Duality

Here, duality means isomorphism between 2 sets of (dummy) var. describing the same object.

$$Z_{\mathcal{O}}[J;a] = \int [d\phi]\, e^{-S_E[\phi;a]+J\mathcal{O}}$$

param. such as T, m etc

Change var. in $\int [d\phi] \ldots$

Example (very simple):

$$Z(\alpha) = \int_0^1 \frac{dx}{\sqrt{1-\alpha^2 x^2}} \qquad (\alpha \in [0,1])$$

$$Z(\alpha) = 1 + \frac{\alpha^2}{6} + \ldots \qquad \alpha \ll 1$$

Change var: $x \to y$: $\alpha x = \sin y$

$$\Rightarrow Z(\alpha) = \frac{1}{\alpha} \int_0^{\arcsin\alpha} dy = \frac{\arcsin\alpha}{\alpha}$$

(Exact solution)

Example (less simple):

$$Z(\alpha) = \int_{-\infty}^{\infty} e^{-\left(x + \frac{\alpha}{x}\right)^2} dx = \sqrt{\pi}\, e^{-4\alpha}$$

$\alpha > 0$

Exercise: show this via var. change.

Example: Poisson summation formula (see e.g. Courant-Hilbert, vol I)

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k),$$

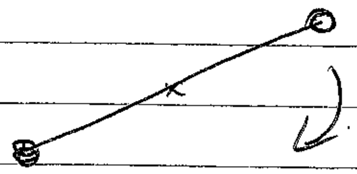
where $\hat{f}$: Fourier transf. of $f \in C^\infty(\mathbb{R})$.

$$\hat{f}(b) = \int_{-\infty}^{\infty} f(a)\, e^{2\pi i b a}\, da$$

With $f(a) = e^{-\beta \pi a^2} \Rightarrow \hat{f}(b) = \beta^{-\frac{1}{2}} e^{-\frac{\pi b^2}{\beta}}$

$$\Rightarrow \boxed{\sum_{n \in \mathbb{Z}} e^{-\pi n^2 \beta} = \frac{1}{\sqrt{\beta}} \sum_{k \in \mathbb{Z}} e^{-\pi k^2/\beta}}$$

<u>Exercise</u>: apply this to Z of the 2-dim rotator



$d_s = 2$

$$-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\varphi^2}\psi = E_\ell\psi$$

$$E_\ell = \frac{\hbar^2}{2I}\ell^2, \quad \ell \in \mathbb{Z}$$

$$\boxed{Z_{rot}(\beta) = \sqrt{\beta}\, Z_{rot}(1/\beta)}$$

<u>Example</u> — Ising model Kramers–Wannier duality

$$\hat{H} = -J\sum_{\langle ij\rangle}\vec{\sigma}_i\cdot\vec{\sigma}_j$$

σ_i - Pauli matrices

Quantum Heisenberg model $\rightarrow$

$\rightarrow$ Ising (Lenz) model

$$H = -J\sum_{\langle ij\rangle} s_i s_j \qquad s_i = \pm 1$$

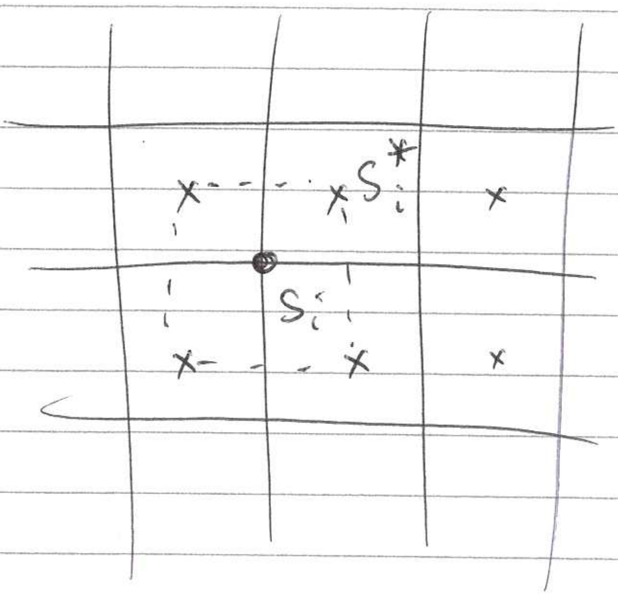
$$Z = \mathrm{tr}\, e^{-\beta \hat{H}} = \sum_{\{s_i = \pm 1\}} e^{K \sum_{\langle ij \rangle} s_i s_j}$$

