On thermodynamics of $\mathcal{N}=4$ SYM (continued)

Entropy density:

$$s = S/V_3 = \frac{2\pi^2}{3} N_c^2 T^3 f(\lambda)$$

$$N_c \to \infty, \quad \forall \lambda$$

$$f(\lambda) = \begin{cases} 1 - \frac{3}{2\pi^2}\lambda + \ldots, & \lambda \ll 1 \\ 3/4 + \mathcal{O}(\lambda^{-3/2}), & \lambda \gg 1 \end{cases}$$

For a d-dimensional CFT we have

$$s = a f(\lambda)\, T^{d-1}$$

$$\Rightarrow s = \left.\frac{\partial P}{\partial T}\right|_{\mu} \Rightarrow P = \frac{1}{d} a f(\lambda) T^d$$

$$\Rightarrow \mathcal{F} = -pV = E - T\cdot S \Rightarrow \varepsilon = Ts - P$$

$$\Rightarrow \varepsilon = \frac{d-1}{d} a f(\lambda) T^d = (d-1) P$$

This is expected since for CFT $T^{\mu}{}_{\mu} = 0$

$$\Rightarrow \mathrm{tr}\begin{pmatrix} -\varepsilon & & & \\ & P & & \\ & & \ddots & \\ & & & P \end{pmatrix} = 0 \Rightarrow \varepsilon = (d-1) P$$

In particular, the speed of sound squared is

$$v_s^2 = \frac{\partial P}{\partial \mathcal{E}} = \frac{1}{d-1}$$

In theories with internal scale(s), e.g. Λ in QCD, the result is more complicated:

$$s = a\, T^{d-1} F(\Lambda/T)$$

with $F \to 1$ for $T \to \infty$ but otherwise a non-trivial function.

Holographic dictionary

$$Z_{\substack{\text{gauge}\\ \text{theory}}}[J] = Z_{\substack{\text{string}\\ \text{theory}}}[J]$$

In particular,

$$Z_{\mathcal{N}=4\ \text{SYM}}[J] = Z_{\substack{\text{type}\\ \text{II B}\\ \text{str. th.}}}[J]$$

SU(N_c) $d=4$ — on $AdS_5 \times S^5$

$\forall\, N_c, \lambda$

$$Z_{\substack{\mathcal{N}=4 \text{ SYM} \\ SU(N_c)\, d=4 \\ N_c \to \infty \\ \lambda \to \infty}} [J] = e^{-S^{\text{type IIB}}_{\substack{\text{SUGRA} \\ \text{on } AdS_5 \times S^5}}[J]} + \text{corrections}$$

$$O\left(\frac{1}{N_c^2}\right), \; O\left(\frac{1}{\lambda^{3/2}}\right)$$

+ corrections

$$O(1/N_c^2), \; O(1/\lambda^\alpha)$$

$$\left\langle e^{\int J \mathcal{O}\, d^4x} \right\rangle_{YM} = e^{-S_{\text{grav}}[J]}$$

$$\Rightarrow \langle \mathcal{O} \dots \mathcal{O} \rangle_{YM}$$

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 .$$

What plays the role of J on the grav. side?

Recall that $g_s = 4\pi g_{YM}^2$ is the expectation value of the dilaton at asympt. infinity,

$g_s = \exp(\langle \Phi_\infty \rangle)$ at ∂AdS.

Deforming gauge theory (changing coupling const) $\Leftrightarrow$ changing the boundary value of a bulk field

More generally,

$$S \to S + \int d^4x \, \phi(x) \, \mathcal{O}(x) ,$$

where $\mathcal{O}$ is a local gauge-inv. operator in YM theory, $\phi(x)$ is a source :

$$\phi(x) = \Phi_{Bulk} \Big|_{\partial AdS} (x) = \lim_{r \to \infty} \Phi_{Bulk} (r, x)$$

Note: this recipe will be refined for massive fields.

Field - operator correspondence

$$\Phi_{Bulk} (r, x) \qquad\qquad \mathcal{O}(x)$$

- quantum numbers of the global symmetries must match
- no general rule, not unique in the $N_c \to \infty$ limit
- DBI action helpful in identifying the couplings
- For conserved currents such as $\mathcal{O} = J^\mu(x)$,

$\mathcal{O} = T^{\mu\nu}(x)$, the appropriate terms are

$$\int A_\mu(x) J^\mu(x)\, d^4x$$

$$\int h_{\mu\nu}(x) T^{\mu\nu}(x)\, d^4x$$

and the bulk fields are the gauge field $A_\mu(x,r)$ and the bulk metric $h_{\mu\nu}(x,r)$.

