-g} d^D x R$$



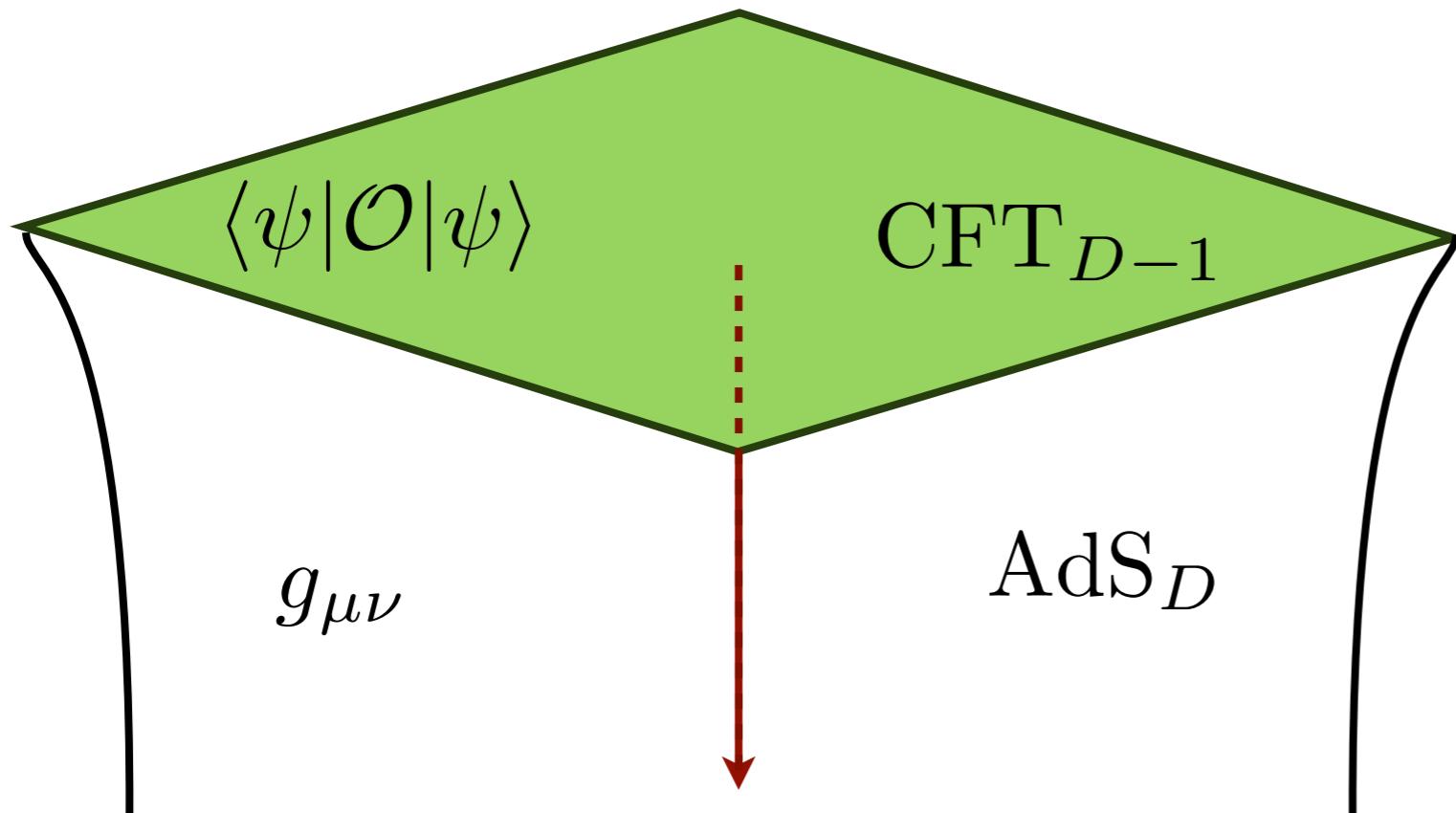
$$S = \frac{\mathcal{A}}{4G}$$

$$\delta S \geq 0$$

$$dM = TdS + \omega dJ$$

AdS/CFT

Maldacena



$$Z_{\text{AdS}_D} = Z_{\text{CFT}_{D-1}}$$

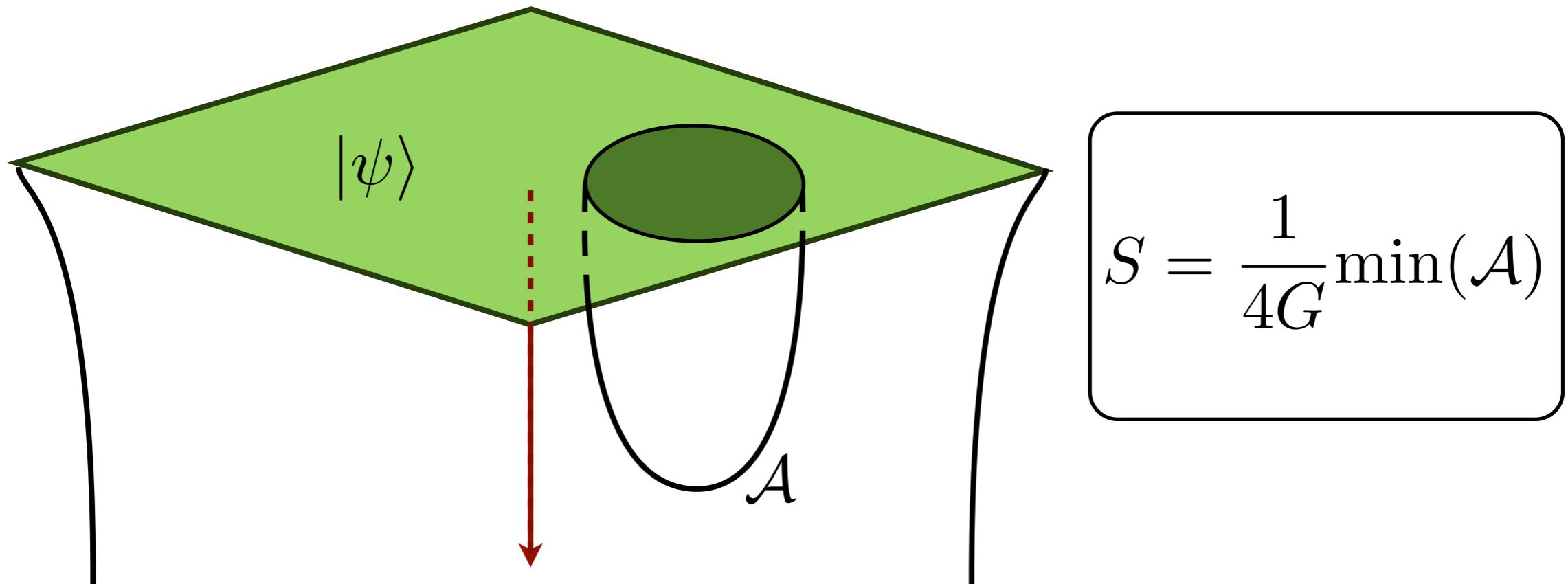
$$N \rightarrow \infty$$

$$\lambda \rightarrow \infty$$

$$Z_{\text{AdS}_D} \approx e^{-I_E[g_{\mu\nu}]}$$

Entanglement in AdS/CFT

Ryu, Takayanagi



Amongst many virtues, this formula makes transparent some properties of entanglement otherwise hidden

Why higher derivatives

- Such corrections are predicted in String Theory (essential to the great success in microscopic accounts of entropy)

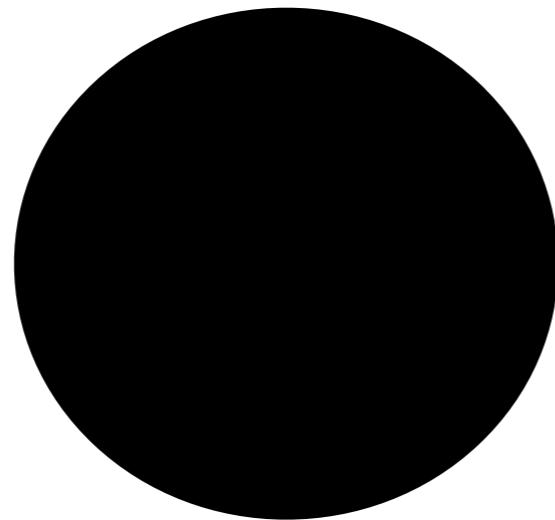
$$I = \frac{1}{16\pi G} \int \sqrt{-g} d^D x \left(R + \alpha' \text{Riem}^2 + \dots \right)$$

- Deformations of AdS/CFT: playground, robustness, universality η/s $c \neq a$
- It is an interesting problem *per se*

Entropy for higher derivative theories

Wald

$$I = \int \sqrt{-g} d^D x \mathcal{L}(R_{\mu\nu\rho\sigma}, \nabla_\mu, g_{\mu\nu})$$



$$\delta S \geq 0 \quad ???$$

$$dM = TdS + \omega dJ$$

$$S = -2\pi \int_W \sqrt{\gamma} d^{D-2} \sigma \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \left. \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \right|_W$$

What this talk is about

- Review of an understanding of the origin of the Ryu-Takayangi proposal
- Use this understanding to derive a formula for higher derivative gravity
- **Warning:** This talk is about a calculation

Overview

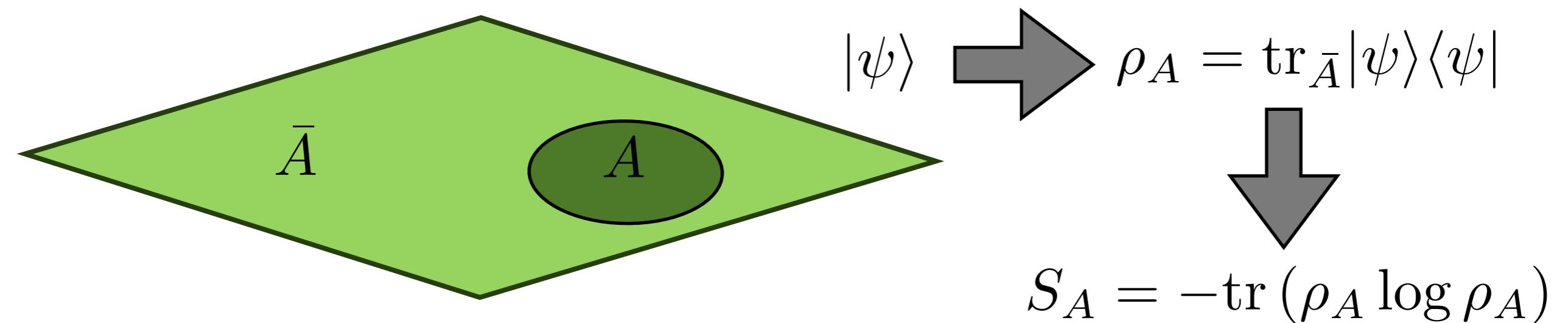
- Generalized entropy
- Replica trick
- Gravity dual of the replica trick

