Holography, Finite-temperature QFT and Hydrodynamics

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Gebze

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Holography, Finite-Temperature QFT and Hydrodynamics

<u>Motivation</u>: we would like to understand physical properties of macroscopic systems based on their microscopic description ("from first principles").

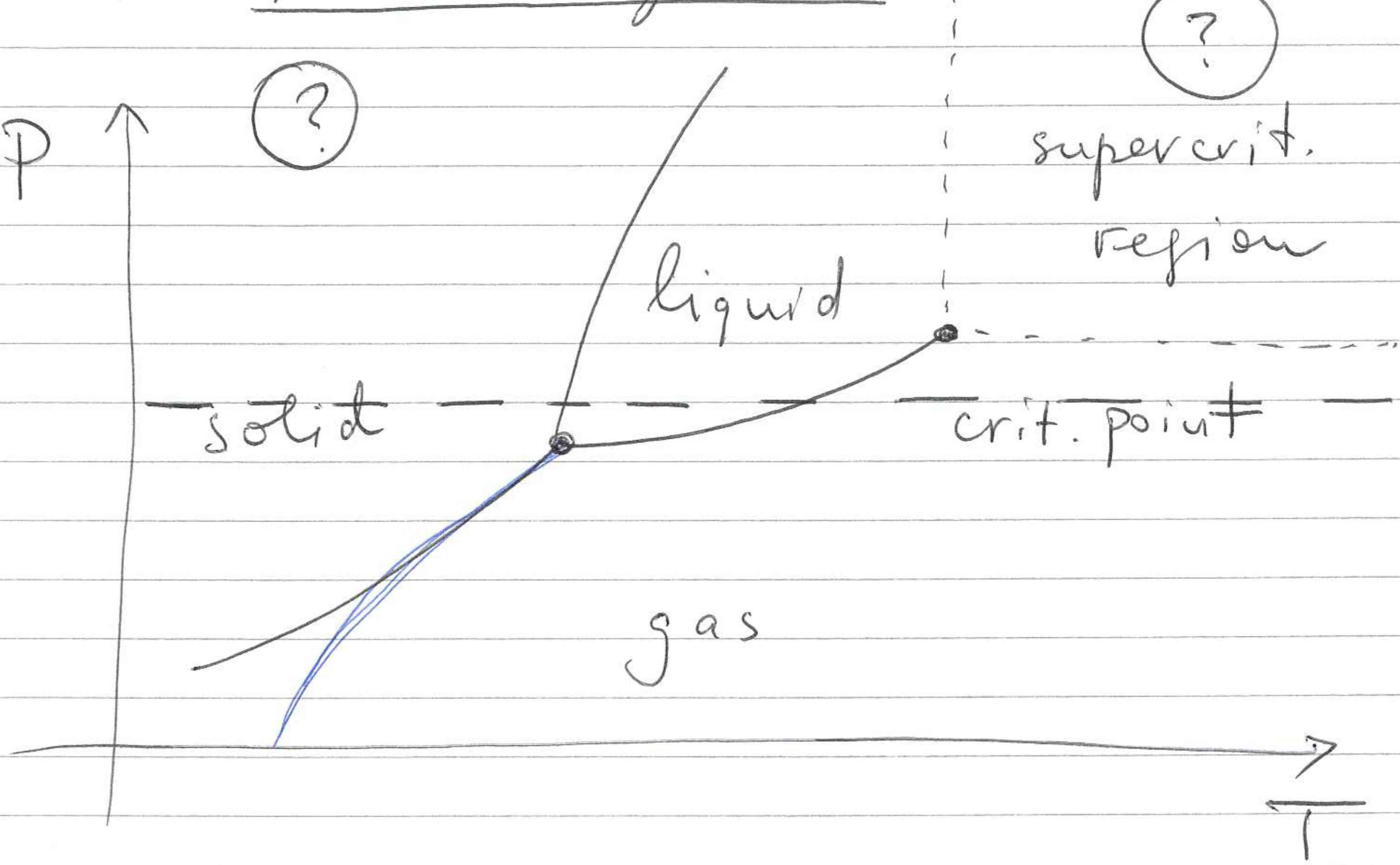
<u>Example</u>: glass of water (or atmosphere of Jupiter)