A note on Euclidean and Minkowski correlation functions (at finite temperature)

Minkowski space: $(-+++)$

$$G^R(k) = -i \int d^4x\, e^{-ikx}\, \theta(t) \langle [\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(0)] \rangle_T$$

$$G^A(k) = i \int d^4x\, e^{-ikx}\, \theta(-t) \langle [\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(0)] \rangle_T$$

Wightman function

$$G(k) = \frac{1}{2} \int d^4x\, e^{-ikx} \langle \hat{\mathcal{O}}(x)\hat{\mathcal{O}}(0) + \hat{\mathcal{O}}(0)\hat{\mathcal{O}}(x) \rangle_T$$

Feynman

$$G^F(k) = -i \int d^4x\, e^{-ikx} \langle T \hat{\mathcal{O}}(x) \hat{\mathcal{O}}(0) \rangle$$

$$G^F(k) = \frac{1}{2}\left(G^R + G^A\right) - iG$$

Using spectral representation of G^R, G (see LL vol IX, Le Bellac, AGD)

$$G(k) = -\coth\frac{\omega}{2T}\ \mathrm{Im}\, G^R(k)$$

$$G^F(k) = \mathrm{Re}\, G^R + i \coth\frac{\omega}{2T}\ \mathrm{Im}\, G^R$$

$$G^A = G^R(-k) = \left(G^R\right)^*$$

Note $T \to 0$ limit:

$$G^F(k) = \mathrm{Re}\, G^R + i\ \mathrm{sgn}\,\omega\ \mathrm{Im}\, G^R$$

Euclidean space:

$$G^E(k_E) = \int d^4x_E\, e^{-ik_E x_E} \langle T_E\, \hat{O}(x_E)\hat{O}(0)\rangle$$

$T \neq 0$: Matsubara propagator, defined at $\omega_E = 2\pi T n$ (for bosons).

$$G^R(2\pi i T n, \vec{k}) = -G^E(2\pi T n, \vec{k}).$$

<u>example:</u> (massless) scalar field in AdS_5

1) Euclidean AdS_5

$$ds_5^2 = \frac{L^2}{z^2}\left(dz^2 + d\vec{x}_E^{\,2}\right)$$

$$S_E = \frac{\pi^3 L^8}{4\varkappa_{10}^2}\int dz\, d^4x \;\; z^{-3}\left[(\partial_z\phi)^2 + (\partial_i\phi)^2 + \frac{m^2L^2}{z^2}\phi^2\right]$$

$$\phi(z,x) = \int\frac{d^4k}{(2\pi)^4}\, e^{ikx} f_k(z)\,\phi_0(k)$$

$$S_E = \frac{\pi^3 L^8}{4\varkappa_{10}^2}\int_\varepsilon^\infty dz \int\frac{d^4k\, d^4k'}{(2\pi)^4}\,\frac{\delta(k+k')}{z^3}\Big[\partial_z f_k\,\partial_z f_{k'} -$$

$$- kk' f_k f_{k'} + \frac{m^2L^2}{z^2} f_k f_{k'}\Big]\phi_0(k)\phi_0(k')$$

The e.o.m. is

$$f_k'' - \frac{3}{z} f_k' - \left(k_E^2 + \frac{m^2L^2}{z^2}\right) f_k = 0$$

$$f_k(z) = A z^2 I_\nu(kz) + B z^2 K_\nu(kz),$$

$\nu = \sqrt{4 + m^2L^2}$ Let $m=0$ now

$$f_k(z) = A z^2 I_2(kz) + B z^2 K_2(kz)$$

B.c. regularity at $z \to \infty$; $f_k(\varepsilon) = 1$:

$$f_k(z) = \frac{z^2 K_2(kz)}{\varepsilon^2 K_2(k\varepsilon)}$$

The action (on shell) is

$$S_E = \frac{\pi^3 L^8}{4\varkappa_{10}^2} \int \frac{d^4k\, d^4k'}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(k+k')\, \phi_0(k)\phi_0(k')$$

$$\times \left. \frac{f_{k'}(z)\, \partial_z f_k(z)}{z^3} \right|_\varepsilon^\infty \equiv$$

$$\equiv \int \frac{d^4k\, d^4k'}{(2\pi)^8} \phi_0(k)\phi_0(k')\, \mathcal{F}(z,k,k') \Big|_\varepsilon^\infty .$$