- Action of regularised cones
- Comments on the new entropy

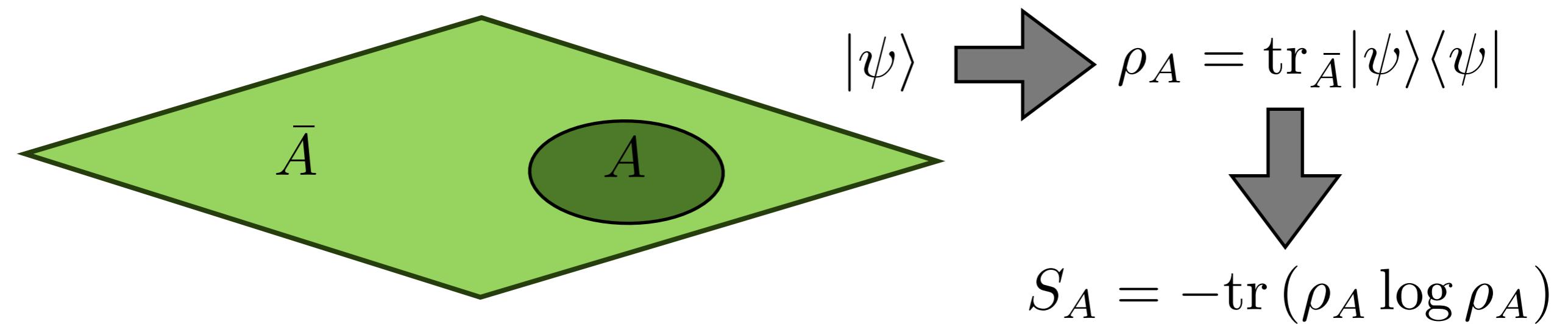
Generalized Entropy

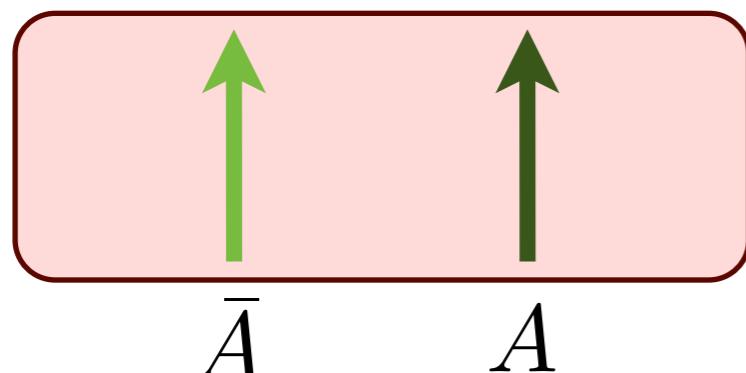
Lewkowycz, Maldacena

Entanglement entropy



Entanglement entropy



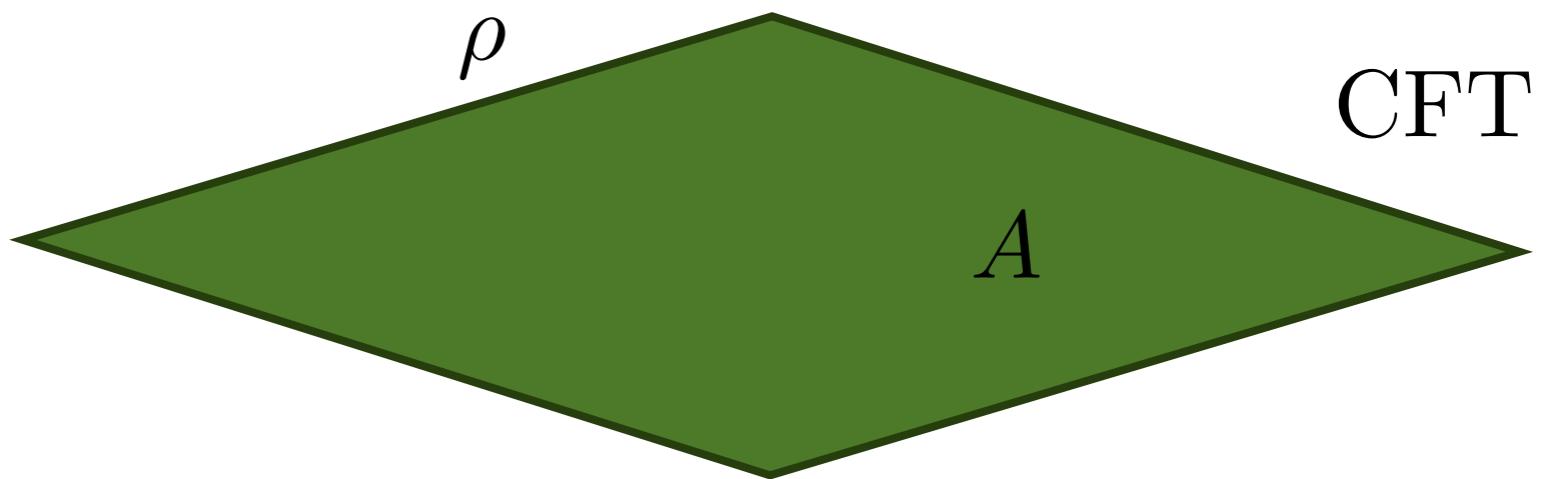
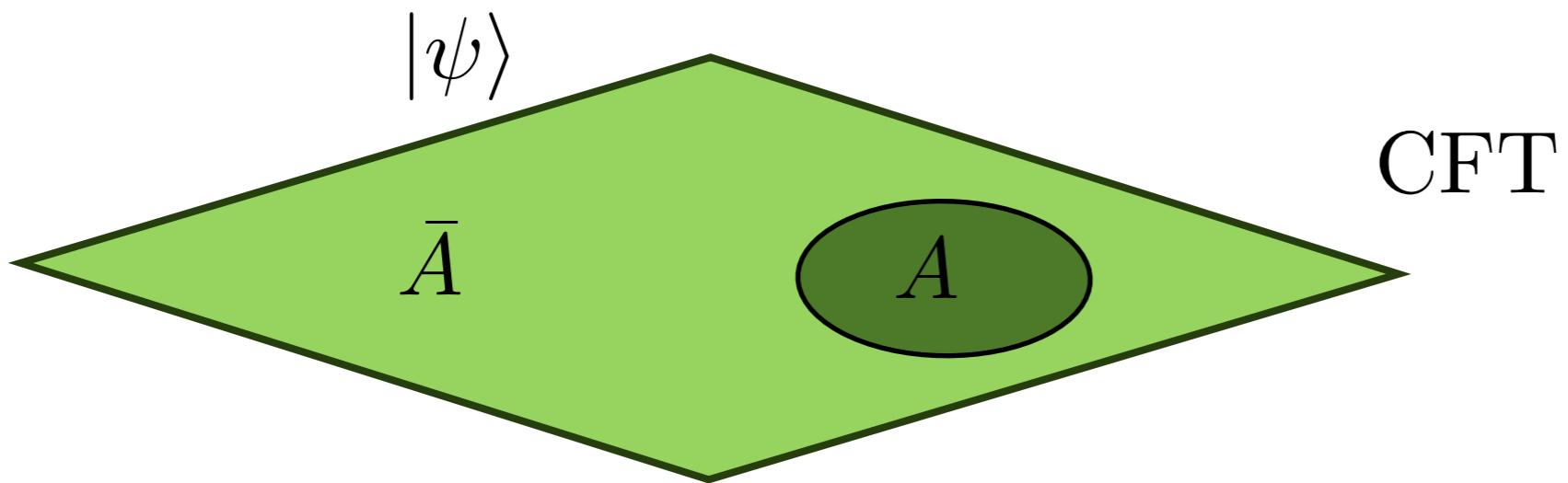

$$|\psi\rangle = \frac{1}{\sqrt{2}} |-\rangle_A |+\rangle_{\bar{A}} + \frac{1}{\sqrt{2}} |+\rangle_A |-\rangle_{\bar{A}}$$

The diagram shows two states represented by red boxes with arrows. The left state, labeled \bar{A} , has two green arrows pointing up. The right state, labeled A , has one green arrow pointing up.