- Thermodynamic properties (phase transitions, specific heat, speed of sound – eq. of state etc)

- Transport properties (viscosity, conductivity, diffusion const. etc)

Description from first principles implies that, given $\mathcal{L}$ (or $\mathcal{H}$), one can compute all thermodynamics ($\Omega = -T \ln Z$, $Z = \mathrm{tr}\,\hat{\rho}$) and transport ($Z_0[J]$)

Phase diagram



H_2O: crit. point $T_c \approx 373.95$ C

$p_c \approx 221$ atm

Linear resistivity of „strange metals" at low T

Landau Fermi-liquid th. $\Rightarrow \rho \sim T^2$ for metals (almost all)

„Strange metals" $\rho \sim T$

Here $\hat{\rho} = e^{-\beta \hat{\mathcal{H}} + \mu_\alpha \hat{Q}^\alpha}$ equilibrium density matrix operator, $\beta = 1/T$

$\hat{Q}_\alpha$: conserved charges : $\dot{\hat{Q}}_\alpha = 0$

Note: in non-relativ. systems we usually have $\hat{Q} = \hat{N}$ (number of particles). In relativ. interacting systems $\hat{N}$ is not conserved, but there are other charges, e.g. $Q_B = \frac{1}{3}(n_q - n_{\bar{q}})$.

Note: $Z_{\mathcal{O}}[J] = \mathrm{tr}\, \hat{\rho}\, e^{J\hat{\mathcal{O}}}$ - generating functional for correlation functions of some operator $\hat{\mathcal{O}}$ of interest. Transport coefficients can be computed from these correlators via Green-Kubo formulae.

$$\langle \hat{\mathcal{O}} \rangle_T = \frac{1}{Z_{\mathcal{O}}} \frac{\delta Z_{\mathcal{O}}[J]}{\delta J}\Big|_{J=0} \quad \text{etc}$$

Can also write $\langle \hat{\mathcal{O}} \rangle_T = \mathrm{tr}\, \hat{\rho} \hat{\mathcal{O}}$.

It seems we have all ~~tools~~ we need.

Linear response theory

$$\hat{H}_0 \to \hat{H}_0 + \delta\hat{H}$$

$$\delta\hat{H} = -\int d^dx\, \lambda_a(t,\bar{x})\,\hat{O}_a(t,\bar{x})$$

$$\delta\langle\hat{O}_a(t,\bar{x})\rangle = -\int dt' \int d^dx'\, G^R_{ab}(t-t',\bar{x}-\bar{x}')\,\lambda_b(t',\bar{x}')$$

Here:

$$G^R_{ab}(t-t',\bar{x}-\bar{x}') = -i\,\theta(t-t')\langle[\hat{O}_a(t,\bar{x}),\hat{O}_b(t',\bar{x}')]\rangle$$

In Fourier space,

$$\delta\langle\hat{O}_a(\omega,\bar{q})\rangle = -G^R_{ab}(\omega,\bar{q})\,\lambda_b(\omega,\bar{q})$$

See e.g. Le Bellac or Kovtun 1205.5040

E.g.: $\hat{O} = \hat{J}^\mu$ or $\hat{T}^{\mu\nu}$ (conserved charges)

Then $\lambda = A_\mu$ or $h_{\mu\nu}$

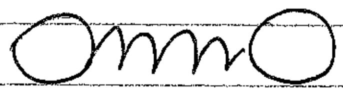
Indeed, this works very well in textbook examples.

Example: N diatomic molecules in $d_s = 3$ dim. (H_2 or N_2 etc)

$$\mathcal{H} = \sum_{i=1}^{N}\left(\frac{p_i^2}{2m} + \mathcal{H}_{internal,\,i}\right) + \cancel{\sum_{i<j} U(i,j)}$$

$$\mathcal{H}_{int} = \mathcal{H}_{rot} + \mathcal{H}_{osc} + \mathcal{H}_{electr} + \dots$$

ideal gas approx.



harmonic osc. approx.