The two-point function of $\hat{O}$ coupled to ϕ is

$$\langle \hat{O}(k)\hat{O}(k') \rangle = Z^{-1} \left. \frac{\delta^2 Z[\phi_0]}{\delta\phi_0(k)\,\delta\phi_0(k')} \right|_{\phi_0 = 0} =$$

$$= -2\mathcal{F}(z,k,k') \Big|_\varepsilon^\infty = -(2\pi)^4 \delta(k+k') \frac{\pi^3 L^8}{2\varkappa_{10}^2} \left. \frac{f_{k'} \partial_z f_k}{z^3} \right|_\varepsilon$$

Note that $\mathcal{F}(z=\infty, k, k') = 0$

Also, recall that $\varkappa_{10} = 2\pi^{5/2} L^4 / N_c$

We find

$$\langle \hat{\mathcal{O}}(k)\, \hat{\mathcal{O}}(k') \rangle = \frac{N_c^2}{8\pi^2} (2\pi)^4 \delta(k+k') \left[\frac{f'(\varepsilon)}{\varepsilon^3} \right],$$

where $\left[\cdots \right] = \lim_{\varepsilon \to 0} \ldots$ and discarding contact terms.

$$\frac{f'}{\varepsilon^3} = -\frac{k^2}{2\varepsilon^2} - \frac{\gamma_E k^4}{4} - \frac{k^4 \ln \varepsilon}{4} - \frac{k^4 \ln k/2}{4} + $$

$$+ O(\varepsilon^2)$$

$$\left[\frac{f'}{\varepsilon^3} \right] = -\frac{k^4}{8} \ln k^2$$

$$\langle \hat{\mathcal{O}}(k)\, \hat{\mathcal{O}}(k') \rangle_E = -\frac{N_c^2}{64\pi^2} k_E^4 \ln k_E^2 \, (2\pi)^4 \delta(k+k')$$

<u>exercise</u>: transform this result back to position space to find $\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle_E \sim \dfrac{N_c^2}{|x-y|^8}$

Hint: use (PBM – 2)

$$\int_0^\infty x^{\alpha-1}\ln x\, J_\nu(cx)\,dx = \frac{2^{\alpha-2}c^{-\alpha}\,\Gamma\left(\frac{\alpha+\nu}{2}\right)}{\Gamma\left(1-\frac{\alpha-\nu}{2}\right)}\times$$

$$\times\left\{\psi\left(\frac{\alpha+\nu}{2}\right)+\psi\left(1-\frac{\alpha-\nu}{2}\right)+2\ln\frac{2}{c}\right\}.$$

Remark: in general, expect $\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}}$.
For $\hat{\mathcal{O}} = \mathrm{tr}\,F_{\mu\nu}^2$, $[\hat{\mathcal{O}}] \sim 1/L^4 \sim 1/L^{\Delta_0}$

$$\Rightarrow \langle\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\rangle \sim \frac{1}{|x-y|^8} \sim \frac{1}{|x-y|^{2\Delta_0}}$$

at weak coupling. AdS/CFT result confirms that $\Delta(\lambda=\infty) = \Delta(\lambda=0) = 4$. This result (due to non-renormal. theorems) was known before in $\mathcal{N}=4$ SYM.

Remark: not all $\mathcal{N}=4$ SYM operators are protected from renormalization. In general, $\Delta(\lambda)$ is a non-trivial function.

Lorentzian signature: $ds_5^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right)$

$$S = -\frac{\pi^3 L^8}{4\varkappa_{10}^2}\int d^4x \int_\varepsilon^\infty \frac{dz}{z^3}\left[(\partial_z\phi)^2 + \eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m^2L^2}{z^2}\phi^2\right]$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1).$$

$$\phi(z,t,\vec{x}) = \int\frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\bar{k}\bar{x}} f_k(z)\,\phi_0(k)$$

$$S = \int\frac{d^4k}{(2\pi)^4}\left[\mathcal{F}(k,\infty) - \mathcal{F}(k,\varepsilon)\right]\phi_0(-k)\,\phi_0(k)$$

$$\mathcal{F}(k,z) = -\frac{\pi^3 L^8}{4\varkappa_{10}^2}\,\frac{f_{-k}\,\partial_z f_k(z)}{z^3}.$$

The e.o.m.

$$f_k'' - \frac{3}{z} f_k' - \left(k^2 + \frac{m^2L^2}{z^2}\right) f_k = 0$$

1) Space-like momenta, $k^2 > 0$

The same calculation as in Euclidean case.

