2775

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 4 Year Course

B1: I. FLOWS, FLUCTUATIONS AND COMPLEXITY AND II. SYMMETRY AND RELATIVITY

TRINITY TERM 2011

Thursday, 23 June, 9.30 am – 12.30 pm

Answer **four** *questions,* **two** *from each section:*

Start the answer to each question in a **fresh book.**

At the end of the examination hand in your answers to Section I and Section II in separate bundles.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section I. (Flows, Fluctuations and Complexity)

The Navier-Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \,,$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector and ν the kinematic viscosity.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

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1. Starting from the Navier-Stokes equations, derive Bernoulli's equation for motion along a streamline in a steady, inviscid flow,

$$\frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho} + gz = \text{constant} ,$$

where z is the vertical coordinate. Provide a physical interpretation of each of the three terms on the LHS in terms of conserved quantities in a small parcel of fluid. [4]

Show that an irrotational and incompressible flow that is independent of y can be represented by a streamfunction $\psi(x, z)$ satisfying Laplace's equation:

$$u = \frac{\partial \psi}{\partial z} , \quad w = -\frac{\partial \psi}{\partial x} ; \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 .$$

An inviscid, incompressible fluid flows steadily in the x-direction over a surface that varies sinusoidally in the direction of flow, so the height of the surface is given by $z_{\mathrm{s}}(x, y) = a\cos(\pi x/b)$, where a and b are constants. Show that the streamfunction

$$\psi = U_0 z - U_0 a \cos\left(\frac{\pi x}{b}\right) \exp\left(-\frac{\pi z}{b}\right)$$

satisfies Laplace's equation, corresponds to uniform flow for large z and approximately satisfies the relevant boundary condition at the surface, which you should state, provided $b \gg a$. Derive an expression for u at the surface. [8]

A simple model of a hydrofoil moving from right to left at speed U_0 in the x-direction through a stationary, incompressible fluid consists of a curved upper surface whose height is given by $z(x, y) = a\cos(\pi x/b)$ for $-b/2 < x < b/2$, $b \gg a$, a flat lower surface and infinite length in the y-direction. Assuming that the x-component of the flow velocity relative to the hydrofoil over the upper surface is $u = U_0(1 + (a\pi/b)\cos(\pi x/b))$, and neglecting flow in the other directions, derive an expression for the pressure on the upper and lower surfaces of the hydrofoil, and hence an expression for the lift force per unit length. [5]

Calculate the circulation around the hydrofoil, and state briefy the implications for the flow away from its immediate vicinity. If $a = 0.1\,\mathrm{m}$, $b = 2.0\,\mathrm{m}$ and the hydrofoil is travelling through water (density $\rho = 10^3\,\mathrm{kg\,m^{-3}}$ and kinematic viscosity $\nu = 1.5 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$) at $2\,\mathrm{m\,s^{-1}}$, use your expression to estimate the lift force per unit length on the hydrofoil. Is the assumption of smooth laminar flow likely to be valid in this instance? [8]

2. A viscous, incompressible, fluid flows steadily between two perspex plates of infinite horizontal extent spanning the planes $z = 0$ and $z = b$ under the influence of a constant pressure gradient, $\partial p/\partial x = -f$. Derive an expression for $\mathbf{u}(z)$ for $0 \le z \le b$ and state the condition on the dynamic viscosity ν for this solution to be valid. [8]

In a pasteurising process, the fluid, initially at room temperature T_0, is irradiated from below with plane-parallel ultraviolet radiation shining vertically upwards through an aperture spanning the region $0 < x < a$ and $-\infty < y < \infty$. The incident radiant energy flux is $F_0\,\mathrm{W\,m^{-2}}$ in the illuminated region, at a wavelength at which the absorption coefficient of the fluid is k and there is negligible absorption in the perspex. Starting from the Beer-Lambert law, show by considering the energy budget of a thin horizontal slab of fluid, that the the instantaneous rate of radiative heating of the fluid, in $\mathrm{W\,m^{-3}}$, is given by

$$q_{\mathrm{R}}(z) = F_0 k\rho e^{-k\rho z}$$

within the irradiated region. [6]