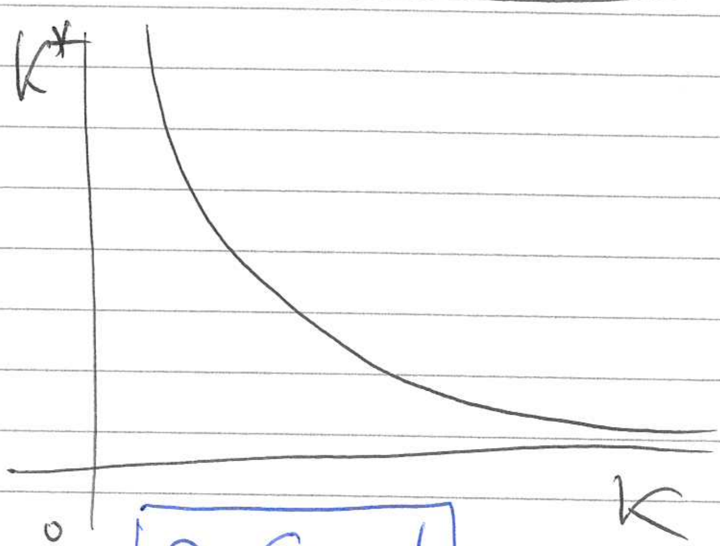
$$K \equiv \beta J$$

$$Z[K] = C(K)\, Z^*(K^*),$$

where $$Z^*(K^*) = \sum_{s_i^* = \pm 1} e^{K^* \sum_{\langle ij \rangle} s_i^* s_j^*}$$

$$\boxed{K^* = -\frac{1}{2} \ln \mathrm{th}\, K} \quad \text{or} \quad \boxed{\sinh 2K \sinh 2K^* = 1}$$





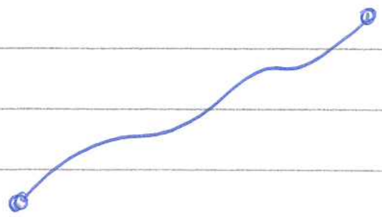
R. Savit

(strong-weak duality)

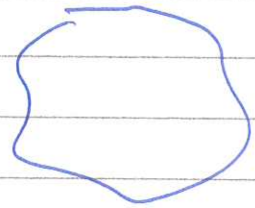
$$K = K_c = K_c^* \Rightarrow \sinh 2K_c = 1 \Rightarrow K_c \approx 0.4407$$
$$= -\frac{1}{2} \ln(\sqrt{2} - 1)$$

Gauge-string duality: origins

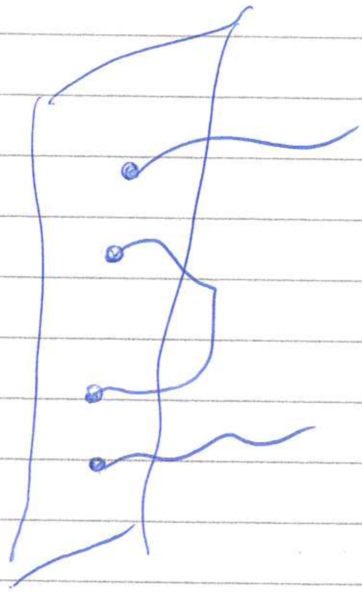
String theory - quantum theory of interacting one- and multi-dim objects



open strings



closed strings



branes

$$S = -T \int d^2\sigma \sqrt{-\det g}$$

Nambu-Goto action in $d=10$

$$T = \frac{1}{2\pi\alpha'} \qquad \alpha' = l_s^2$$

Spectrum:

<u>Closed strings</u>

$m_s \sim 1/l_s$

massless

$g_{\mu\nu}, \ \phi, \ C, \ B_{\mu\nu}, \ C^+_{\mu\nu\lambda\sigma}, \ \psi_{\mu,\alpha}, \ \lambda_\alpha$