$$Z = Z_0 \, Z_{internal}^{N}$$

(Z_0 : monoatom ideal gas)

$$Z_{int} = Z_{rot}\, Z_{osc}\, Z_{electr}$$

$$Z_{osc} = \mathrm{tr}\,\hat{\rho} = \sum_n \langle n | e^{-\beta \hat{\mathcal{H}}_{osc}} | n \rangle$$

$$\hat{\mathcal{H}}_{osc}|n\rangle = E_n |n\rangle \qquad E_n = \hbar\omega\,(n + 1/2)$$

$n = 0, 1, \dots$

$$Z_{osc} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}\, e^{-\beta\hbar\omega/2} = \frac{1}{2\sinh\frac{\beta\hbar\omega}{2}}$$

Z_{rot} : $E_{lm} = \frac{\hbar^2}{2I} l(l+1)$

$$l = 0, 1, 2 \ldots$$

$$m = -l, \ldots l$$

$$Z_{rot} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-\frac{\hbar^2}{2I}\beta l(l+1)} =$$

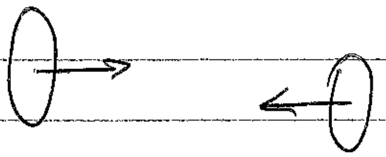
$$= \sum_{l=0}^{\infty} (2l+1) e^{-\frac{\hbar^2}{2I}\beta l(l+1)}$$

This is already rather complicated...

Taking into account interactions is even more difficult. Can be systematically done for dilute systems (kinetic theory), essentially ignoring 3- and higher-body interactions, but not for liquids or dense gases (or plasmas)

How about quark-gluon plasma?

Heavy ion collisions (RHIC, LHC, NICA, FAIR) produce QGP in the lab...

 $^{197}_{79}Au$ or $^{207}_{82}Pb$

Crude estimate of energy density:

- Energy of the beam (RHIC) $\sim 200\ GeV/nucl$
- Number of nucleons ~ 200
- $R_{Au} \sim 7\,fm$ $(1\,fm = 10^{-13}\,cm)$

$$\mathcal{E}_{TOT} \sim 200\ \frac{GeV}{nucl} \cdot 200\ nucl \sim 4 \cdot 10^4\ GeV$$

$$\epsilon \sim \mathcal{E}_{TOT}/V_3 \sim \frac{40\,000\ GeV}{\frac{4}{3}\pi\, 7^3\, fm^3} \sim 30\ GeV/fm^3$$

More precise estimates give $\epsilon \sim 5\ GeV/fm^3$

- Properties of nuclear matter -? Phase trans.-? Equation of state?

- Also, need transport properties: nuclear „droplet" can be described by relativ. Navier-Stokes eqs. The eqs. are the same for different systems ~~but~~ differ by values of transport coeffi in them.

Thermodynamics and transport in QCD from first principles

$$\mathcal{L}_{QCD} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu}_{a} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$$

$Z = \mathrm{tr}\, e^{-\beta\hat{H} + \mu\hat{Q}}$: can be represented as functional integral

$\Omega = -T \ln Z \Rightarrow$ all TD is known

e.g. $p = -\left.\frac{\partial\Omega}{\partial V}\right|_{T,\mu}$ (pressure)

$S = -\left.\frac{\partial\Omega}{\partial T}\right|_{V,\mu}$ (entropy)

Parameters of QCD: $N_c = 3$ ($SU(N_c)$ colour group)

$N_f = 6$ (number of quarks)

g_{YM} is not a fixed parameter - changes with energy: $\alpha_s = g_{YM}^2/4\pi \to 0$ at $E \to \infty$ (asymptotic freedom)

More precisely, with $E/\Lambda_{QCD} \to \infty$.