Assuming a is sufficiently small that the heating has a negligible impact on the background flow, that the fluid has a specific heat capacity c, and that the thermal conductivity and expansivity of both fluid and perspex are negligible, derive an expression for the temperature $T(z)$ at $x = a$ in the body of the fluid away from the boundaries. Explain how the processes you have neglected would cause your expression to break down near the boundaries. [6]

If the thermal conductivity of the fluid, κ, is small but not zero (but you can still neglect conduction through the perspex), the heating of a thin horizontal slab of fluid due to conduction from slabs above and below it is given by

$$q_{\mathrm{C}} = \kappa \frac{\partial^2 T}{\partial z^2}\,.$$

By comparing q_{C} computed from your expression for $T(z)$ at $x = a$ with q_{R} near the lower boundary, assuming $k\rho b$ is small enough and b/a is large enough that $e^{-k\rho z} \simeq 1$ and $b - z \simeq b$ in this region, find the range of values of z for which conduction plays an important role in determining $T(z)$ near the lower boundary. [5]

3. Explain the meaning of the terms *fixed point*, *unstable spiral* and *attractor* in the context of a phase-space description of a dynamical system. [3]

The Rössler attractor is generated by the dynamical system

$$\begin{aligned}\dot{x} &= -y - z \ , \\ \dot{y} &= x + ay \ , \\ \dot{z} &= a + z(x - c) \ ,\end{aligned}$$

where a and c are positive constants. Find the two fixed points of this system. [4]

Write down the Jacobian, $\mathbf{J}$, of the Rössler system. Assuming that $c \gg 1$ and $a \ll 1$, and that the properties of Jacobian at the origin are representative of behaviour near the fixed point nearest the origin, show that any perturbation in the z-direction contracts rapidly in this region, confining trajectories near the x-y plane. Neglecting any coupling between z and the other two dimensions, assess the stability of the fixed point nearest the origin and sketch the projection onto the x-y plane of a trajectory initialised near the origin. State when the confinement near the x-y plane breaks down. [9]

Using the properties of $(\mathbf{J}+\mathbf{J}^T)/2$, or otherwise, find the local Lyapunov exponents (rates of error growth or decline) as a function of position for the Rössler system. Show that perturbations grow exponentially in two directions, and shrink in the third direction, provided $z \neq 1$. Find the orientation of the directions of fastest perturbation growth and decline when the trajectory crosses the plane $x = c$ assuming $|z - 1| > 2a$. [9]

4. In a simplified theoretical model of muscle, each of N myosin heads attached to a thick filament has a periodic interaction with an actin filament with period Δ. A head can bind actin with rate constant $k_{\rm on}(x)$, where x measures the position of the head relative to the actin filament:

$$\begin{aligned} k_{\rm on}(x) &= k_1, & 0 \le x < x_1 < x_2 \ ; \\ &= 0, & x_1 \le x < \Delta \ ; \\ k_{\rm on}(x+m\Delta) &= k_{\rm on}(x), & m \text{ integer} \ . \end{aligned}$$

A bound head can unbind with rate constant $k_{\rm off}$:

$$\begin{aligned} k_{\rm off}(x) &= 0, & 0 \le x < x_2 \ ; \\ &= k_2, & x_2 \le x < \Delta \ ; \\ k_{\rm off}(x+m\Delta) &= k_{\rm off}(x), & m \text{ integer} \ , \end{aligned}$$

where k_1 and k_2 are constants. Each bound head exerts a constant force f_0 driving the thick filament in the positive x direction. If the thick filament slides at velocity u, the probability density $P(x,t)$ that a given head is bound to the actin filament obeys the following reaction-diffusion equation:

$$\frac{\partial P(x,t)}{\partial t} = k_{\rm on}(x)\,(1 - P(x,t)) - \ k_{\rm off}(x) P(x,t) - u\frac{\partial P(x,t)}{\partial x} \ .$$