$$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

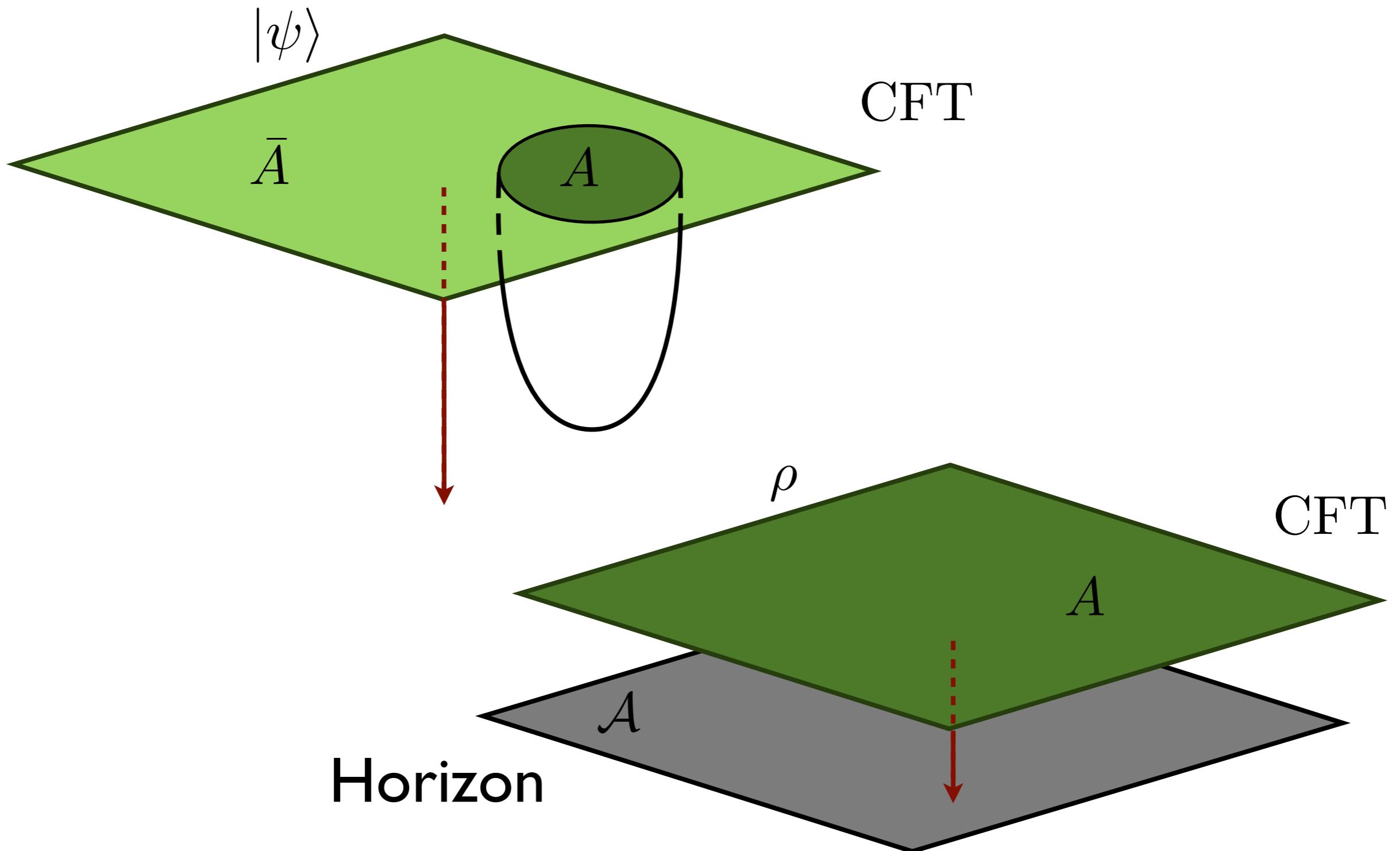
$$S_A = \log 2$$

Setup



Setup

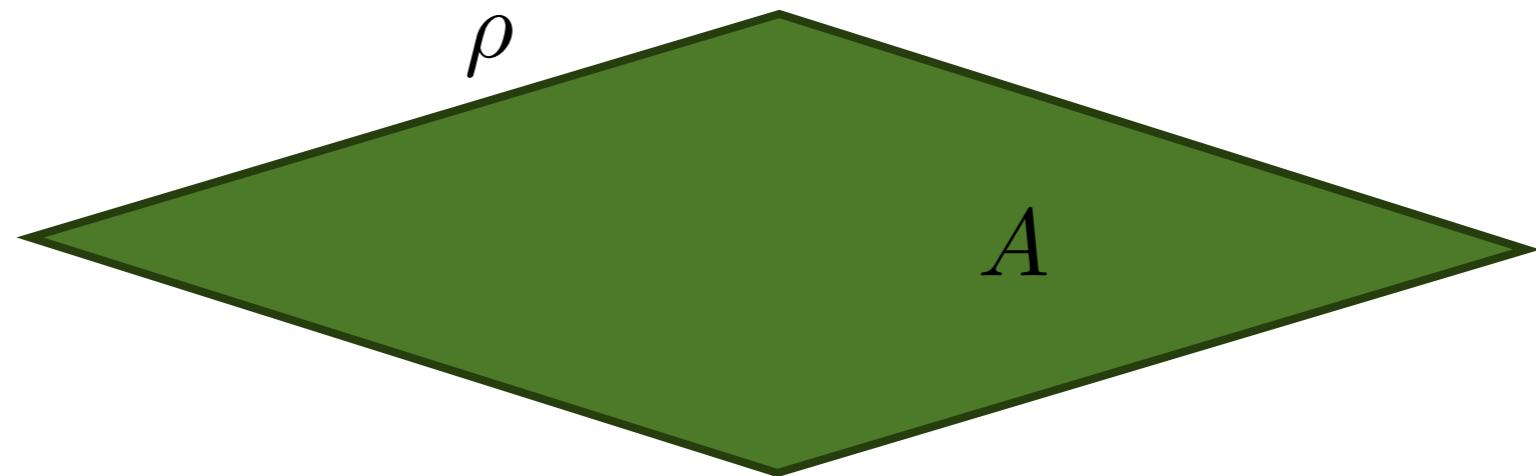
Casini, Huerta, Myers



Replica trick

Entanglement Entropy

$$S = -\text{tr}(\rho \log \rho)$$

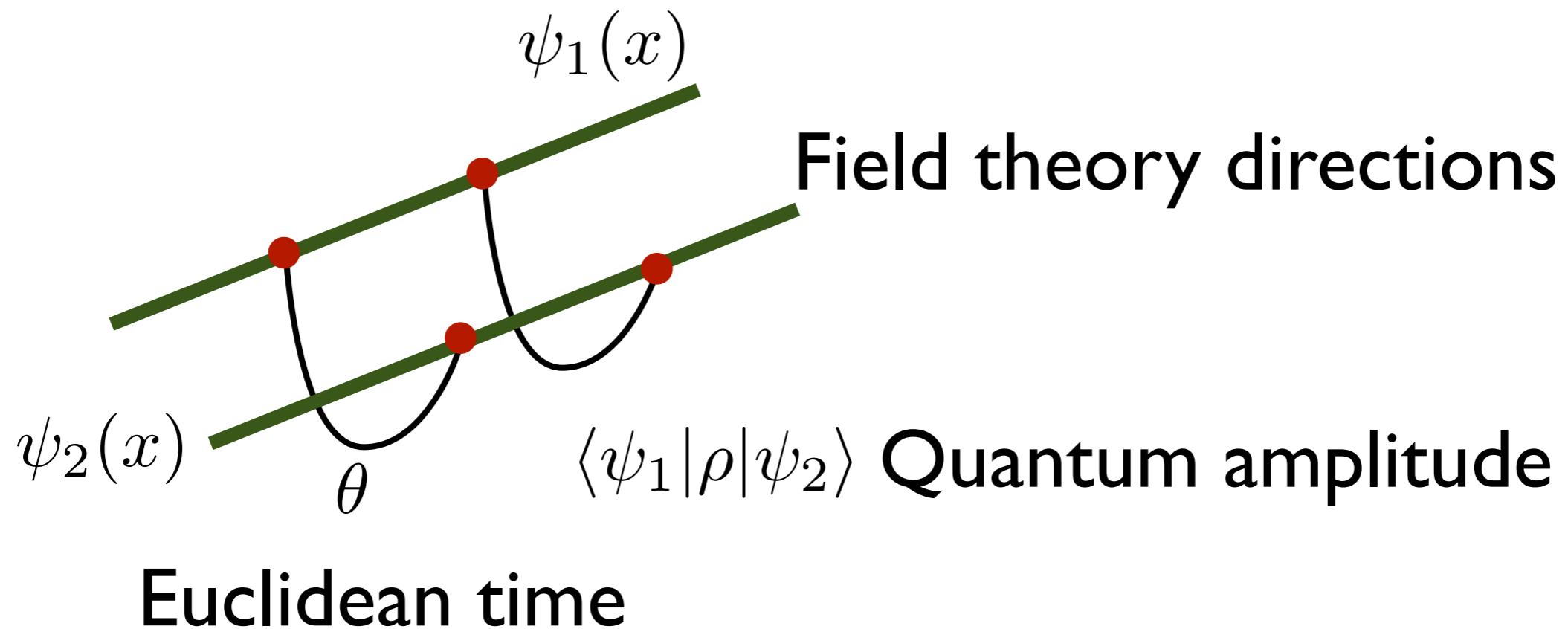


Trick: Consider Rényi entropies

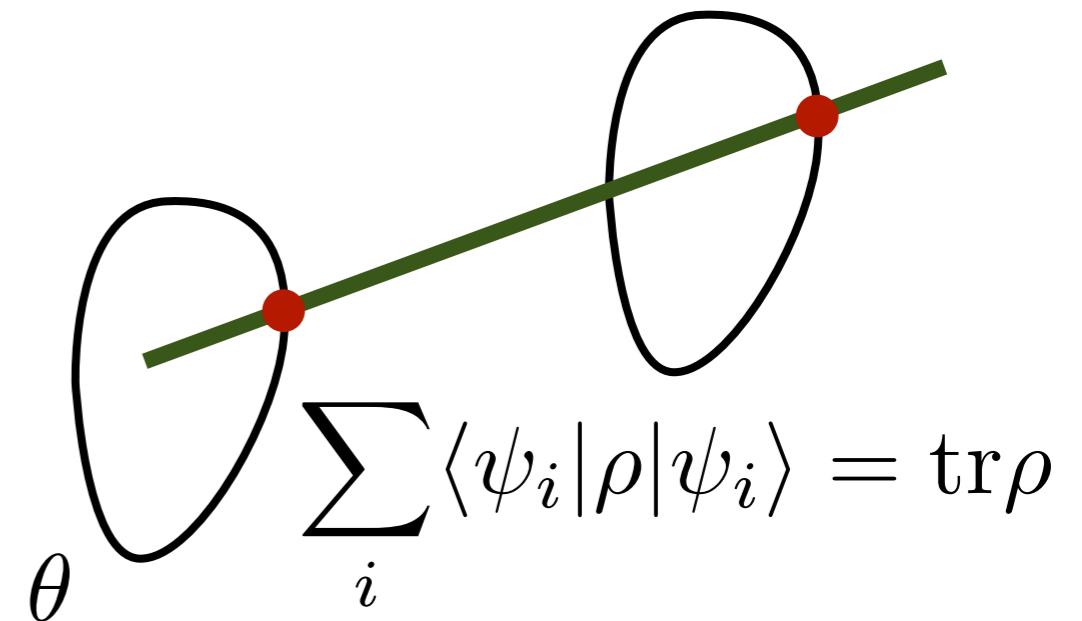
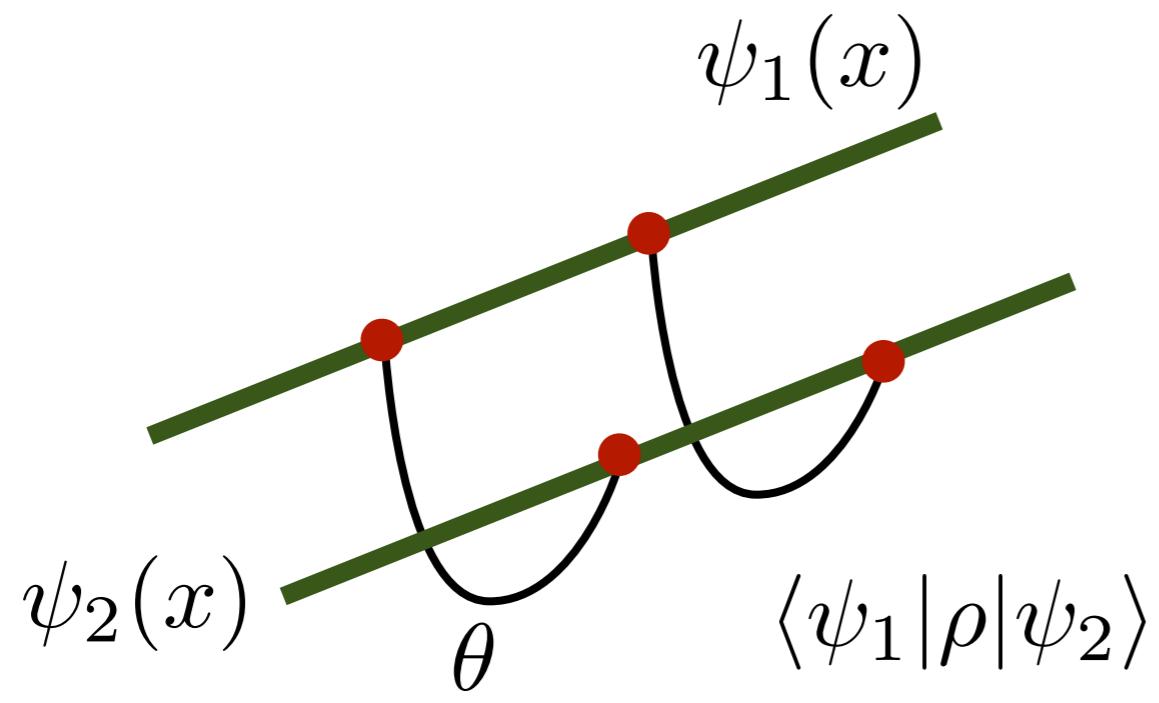
$$S_n = \frac{\log(\text{tr} \rho^n)}{1 - n}$$

$$S = \lim_{n \rightarrow 1} S_n = - \partial_n \log \text{tr} (\rho^n) \Big|_{n=1}$$

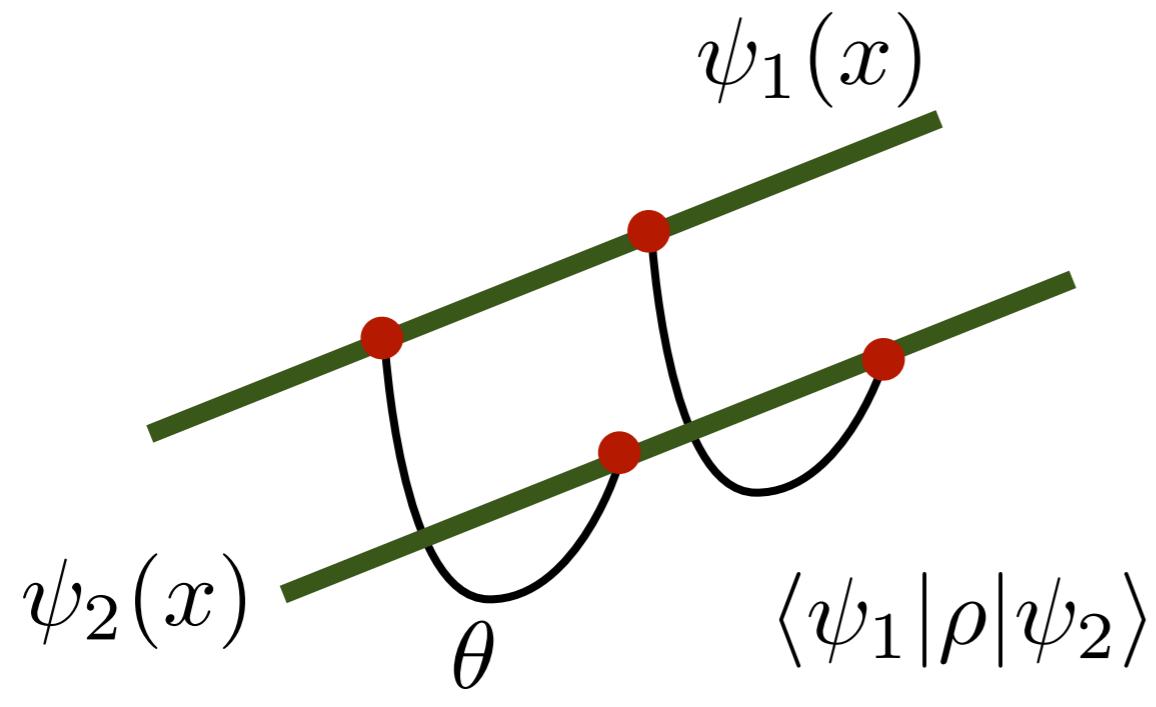
Path Integral calculation



Path Integral calculation



Path Integral calculation



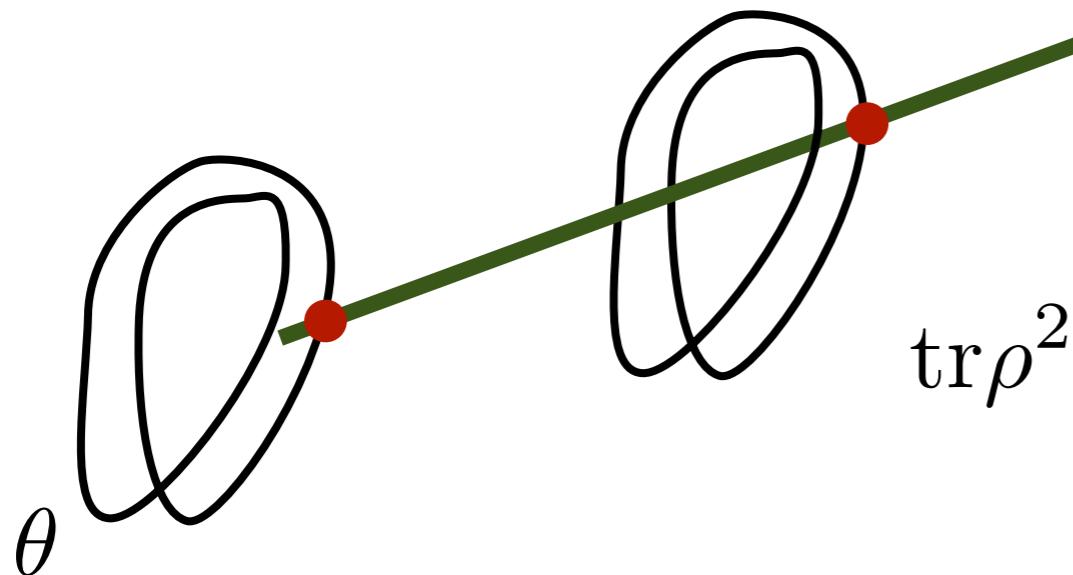
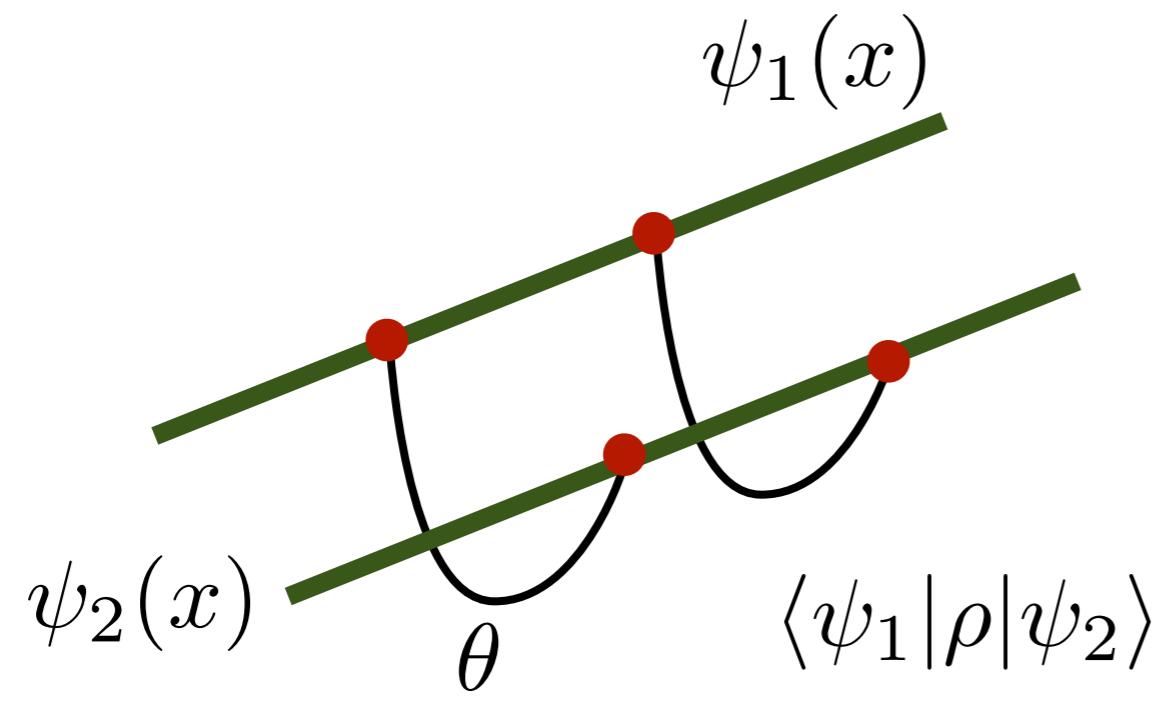
A diagram illustrating the trace of a density operator as a sum of path integrals. It shows a single green line with two red dots representing the state ψ_i . Two black ovals, representing loops, are attached to the line at these points. The angle between the line and the loops is labeled θ . The expression for the trace is given as:

$$\sum_i \langle \psi_i | \rho | \psi_i \rangle = \text{tr} \rho$$

A diagram illustrating the trace of the square of a density operator as a double sum of path integrals. It shows two green lines with red dots representing states ψ_i and ψ_j . Each line has a corresponding black oval loop attached to it. The angle between the lines is labeled θ . The expression for the trace is given as:

$$\text{tr} \rho^2 = \sum_i \sum_j \langle \psi_i | \rho | \psi_j \rangle \langle \psi_j | \rho | \psi_i \rangle$$