~~Open strings: similar story but with the A_μ analogue~~

Closed string eom (low energy, $E \ll \frac{1}{l_s}$):

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \dots + \frac{1}{96}\tilde{F}_{\mu\lambda\rho\sigma\kappa}\tilde{F}_{\nu}{}^{\lambda\rho\sigma\kappa}$$

Solutions: Black 3-brane solution

$$ds^2_{10} = H^{-1/2}(r)\left[-f\,dt^2 + dx^2 + dy^2 + dz^2\right] +$$

$$+ H^{1/2}(r)\left[\frac{dr^2}{f} + r^2 d\Omega_5^2\right]$$

$$H = 1 + L^4/r^4 \qquad f = 1 - r_0^4/r^4$$

Open string theory (low energy, $E \ll \frac{1}{l_s}$)

Massless fields: $A_\mu,\ \Phi_I\ \ \lambda^a_\alpha$ $\quad a = 1,..4$

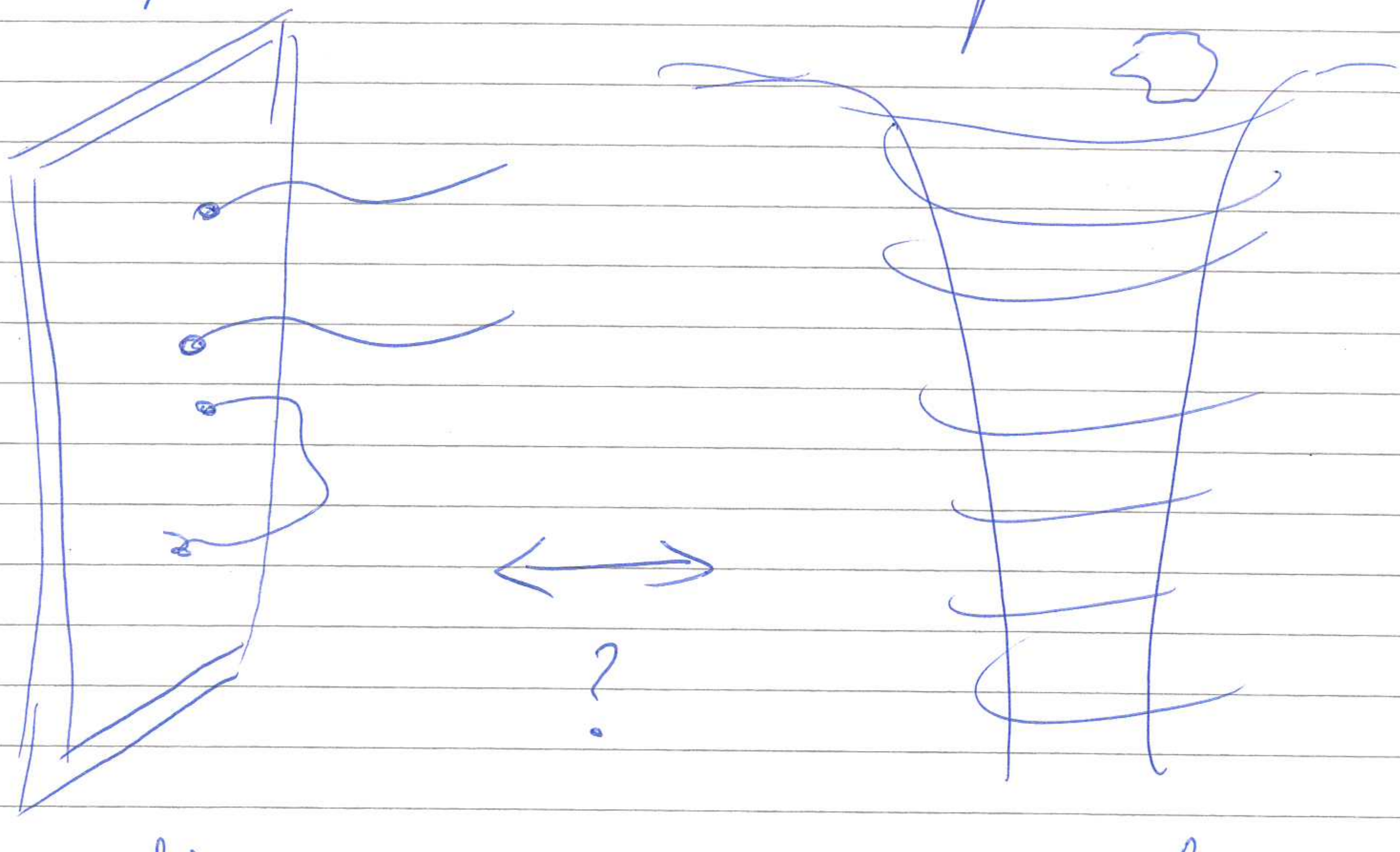
$I = 1,..6$ $\quad \alpha = 1,2$

$SU(N_c)$

$$\mathcal{L}_{YM} = -\frac{1}{g_{YM}^2} \mathrm{tr}\left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^I D^\mu \phi^I + [\phi^I, \phi^J]^2 \right) + \text{fermions} + O(E/l_s)$$

Open String picture

Closed string picture



?

10 dim

10 dim

N_c branes

g_{YM}

L

l_s, g_s

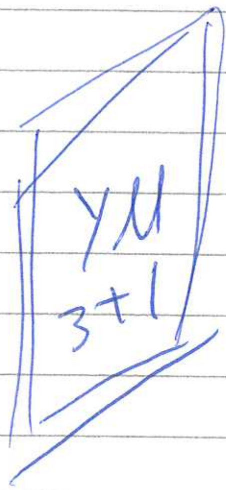
Is it the same object?

Do „scattering experiments":

Klebanov, 1997 hep-th/9702076

near-hor. region

$= AdS_5 \times S_5 \iff$



$$\begin{cases} \dfrac{L^4}{l_s^4} = g_{YM}^2 N_c \equiv \lambda & \text{'t Hooft coupling} \\[2ex] g_s = g_{YM}^2 / 4\pi \end{cases}$$