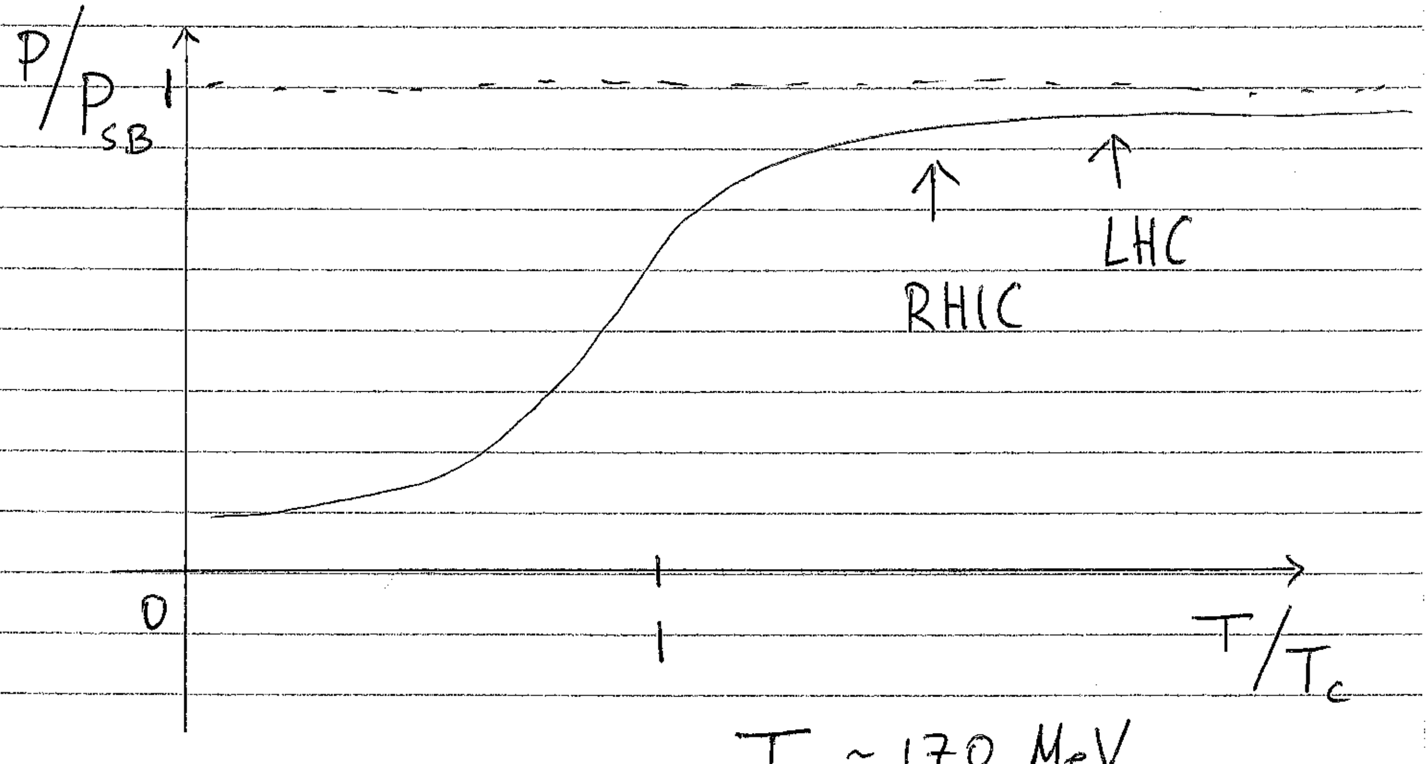
Thus, expect free (ideal) gas of quarks and gluons at $T \to \infty$