Path Integral calculation

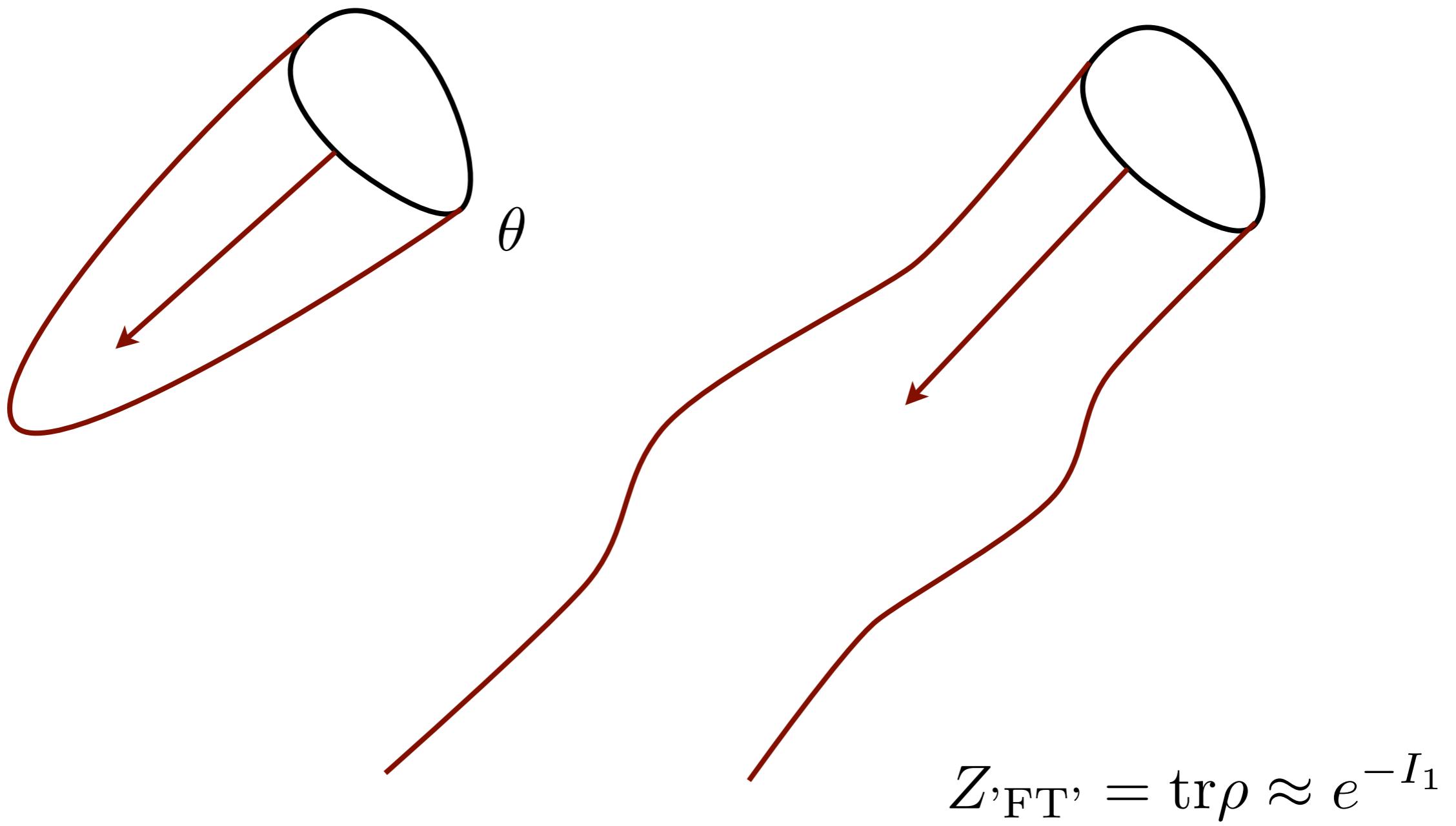


$$\theta \sum_i \langle \psi_i | \rho | \psi_i \rangle = \text{tr} \rho$$

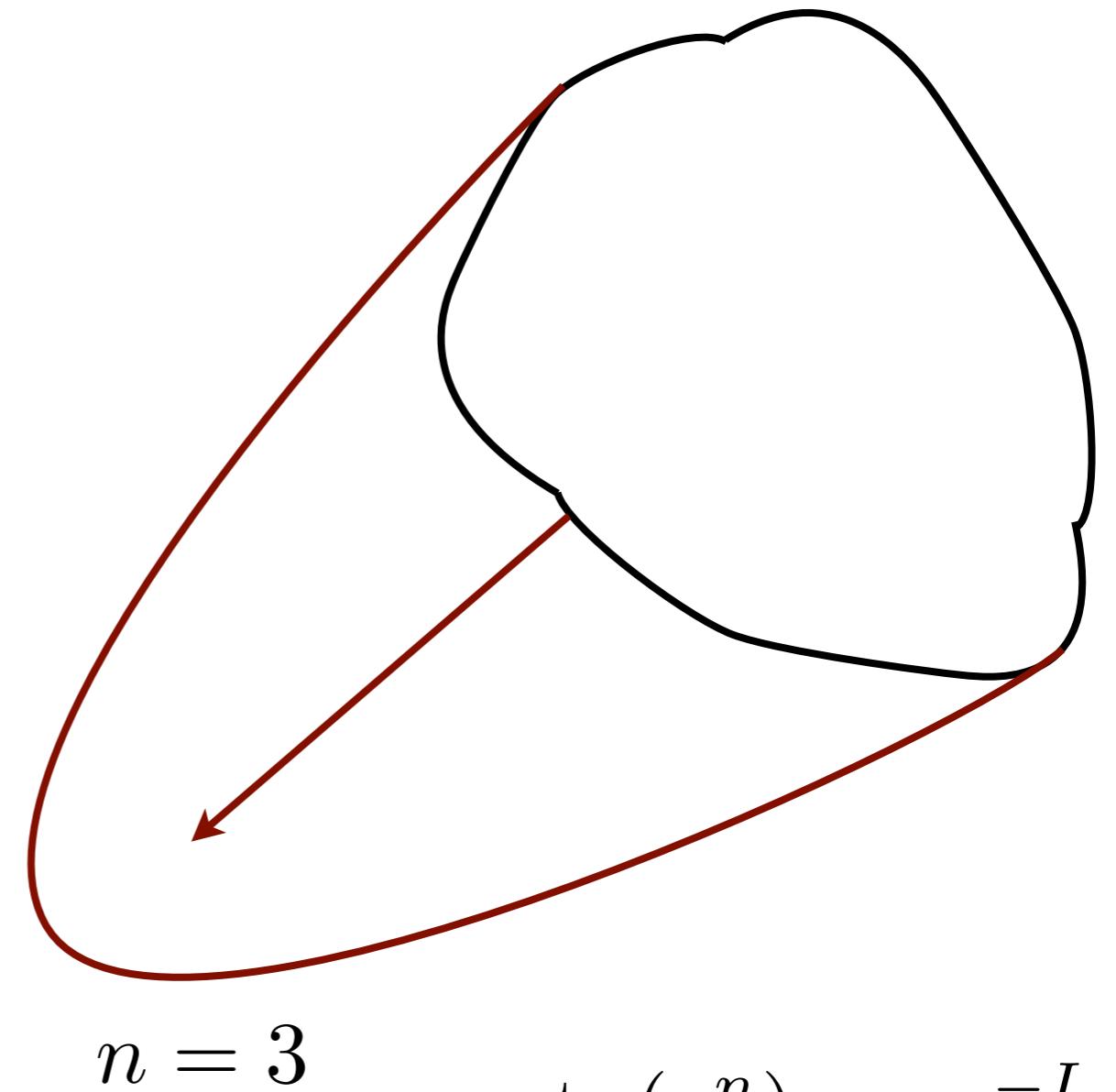
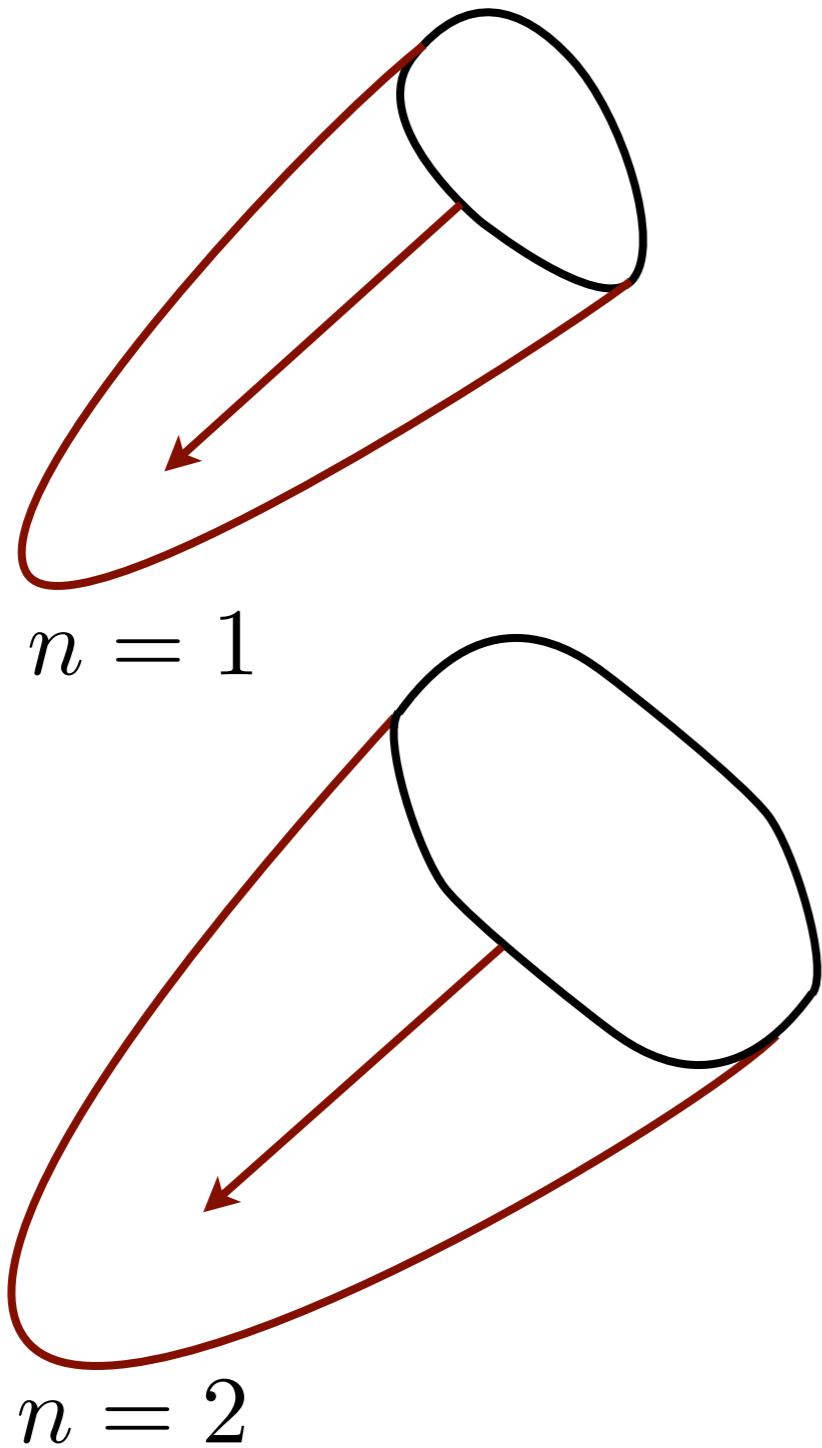
A diagram showing the decomposition of a path integral. It consists of two parts separated by an equals sign (=). The left part shows a green line with red dots and black loops, with a label θ below it. The right part shows a green line with red dots and black loops, also with a label θ below it.