$$Z_{\substack{YM \\ d=4}}[J] = \tilde{Z}_{\substack{\text{IIB String} \\ \text{on } AdS_5 \times S^5}}[\tilde{J}]$$

$$\Big\downarrow \; N_c \to \infty, \; \lambda \to \infty \qquad\qquad \Big\downarrow \; \frac{L}{\ell_s} \gg 1, \; L/\ell_p \gg 1$$

$$Z_{\substack{YM \\ d=4}}[J] = e^{-S_{grav}[\tilde{J}]} + \dots$$

(in this limit)

$\Rightarrow$ Recipe to compute $G^R(\omega, \bar{q})$ in YM theory in the limit $N_c \to \infty$, $\lambda \to \infty$

(+ corrections)

We find from gravity side

$$G^R_{xy,xy} = -\frac{\pi^2 N_c^2 T^4}{4}\left(i\hat{w} - \hat{w}^2 + \hat{q}^2 + \hat{w}^2 \ln 2 - \frac{1}{2}\right)$$

$$+ O(\hat{w}^3, \hat{w}\hat{q}^2) \qquad (*)$$

$$\hat{w} = \omega/2\pi T \qquad \hat{q} = q/2\pi T$$

But recall linear response eq

$$\delta T_{xy} = -G^R_{xy,xy}(\omega, q)\, h_{xy} \qquad (**)$$

and $\delta T_{xy} = -P h_{xy} - \eta \partial_t h_{xy} + \eta \tau_\pi \partial^2_{tt} h_{xy}$
$+ \ldots$

Compare $*$ and $**$:

$$P = \frac{\pi^2}{8} N_c^2 T^4 \qquad \eta = \frac{\pi}{8} N_c^2 T^3$$

$$\tau_\pi = \frac{2 - \ln 2}{2\pi T} \quad \text{etc}$$

in $\mathcal{N}=4$ SYM at $N_c \to \infty$, $\lambda \to \infty$

$$P \rightarrow s = \frac{\pi^2}{2} N_c^2 T^3$$

$$\Rightarrow \eta/s = 1/4\pi \quad \text{or} \quad \eta/s = \hbar/4\pi k_B$$

(+ corrections $1/\lambda^{3/2}$, $1/N_c^2$)