Exercise: show that

$$P_{SB} = \frac{8\pi^2}{45}\left(1 + \frac{21}{32}N_f\right)T^4 + O(m^2T^2)$$

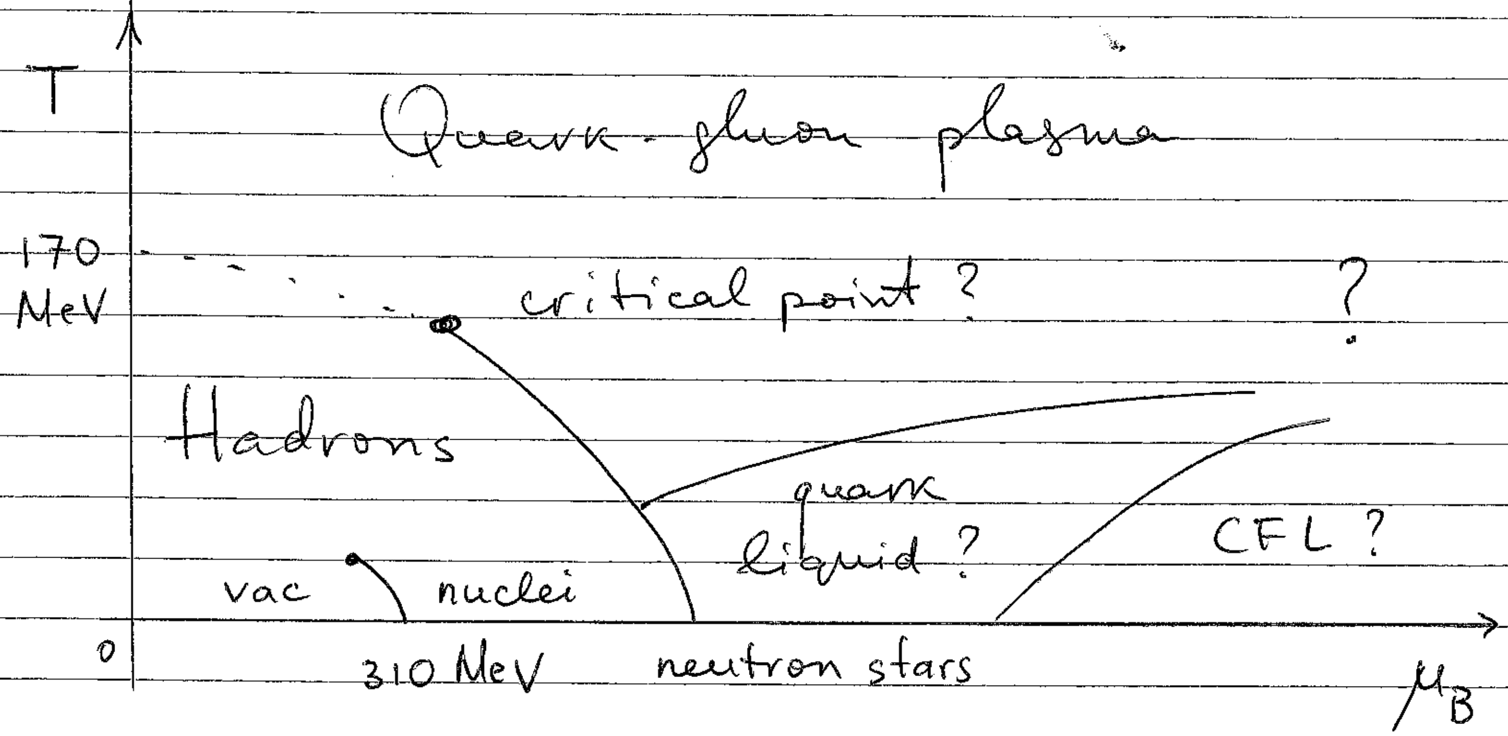
Stefan-Boltzmann (ideal gas).

For $\alpha_s \ll 1$, can compute $P = P_{SB} + O(\alpha)$ by perturb. theory (this is difficult - see books by Kapusta-Gale and Laine-Vuorinen). For $\alpha_s \gtrsim 1$, can use Lattice QCD



$T_c \sim 170$ MeV

QCD phase diagram



Transport in QCD from firsts principles

Effective theory for QCD at $T, \mu \neq 0$ at large dist. and large times = fluid dynamics

Equilibrium: E = const, Q = const

Near-eq: $\epsilon(t, \bar{x}) = E/V \quad q(t, \bar{x}) = Q/V$

densities of conserved charges

More precisely, $\langle \hat{T}^{\mu\nu} \rangle \equiv T^{\mu\nu}(t, \bar{x})$ - non-triv. function

exp. value in any state, e.g. non-eq

In equil., $\langle \hat{T}^{\mu\nu} \rangle_{T,\mu} = \mathrm{tr}\, \hat{\rho} \hat{T}^{\mu\nu} =$

$$= T^{\mu\nu}_{eq} = \begin{pmatrix} \epsilon & & 0 \\ & P & \\ 0 & & P_P \end{pmatrix} \quad (1)$$

For isotropic homog. system in its ref frame

If the system moves with $\overset{\text{constant}}{u^\mu} = (\gamma c, \gamma \vec{v})$,

$$T^{\mu\nu}_{eq} = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu}, \quad (2)$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, $c = 1$,

$$\Delta^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu \qquad (u_\mu \Delta^{\mu\nu} = 0)$$

<u>Exercise</u>: Derive (2) using (1)

<u>Exercise</u>: Show that $\Delta_{\mu\nu} \Delta^{\nu}{}_{\sigma} = \Delta_{\mu\sigma}$.