A diagram showing the decomposition of a path integral. It consists of two parts separated by an equals sign (=). The left part shows a green line with red dots and black loops, with a label θ below it. The right part shows a green line with red dots and black loops, also with a label θ below it.

Gravity dual



Gravity dual



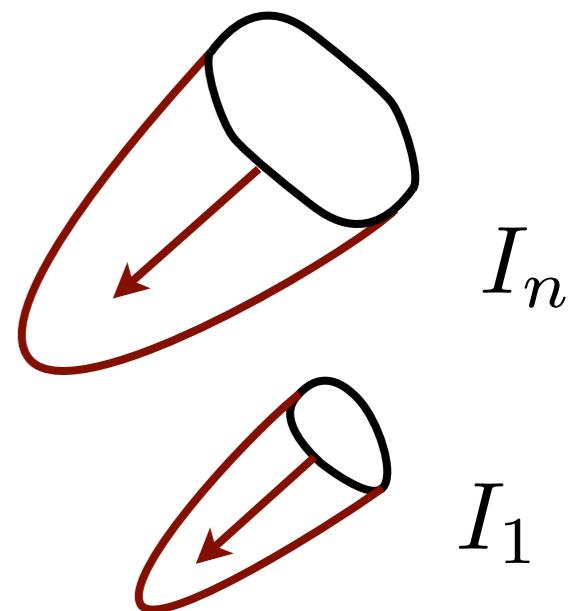
$$\frac{\text{tr}(\rho^n)}{(\text{tr}\rho)^n} \approx \frac{e^{-I_n}}{e^{-nI_1}}$$

Generalized entropy

Entanglement entropy:
Replica trick $S = \lim_{n \rightarrow 1} S_n = -\partial_n \log \frac{\text{tr}(\rho^n)}{(\text{tr}\rho)^n} \Big|_{n=1}$

Gravity saddlepoint

$$\frac{\text{tr}(\rho^n)}{(\text{tr}\rho)^n} \approx \frac{e^{-I_n}}{e^{-nI_1}}$$

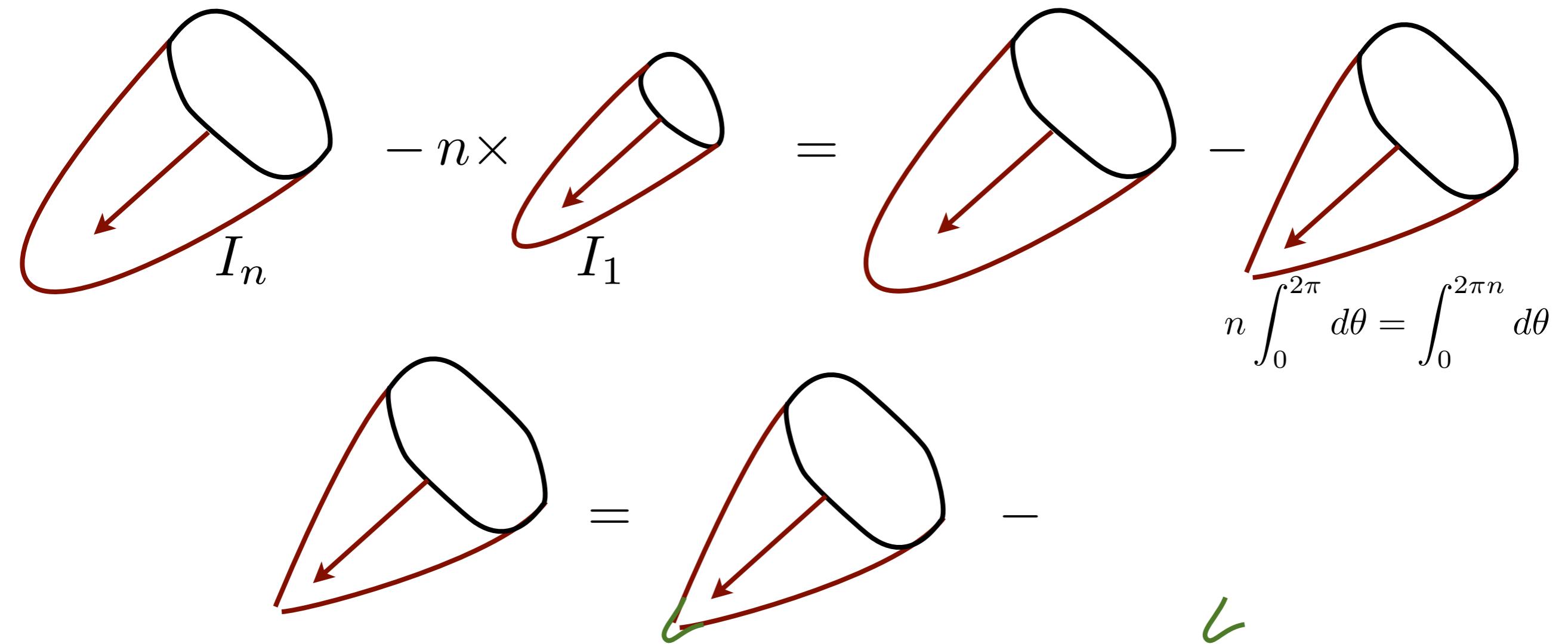


Gravitational entropy

$$S = \partial_n (I_n - nI_1) \Big|_{n=1}$$

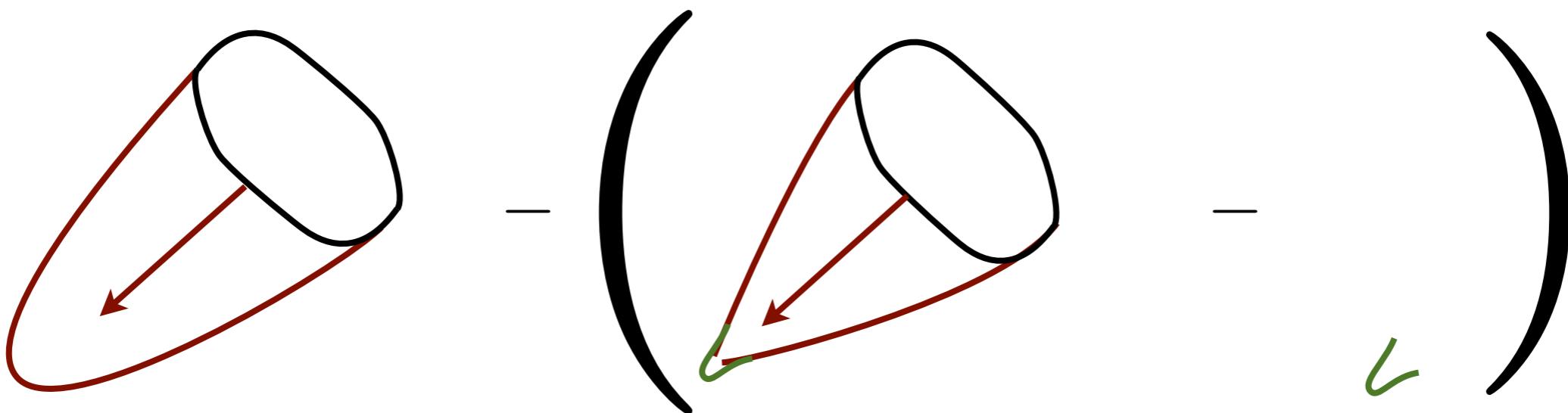
Replica manipulations

$$S = \partial_n (I_n - nI_1)|_{n=1}$$



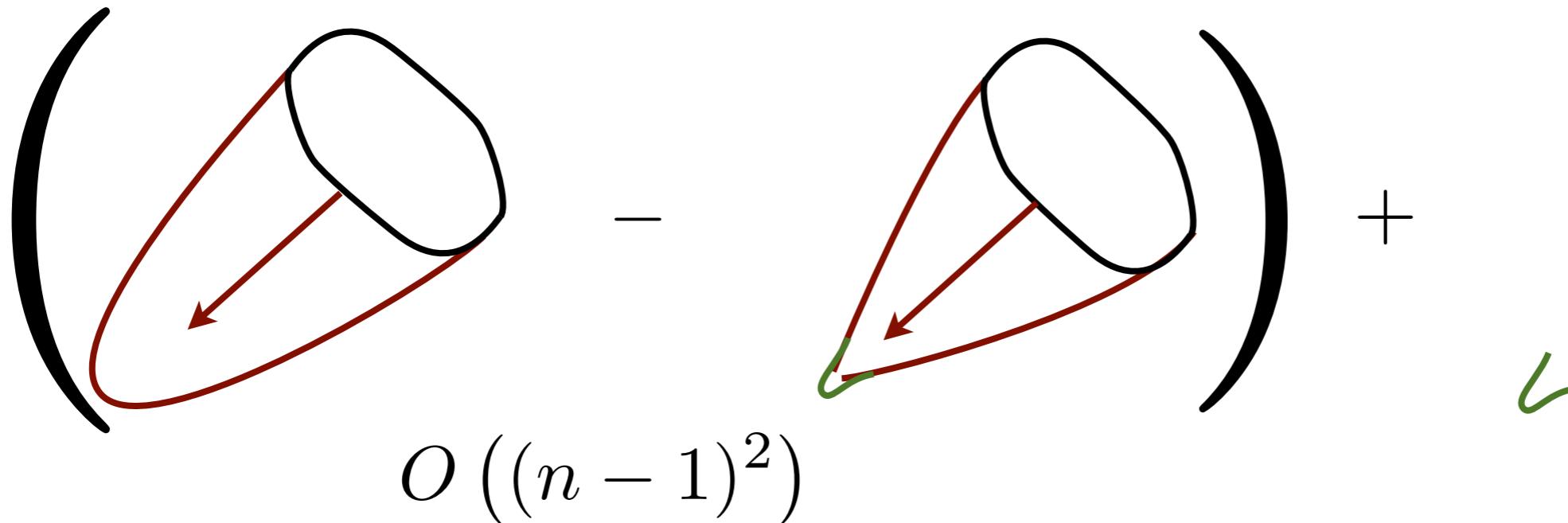
Replica manipulations

$$S = \partial_n (I_n - nI_1)|_{n=1}$$



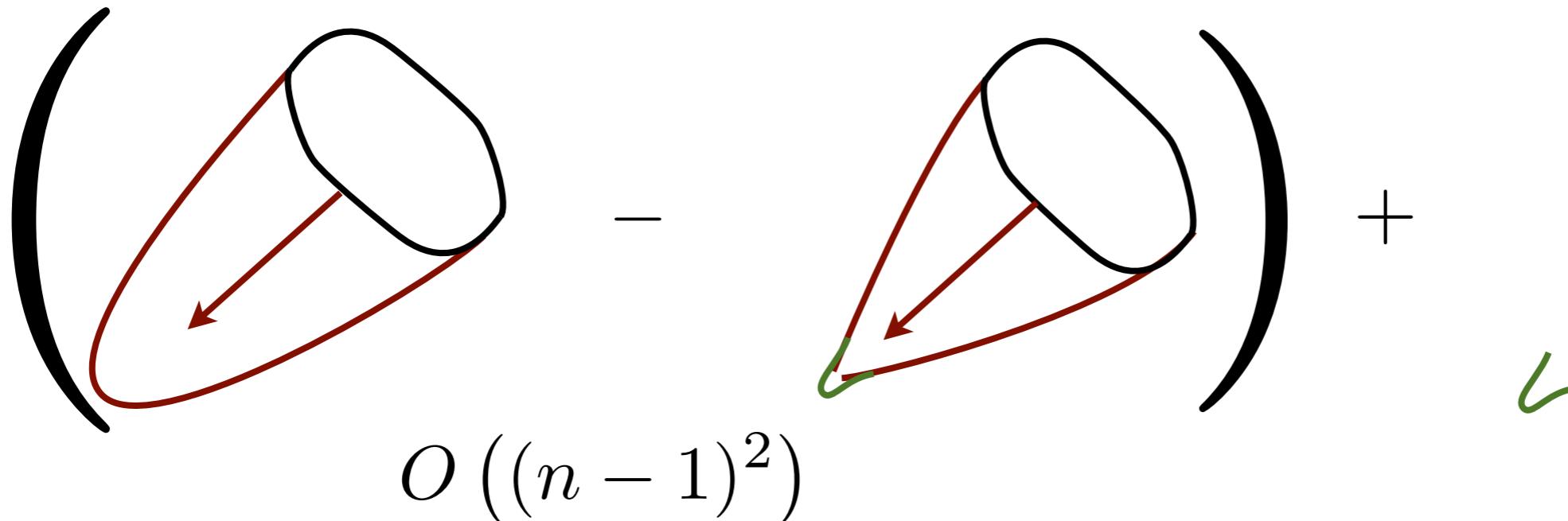
Replica manipulations

$$S = \partial_n (I_n - nI_1) \Big|_{n=1}$$



Replica manipulations

$$S = \partial_n (I_n - nI_1)|_{n=1}$$



$$S = \partial_n (I[\text{L}])|_{n=1}$$

Recap

- Replica trick: EE and loops in eucl. time
- Assume ‘holography’ $Z_{\text{FT}} \approx e^{-I_n}$
- Entanglement entropy becomes:

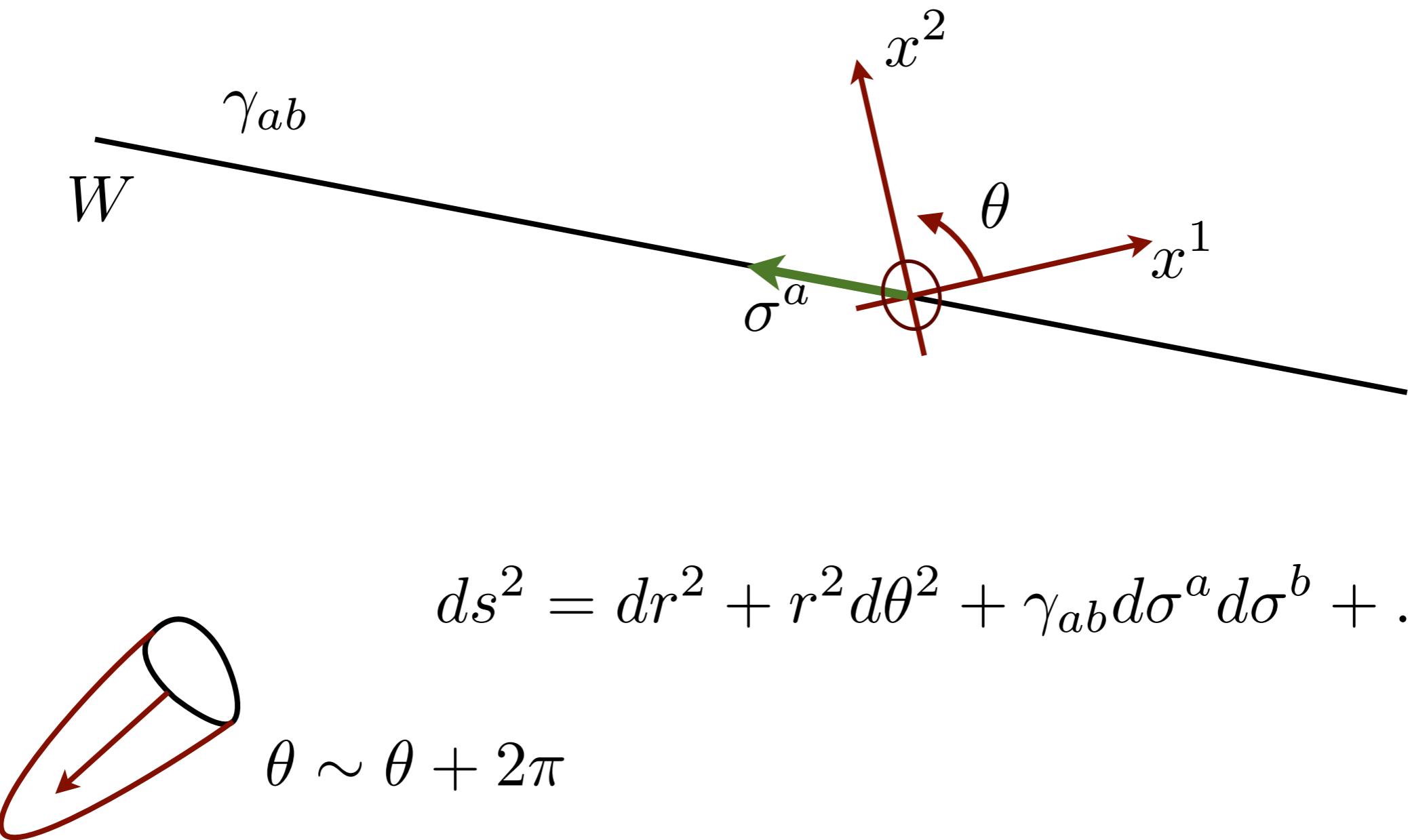
$$S = \partial_n (I[\mathcal{L}])|_{n=1}$$

The calculation

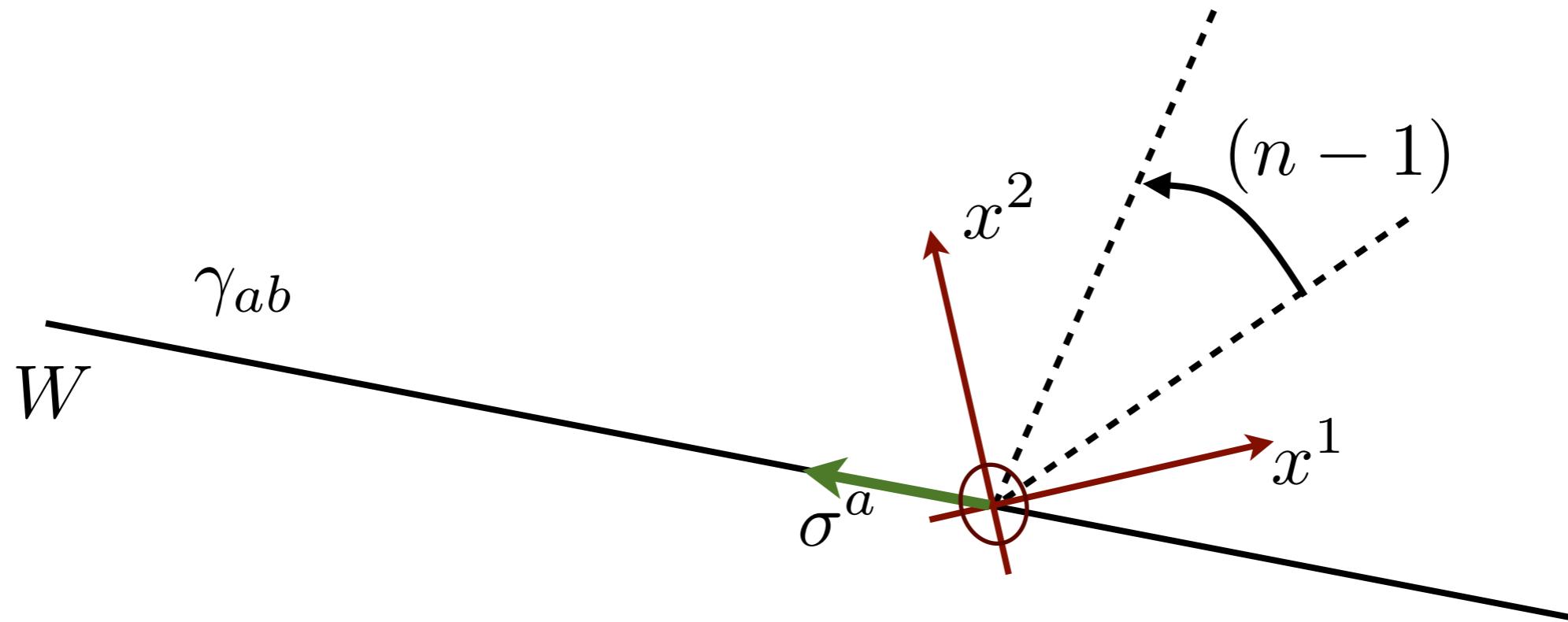
$$S = \partial_n (I[\mathcal{L}])|_{n=1}$$

First, assume euclidean
time independence

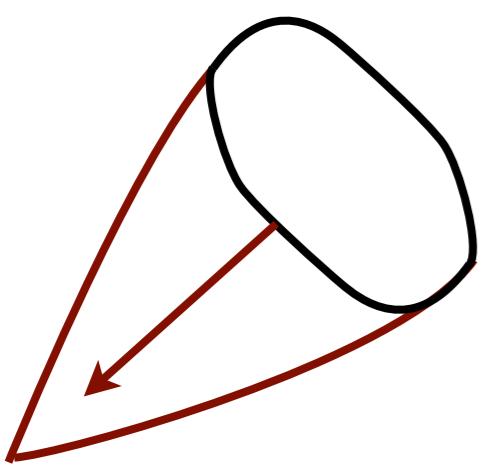
Adapted coordinates



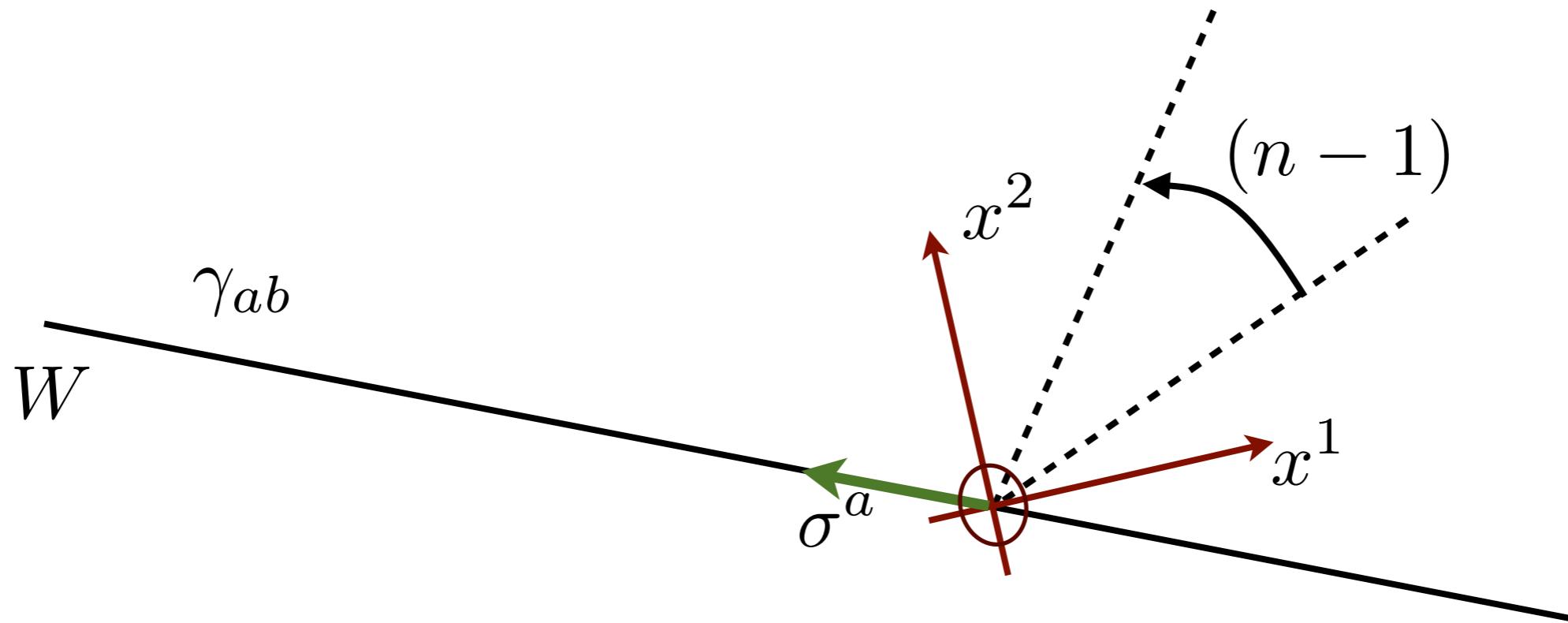
Adapted coordinates



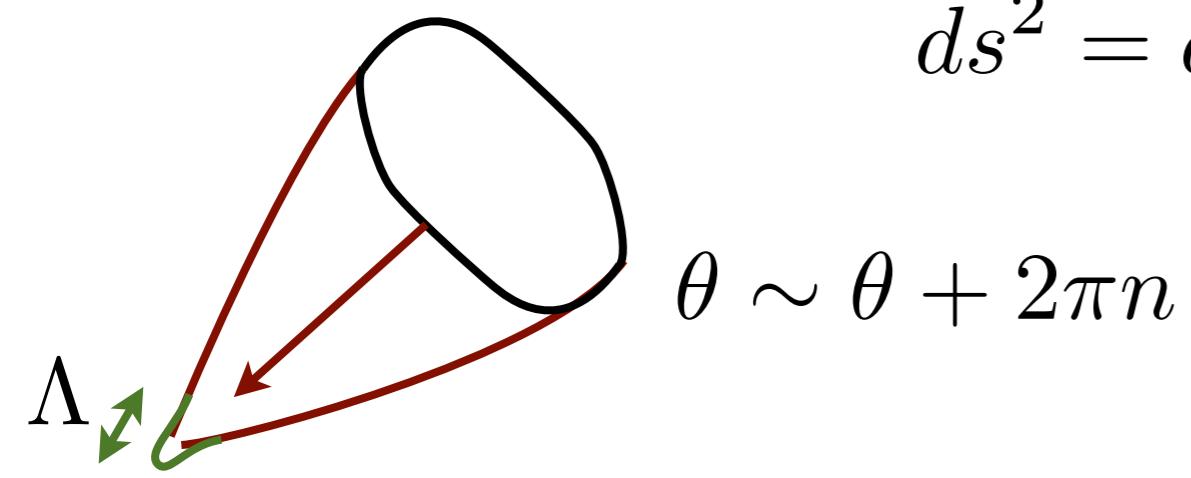
$$ds^2 = dr^2 + r^2 d\theta^2 + \gamma_{ab} d\sigma^a d\sigma^b + \dots$$