Calculate and sketch $P_{\rm ss}(x)$, the steady-state probability density, averaged over many periods, that the head is bound, in the limit $u \ll k_2(\Delta - x_2)$ and $N \gg 1$ (you may ignore fluctuations in u). You may restrict your sketch to the interval $0 \le x < \Delta$. [10]

If, additionally, $x_1 \ll x_2$ and $u \ll k_2 x_2$, show that the average force exerted by each motor head is

$$\langle f \rangle \simeq f_0 \frac{x_2}{\Delta} \ \left\{ 1 - \exp\left(-\frac{k_1 x_1}{u} \right) \right\}$$

stating the approximations that you make. [4]

The sliding velocity of the thick filament is related to the force applied to the muscle by an external load $F_{\rm ext}$ by

$$F_{\rm ext} + N\langle f \rangle + \gamma u = 0 \ ,$$

where γ is the drag coefficient of the whole filament. On a single set of axes, sketch graphs of the drag force, γu, and of the average total force generated by all attached myosin heads, $N\langle f \rangle$, as functions of u. Calculate and indicate on this sketch the maximum force exerted by the muscle on the load, $F_{\rm max}$. Indicate by graphical construction, without calculation, the speeds of muscle contraction when $F_{\rm ext} = 0$ and when $F_{\rm ext} = -F_{\rm max}/2$. [11]

Section II. (Symmetry and Relativity)

5. Define the terms *proper time*, τ, *rapidity*, ρ, and *proper acceleration*, a_0. Show that the acceleration a of an object observed in a frame moving at speed $v = \beta c$ relative to the object in the same direction as its proper acceleration is given by

$$a = \frac{a_0}{\gamma^3} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad .$$

Hence, or otherwise, show that $d\rho/d\tau = a_0/c$ for any object moving in a straight line and explain briefly why this makes rapidity a useful concept. [10]

A rocket, initially at rest with rest-mass M_0, propels itself by converting rest-mass into photons at a constant fractional rate α, so $dM(\tau)/d\tau = -\alpha M(\tau)$. If all these photons are emitted rearwards, derive expressions for (a) the acceleration of the rocket as a function of time as observed by astronauts on the rocket and (b) the speed of the rocket relative to the launch pad as a function of time in the rest-frame of the launch pad. [8]

A second space-craft, also of rest-mass M_0 and initially at rest, is propelled by reflecting a plane-parallel beam of photons, generated at a rate αM_0 from a stationary source mounted on the launch pad, off a perfectly reflecting mirror mounted on the rear of the space-craft. Derive an expression for the acceleration of this second space-craft as observed by astronauts on the space-craft as a function of its velocity relative to the launch pad. Which space-craft would you expect to reach a distant star to which they are travelling first? [7]

[You may assume without proof that $\frac{d}{dv}(\gamma v) = \gamma^3$.]

6. A particle of rest-mass m_0 and initial energy E_0 decays into two particles 1 and 2 of rest-masses m_1 and m_2 respectively. Derive expressions for the energy and momentum of decay product 2 in the centre-of-momentum frame and of the speed βc of this frame relative to the laboratory. [4]

Derive an expression for the energy of particle 2 in the laboratory frame when (a) all trajectories are parallel to the line of flight of the original particle and particle 2 is emitted in the forward direction; and (b) the trajectories of the decay products are perpendicular to the line of flight of the original particle in the centre-of-momentum frame. In case (b), derive an expression for the angle at which particle 2 emerges in the laboratory frame relative to the line of flight of the original particle in terms of β and the energy and momentum in the centre-of-momentum frame. [12]