2) Time-like momenta, $k^2 < 0$

Let $q = \sqrt{-k^2} = |k|$

The solution to the e.o.m. is

$$f_k(z) = \begin{cases} \dfrac{z^2 H_\nu^{(1)}(qz)}{\varepsilon^2 H_\nu^{(1)}(q\varepsilon)}, & \omega > 0 \\[2ex] \dfrac{z^2 H_\nu^{(2)}(qz)}{\varepsilon^2 H_\nu^{(2)}(q\varepsilon)}, & \omega < 0 \end{cases}$$

$\underline{H_\nu^{(1,2)}(z) = J_\nu \pm i Y_\nu}$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z - \nu\pi/2 - \pi/4)}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i(z - \nu\pi/2 - \pi/4)}$$

<u>Retarded correlators</u> $\rightsquigarrow$

b.c.

$$\sim e^{-i\omega t + iqz}$$

$z = \infty$

$$\omega\, dt + q\, dz = 0 \Rightarrow \frac{dz}{dt} = \frac{\omega}{q} > 0$$

for $\omega > 0 \Rightarrow H_\nu^{(1)}(qz)$ for $\omega > 0$.

(Incoming wave b.c.).

Explicitly, combining space-like and time-like results, we find

$$G^R(k) = \frac{N_c^2 k^4}{64\pi^2}\left(\ln|k^2| - i\pi\,\theta(-k^2)\,\mathrm{sgn}\,\omega \right)$$

<u>exercise</u>: show that the appropriate analytic continuation of $G_E(k)$ gives $G^R(k)$.

More general case: AdS_{d+1} : $ds^2_{d+1} = \frac{L^2}{z^2}\left(dz^2 + \underbrace{\eta_{\mu\nu}dx^\mu dx^\nu}_{d\text{-dim.}}\right)$

Then the e.o.m. for a min. coupled scalar is:

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi\right) - k^2z^2\phi - m^2L^2\phi = 0,$$

$$k^2 = -\omega^2 + \vec{k}^2.$$

Near the boundary $(z \to 0)$:

$$\phi(z,k) = A(k)\,z^{d-\Delta} + \ldots + B(k)\,z^{\Delta} + \ldots,$$

$$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2L^2},$$

ν is real for $m^2L^2 \geq -d^2/4$

(Breitenlohner - Freedman bound, see

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AdS inner product:

$$(\phi_1, \phi_2) = -i \int_{\Sigma_t} dz\, d^dx \sqrt{-g}\, g^{tt} \left(\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*\right)$$

The modes with masses squared above the BF bound are split into two sets

$$-d^2/4 \leq m^2L^2 \leq -d^2/4 + 1 \qquad (1)$$

$$-d^2/4 + 1 \leq m^2L^2 \qquad (2)$$

In set (1), both modes ($\sim z^{d-\Delta}$ and $\sim z^{\Delta}$) are normalizable w.r.t. inner product. In set (2), the modes $\sim z^{d-\Delta}$ are non-normalizable ones, and $\sim z^{\Delta}$ are normalizable. Here we consider set (2) only (but the story of set (1) is known).

<u>The non-normalizable term is a source for $\mathcal{O}(x)$ in the boundary action</u>

$$S_{\partial M} \to S_{\partial M} + \int d^d x \, J(x)\, \mathcal{O}(x),$$

with $J(x) = A(x)$ (recall the dilaton example before):

$$J(x) = \phi\Big|_{\partial M} = \lim_{z \to 0} z^{\Delta - d} \phi(z, x).$$

Normalizable modes: quantization in curved space-time (see Birrell-Davies) $\Rightarrow$ Hilbert space (bulk) — should be identified with $\mathcal{H}$-space of the boundary theory. <u>Normalizable modes = states of the boundary theory.</u> One can show that

$$\langle \mathcal{O}(x) \rangle_J = 2\nu B(x).$$

<u>Remark</u>: for other wave equations (p-forms)

$$m^2 L^2 = (\Delta - p)(\Delta + p - d)$$