Adapted coordinates

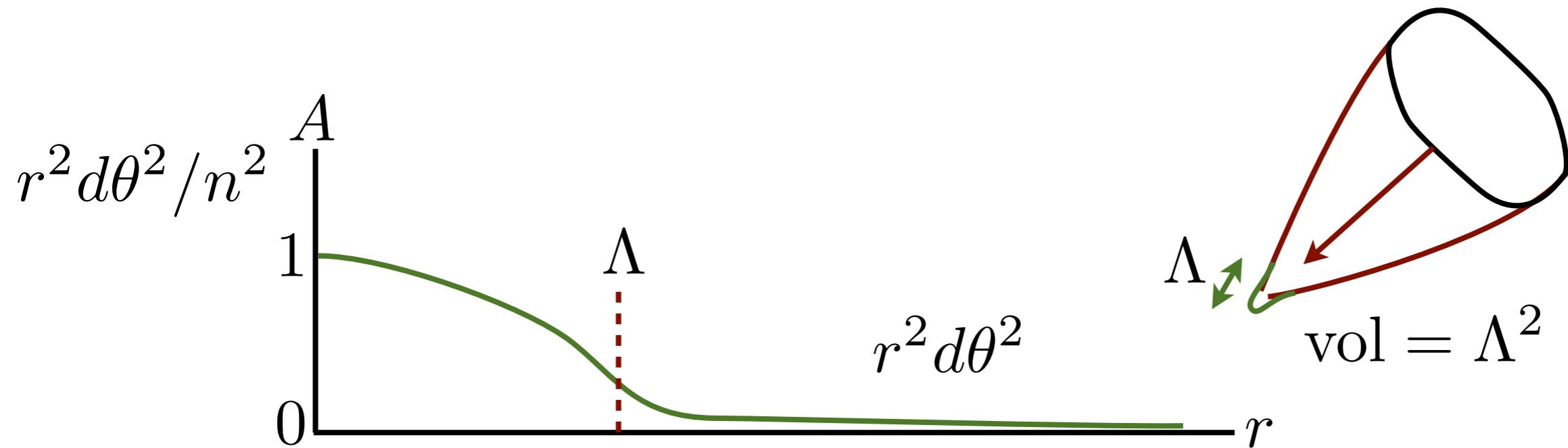


$$ds^2 = dr^2 + r^2 d\theta^2 + \gamma_{ab} d\sigma^a d\sigma^b + \dots$$



Regulated cones

$$ds^2 = dr^2 + r^2 \left(1 - \frac{n-1}{n} A(r^2/\Lambda^2) \right)^2 d\theta^2 + \gamma_{ab} d\sigma^a d\sigma^b + \dots$$



$$\theta \sim \theta + 2\pi n$$

Entropy

$$S = \partial_n \left(I[\textcolor{brown}{L}] \right) |_{n=1}$$

$$\delta R_{\mu\nu\rho\sigma}=\frac{1}{\Lambda^2}\frac{n-1}{n}\frac{1}{y}\frac{d^2(yA(y^2))}{dy^2}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}$$

$$r=\Lambda y$$

$$\delta I = \frac{\delta I}{\delta \mathrm{Riem}} \delta \mathrm{Riem}$$

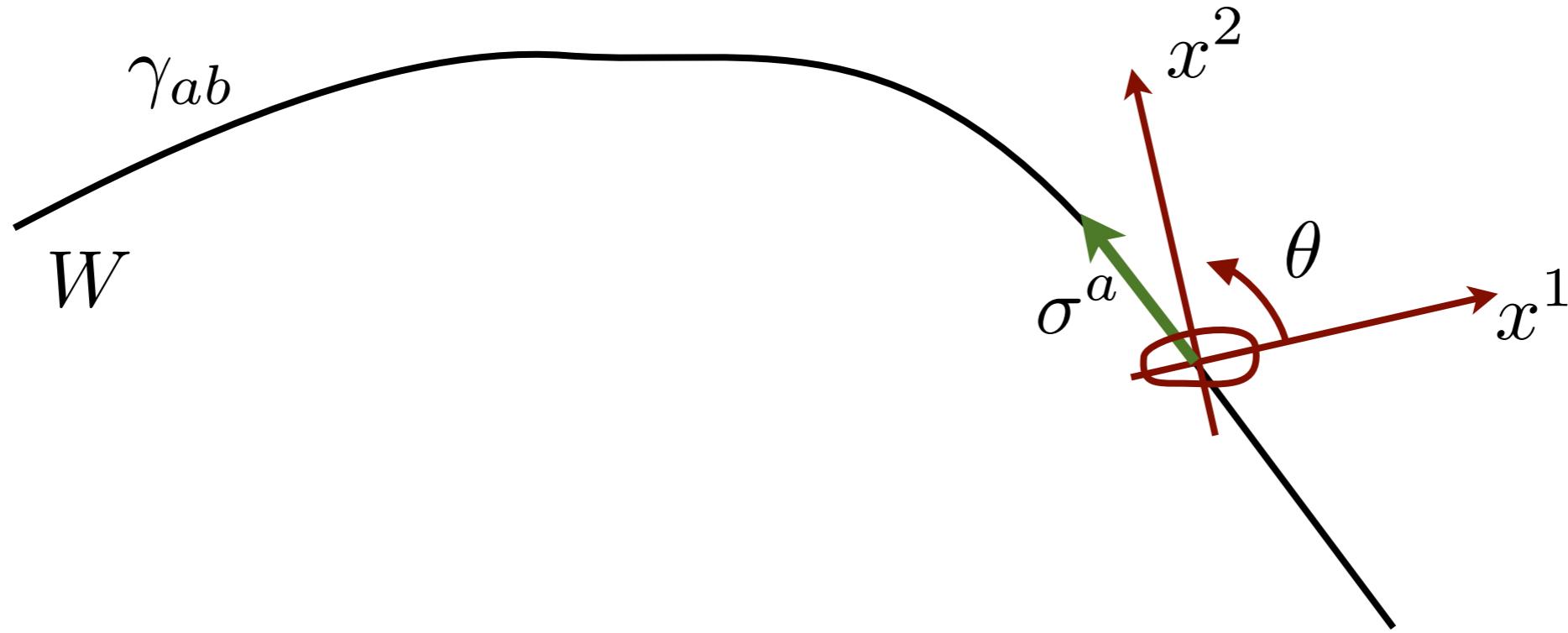
$$\Lambda \rightarrow 0$$

$$S=-2\pi\int_W\sqrt{\gamma}d^{D-2}\sigma\,\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\left.\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\right|_W$$

Comments

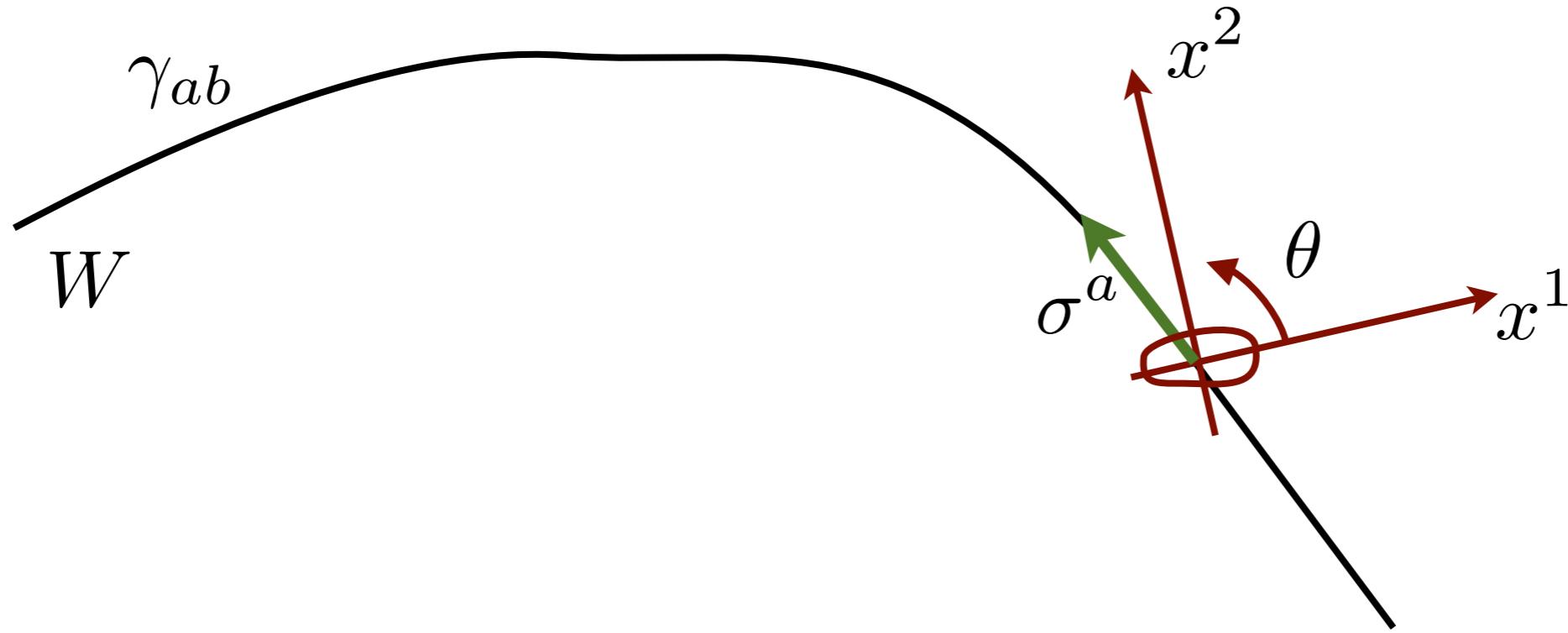
- Wald's entropy
- Regulation independent
- Assumed Euclidean time stationarity: No extrinsic curvature

More generally...



$$ds^2 = [\gamma_{ab} - 2(K_{ab}{}^1 r \cos \theta + K_{ab}{}^2 r \sin \theta)] d\sigma^a d\sigma^b + dr^2 + r^2 d\theta^2 + \dots$$

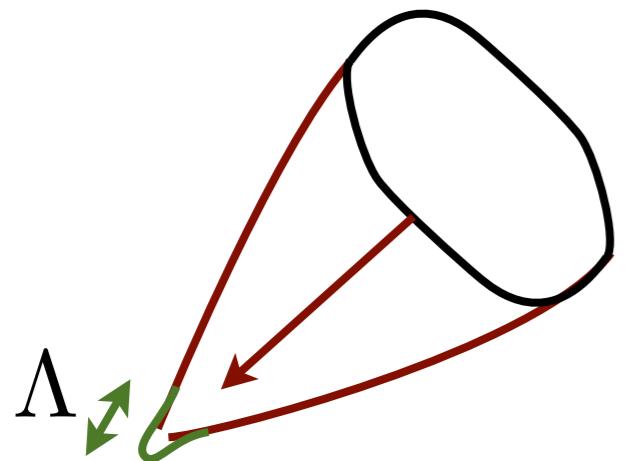
More generally...



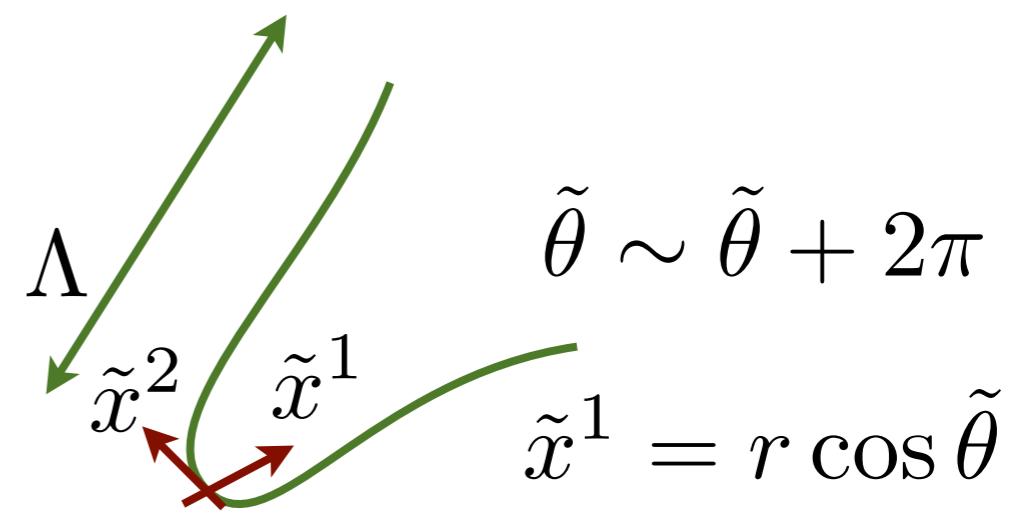
$$ds^2 = \left[\gamma_{ab} - 2 \left(K_{ab}{}^1 r \cos \theta + K_{ab}{}^2 r \sin \theta \right) \left(\frac{r}{\Lambda} \right)^{(n-1)B(r^2/\Lambda^2)} \right] d\sigma^a d\sigma^b$$
$$+ dr^2 + r^2 \left(1 - \frac{n-1}{n} A(r^2/\Lambda^2) \right)^2 d\theta^2 + \dots$$