A beam of K^+ kaons, each with an energy of 8 GeV, enters a detector array in which they each decay to a μ^+ muon and a ν_μ neutrino: you may ignore other decay paths. Assuming the rest-mass of the neutrino is negligible, calculate the energy of the emerging neutrinos for (a) decay parallel to and (b) decay perpendicular to the kaon beam-line in the centre-of-momentum frame. In case (b), calculate the angle at which neutrinos are emitted relative to the kaon beam-line. [6]

How would you expect these observations to change if the rest-mass of the ν_μ neutrino were $200\,\mathrm{keV}/c^2$? Discuss the implications for the use of this kind of experiment to measure the rest-mass of the neutrino. [3]

7. A rod of length 2ℓ, stationary in frame S, is oriented along the x-axis. Show how the universality of the speed of light implies that the length of the rod as measured by an observer in frame S' moving with a speed $v = \beta c$ in the x-direction relative to frame S is given by $2\ell\sqrt{1-\beta^2}$. [6]

What is meant by the terms *pure force* and *proper force*? If $\mathbf{f}$ is a three-force in rest frame S acting on an object that is stationary in S, derive expressions for $\mathbf{f}'$, the three-force acting on the same object in frame S', in the cases that (a) $\mathbf{f}$ is oriented in the x-direction in S and (b) $\mathbf{f}$ is oriented in the y-direction in S. [8]

Suppose the rod is replaced by an ideal spring such that the proper force on the ends of the spring is $f = \kappa \Delta x$, where κ is the spring constant and Δx is the spring extension, both observed in frame S. Using your results, or otherwise, determine the spring constant κ' as measured by an observer in frame S' when the spring is (a) aligned with and (b) orthogonal to the direction of motion. Comment on your result. [5]

Suppose the proper force is provided by electrostatic repulsion of two point charges attached to the ends of the spring. Sketch the field lines around a point charge both at rest and moving at a relativistic speed in the x-direction. Explain how your sketch relates to your results concerning the behaviour of the ideal spring. [6]

8. Define the terms *four-force* and *four-velocity.* If the four-force F^a on a particle carrying a proper charge q moving with four-velocity U_b is given by

$$F^a = q\mathcal{F}^{ab}U_b \quad ,$$

where $\mathcal{F}^{ab}$ is a second-rank tensor, show that a necessary and sufficient condition that the force does not affect the rest-mass of the particle is that the tensor $\mathcal{F}^{ab}$ must be anti-symmetric. [6]

The Faraday tensor, $\mathcal{F}^{ab}$, is given by:

$$\mathcal{F}^{ab} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Show that, if $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ are the electric and magnetic fields experienced by a particle in a given reference frame, and $\mathbf{E}' = (E'_x, E'_y, E'_z)$ and $\mathbf{B}' = (B'_x, B'_y, B'_z)$ these fields in a reference frame moving in the positive x-direction at speed v relative to the first, then

$$\begin{aligned} E'_x &= E_x \\ E'_y &= a_1 E_y + a_2 B_x + a_3 B_z \\ B'_x &= B_x \\ B'_y &= a_1 B_y + \frac{a_2 E_x - a_3 E_z}{c^2} \end{aligned}$$

and find the constants a_1, a_2 and a_3. [9]

Show that $\mathcal{F}_{ab}\mathcal{F}^{ab}$ is proportional to $B^2 - E^2/c^2$ and explain why you would expect this quantity to be Lorenz invariant. [4]

A space-ship of rest-mass m_0, travelling with a velocity of βc in the x-direction, acquires a charge q from interstellar dust before entering a uniform interstellar magnetic field oriented in the z-direction, $\mathbf{B} = (0, 0, B)$. Derive an expression for electric and magnetic field in the rest-frame of the rocket immediately after entering the field. Hence, or otherwise, find the proper three-force on and proper acceleration of the rocket immediately after entering the field. [6]