<u>Note</u>: $\Delta^{\mu\nu}$ serves as a projector, so

$$T^{\mu\nu}_{eq} = T^{\mu\nu}_{\parallel} + T^{\mu\nu}_{\perp} = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu}$$

We now assume $u^\mu = u^\mu(x)$. Then $T^{\mu\nu}$ will contain derivatives of $u^\mu(x)$.

Assume the gradients $\partial_\mu u_\nu$ are in <u>some sense</u> small $\to$ grad. expansion

$$T^{\mu\nu} = T_0^{\mu\nu} + T_1^{\mu\nu} + \mathcal{O}(\partial^2)$$

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu + p \Delta^{\mu\nu}$$

$$T_1^{\mu\nu} = -\eta \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\lambda u^\lambda \right)$$
$$- \zeta \Delta^{\mu\nu} \partial_\lambda u^\lambda$$

Note: this can be generalised to curved space-time by replacing $\eta_{\alpha\beta} \to g_{\alpha\beta}$ (and $\partial_\alpha \to \nabla_\alpha$).

This is known as k-th order grad. expansion in hydrodynamics. Also as constitutive relation (see below).

Combining $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu} = T_0^{\mu\nu} + \ldots$ gives (relativistic) Euler ($k=0$), Navier-Stokes ($k=1$), Burnett ($k=2$) etc equations

Exercise: derive rel. Navier-Stokes eqs this way. Consider non-rel. limit.

Similar eqs can be written in case of a conserved current J^μ:

$$\partial_\mu j^\mu = 0$$

$$j^\mu = j_0^\mu + j_1^\mu + \dots = \underbrace{\rho u^\mu}_{\text{convection}} \underbrace{- D \Delta^{\mu\nu} \partial_\nu \rho}_{\text{diffusion}} + \dots$$

In fluid's rest frame, $j^k = -D \partial_k \rho$
– Fick's law of diffusion

Hydro $\Leftrightarrow$ d.o.f. = densities of conserved charges (e.g. ρ). E.o.m. =
= conservation laws (e.g. $\partial_\mu j^\mu = 0$)
$\oplus$ constitutive relat. (e.g. $j^k = -D \partial_k \rho + \dots$)
$\Rightarrow \partial_t \rho + D \nabla^2 \rho = 0$ (diffusion eq)

Note: hydro of 1st and 2nd order is known; 3rd order is partially known. Open (simple) question – number of transport coeffi at $\forall$ order k = ?

Now recall linear ~~r~~esponse formula (with $\hat{O} = \hat{T}_{xy}$ and $\vec{q}$ along z)

$$\delta T_{xy}(\omega, \bar{q}) = - G^R_{xy,xy}(\omega, \bar{q})\, h_{xy}(\omega, \bar{q}),$$

where h_{xy} is the metric perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

with $h_{xy} \neq 0$ only (grav. wave of certain polarisation). 0704.0240

But for small $\omega, |\bar{q}|$, we have explicit expressions for $\delta T_{xy} = T_{xy} - \underset{0}{\underset{\shortparallel}{T^{eq}_{xy}}}$

$$\delta T_{xy} \overset{-Ph_{xy}}{=} -2\eta \Gamma^t_{xy} \overset{-Ph_{xy}}{=} -\eta\, \partial_t h_{xy} + \eta \tau_\Pi \partial^2_{tt} h_{xy} + \ldots$$

With $h_{xy} \sim e^{-i\omega t}$ (and $\bar{q} = 0$);

$$i\omega\eta h_{xy} = - G^R_{xy,xy}(\omega, 0)\, h_{xy}$$

More general 1610.01081

$$\Rightarrow G^R_{xy,xy}(\omega, 0) = -i\eta\omega + O(\omega^2)$$

$$\Rightarrow \eta = - \lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} G^R_{xy,xy}(\omega, 0)$$

Kubo