Why the new terms

$$K_{ab}^{-1} r \cos \theta \left(\frac{r}{\Lambda} \right)^{(n-1)B(r^2/\Lambda^2)} \approx K_{ab}^{-1} r \cos \theta \left(\frac{r}{\Lambda} \right)^{(n-1)}$$



$$\theta = n\tilde{\theta}$$

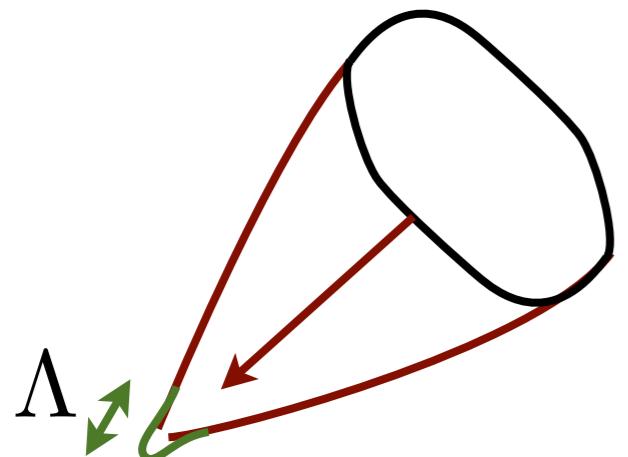


$$\tilde{\theta} \sim \tilde{\theta} + 2\pi$$

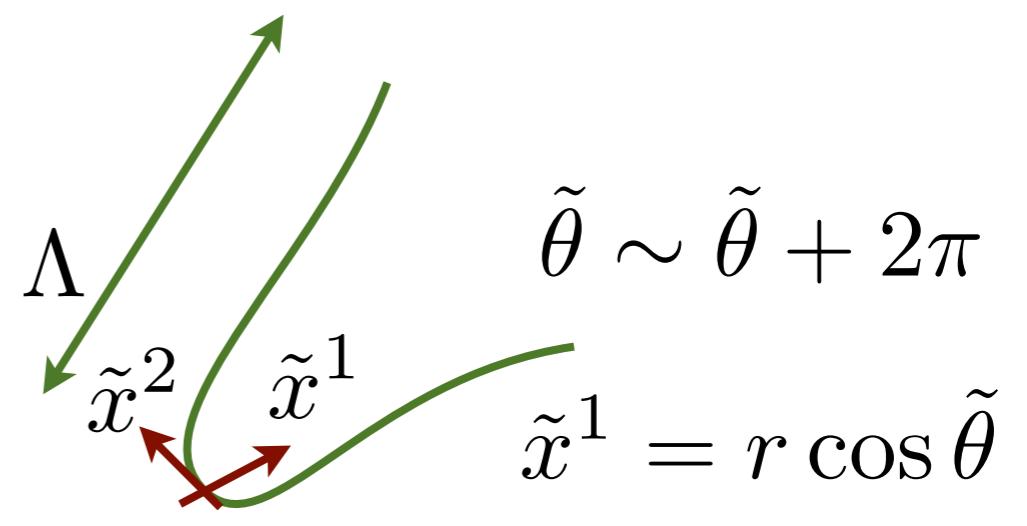
$$\tilde{x}^1 = r \cos \tilde{\theta}$$

Why the new terms

$$K_{ab}^{-1} r \cos \theta \left(\frac{r}{\Lambda} \right)^{(n-1)B(r^2/\Lambda^2)} \approx K_{ab}^{-1} r \cos \theta \left(\frac{r}{\Lambda} \right)^{(n-1)}$$



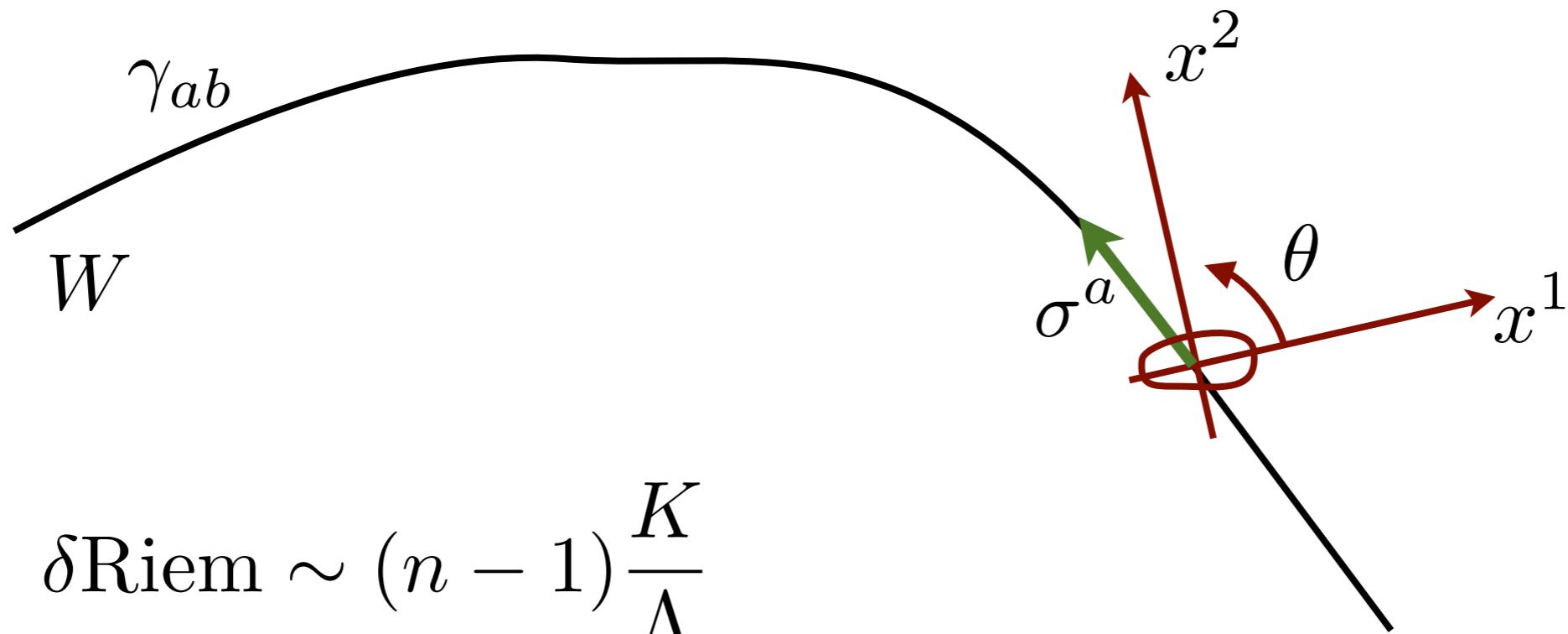
$$\theta = n\tilde{\theta}$$



$$K_{ab}^{-1} \frac{r^n \cos(n\tilde{\theta})}{\Lambda^{n-1}} \rightarrow K_{ab}^{-1} \frac{(\tilde{x}^1)^2 - (\tilde{x}^2)^2}{\Lambda}$$
$$n = 2$$

An analogous analysis applies to terms beyond extrinsic curvature (higher powers of r)

Entropy contributions



$$\delta \text{Riem} \sim (n - 1) \frac{K}{\Lambda}$$

$\Lambda \rightarrow 0$

$$\delta S \sim \int \sqrt{\gamma} d^{D-2} \sigma \frac{\partial^2 \mathcal{L}}{\partial \text{Riem}^2} K^2$$

The subtlety

$$S = \partial_n (I[\mathcal{L}])|_{n=1}$$

$$\lim_{n \rightarrow 1} \partial_n \int_0^\infty dy (n-1)^2 y^{2n-3} e^{-y^2} A(y^2)$$

$$\lim_{n \rightarrow 1} \partial_n \left((n-1)^2 \frac{\Gamma(n-1)}{2} \right) = \frac{1}{2}$$

$$\delta S \sim \int \sqrt{\gamma} d^{D-2} \sigma \frac{\partial^2 \mathcal{L}}{\partial \text{Riem}^2} K^2$$

Comments on the new formula

$$S \sim \int_W \sqrt{\gamma} d^{D-2} \sigma \left(\frac{\partial \mathcal{L}}{\partial \text{Riem}} + \frac{\partial^2 \mathcal{L}}{\partial \text{Riem}^2} K^2 \right)$$

- It reduces to Wald's for stationary cases (trivially)
- For Lovelock Gravity, it gives the Jacobson-
Myers entropy functional
 - de Boer et al
 - Myers et al
 - Fursaev et al
- Disagrees with Wald-Iyer's

Final remarks

- First principles derivation of Ryu-Takayanagi
- Euclidean space essential: Lorentzian?
- Multiple regions?
- Holographic emergence of space?
- Non-stationary entropy? Second law?

The final formula

$$S = \int_W \sqrt{\gamma} d^{D-2}\sigma \left(\boxed{\delta^{(1)} R_{\mu\nu\rho\sigma} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}} + \boxed{\delta^{(2)} R_{\mu\nu\rho\sigma\tau\pi\xi\zeta} \frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\rho\sigma} \partial R_{\tau\pi\xi\zeta}}} \right) \Big|_W$$

Wald

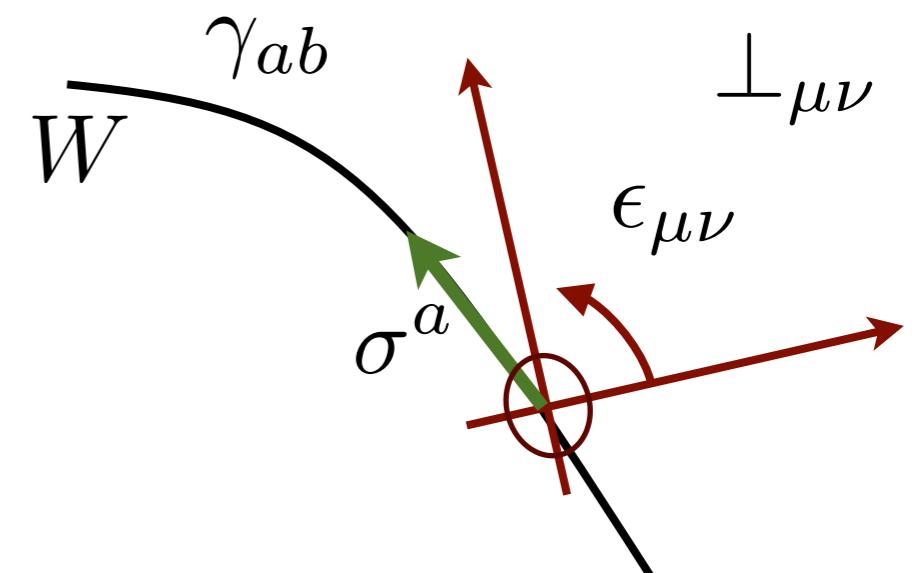
$$\delta^{(1)} R_{\mu\nu}{}^{\rho\sigma} = -2\pi \epsilon_{\mu\nu} \epsilon^{\rho\sigma}$$

New

$$\delta^{(2)} R_{\mu\nu}{}^{\rho\sigma}{}_{\tau\pi}{}^{\xi\zeta} = 4\pi \left(K_{[\mu}{}^{[\rho|i|} \perp_{\nu]}{}^{\sigma]}{}_{[\pi}{}^{[\zeta} K_{\tau]}{}^{\xi]j} \perp_{ij} + K_{[\mu}{}^{[\rho|k|} \tilde{\perp}_{\nu]}{}^{\sigma]}{}_{[\pi}{}^{[\zeta} K_{\tau]}{}^{\xi]l} \epsilon_{kl} \right)$$

$$\perp_{\nu\sigma\pi\zeta} = \perp_{\nu\pi} \perp_{\sigma\zeta} + \perp_{\nu\zeta} \perp_{\pi\sigma} - \perp_{\nu\sigma} \perp_{\pi\zeta}$$

$$\tilde{\perp}_{\nu\sigma\pi\zeta} = \perp_{\nu\pi} \epsilon_{\sigma\zeta} + \perp_{\sigma\zeta} \epsilon_{\nu\pi}$$

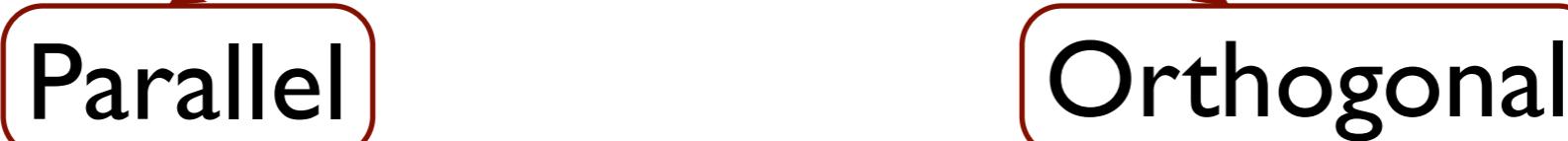


A further refinement

Dong

$$\int \sqrt{\gamma} d^{D-2} \sigma K^2 \frac{\partial^2 \mathcal{L}}{\partial \text{Riem}^2} \rightarrow \int \sqrt{\gamma} d^{D-2} \sigma K^2 \sum_{\alpha} \frac{1}{1 + q_{\alpha}} \frac{\partial^2 \mathcal{L}}{\partial \text{Riem}^2} \Big|_{\alpha}$$

$$q_{\alpha} = \frac{1}{2} (\# \text{ of } K_{abi} \text{ s}) + \frac{1}{2} (\# \text{ of } R_{aijk} \text{ s}) + (\# \text{ of } R_{ab(ij)} \text{ s})$$



$$\lim_{n \rightarrow 1} \partial_n \int_0^\infty dy (n-1)^2 y^{2n-3} e^{-y^2} (y^{n-1} K)^t = \frac{(K)^t}{1+t/2}$$