Symmetry and Relativity
2011 Exam
Solution Notes

Section II. (Symmetry and Relativity)

5. Define the terms *proper time*, τ, *rapidity*, ρ, and *proper acceleration*, a_0. Show that the acceleration a of an object observed in a frame moving at speed $v = \beta c$ relative to the object in the same direction as its proper acceleration is given by

$$a = \frac{a_0}{\gamma^3} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad .$$

Hence, or otherwise, show that $d\rho/d\tau = a_0/c$ for any object moving in a straight line and explain briefly why this makes rapidity a useful concept. [10]

A rocket, initially at rest with rest-mass M_0, propels itself by converting rest-mass into photons at a constant fractional rate α, so $dM(\tau)/d\tau = -\alpha M(\tau)$. If all these photons are emitted rearwards, derive expressions for (a) the acceleration of the rocket as a function of time as observed by astronauts on the rocket and (b) the speed of the rocket relative to the launch pad as a function of time in the rest-frame of the launch pad. [8]

A second space-craft, also of rest-mass M_0 and initially at rest, is propelled by reflecting a plane-parallel beam of photons, generated at a rate αM_0 from a stationary source mounted on the launch pad, off a perfectly reflecting mirror mounted on the rear of the space-craft. Derive an expression for the acceleration of this second space-craft as observed by astronauts on the space-craft as a function of its velocity relative to the launch pad. Which space-craft would you expect to reach a distant star to which they are travelling first? [7]

[You may assume without proof that $\frac{d}{dv}(\gamma v) = \gamma^3$.]

6. A particle of rest-mass m_0 and initial energy E_0 decays into two particles 1 and 2 of rest-masses m_1 and m_2 respectively. Derive expressions for the energy and momentum of decay product 2 in the centre-of-momentum frame and of the speed βc of this frame relative to the laboratory. [4]

Derive an expression for the energy of particle 2 in the laboratory frame when (a) all trajectories are parallel to the line of flight of the original particle and particle 2 is emitted in the forward direction; and (b) the trajectories of the decay products are perpendicular to the line of flight of the original particle in the centre-of-momentum frame. In case (b), derive an expression for the angle at which particle 2 emerges in the laboratory frame relative to the line of flight of the original particle in terms of β and the energy and momentum in the centre-of-momentum frame. [12]

A beam of K^+ kaons, each with an energy of 8 GeV, enters a detector array in which they each decay to a μ^+ muon and a ν_μ neutrino: you may ignore other decay paths. Assuming the rest-mass of the neutrino is negligible, calculate the energy of the emerging neutrinos for (a) decay parallel to and (b) decay perpendicular to the kaon beam-line in the centre-of-momentum frame. In case (b), calculate the angle at which neutrinos are emitted relative to the kaon beam-line. [6]

How would you expect these observations to change if the rest-mass of the ν_μ neutrino were 200 keV/c^2? Discuss the implications for the use of this kind of experiment to measure the rest-mass of the neutrino. [3]

7. A rod of length 2ℓ, stationary in frame S, is oriented along the x-axis. Show how the universality of the speed of light implies that the length of the rod as measured by an observer in frame S' moving with a speed $v = \beta c$ in the x-direction relative to frame S is given by $2\ell\sqrt{1-\beta^2}$. [6]

What is meant by the terms *pure force* and *proper force*? If $\mathbf{f}$ is a three-force in rest frame S acting on an object that is stationary in S, derive expressions for $\mathbf{f}'$, the three-force acting on the same object in frame S', in the cases that (a) $\mathbf{f}$ is oriented in the x-direction in S and (b) f is oriented in the y-direction in S. [8]

Suppose the rod is replaced by an ideal spring such that the proper force on the ends of the spring is $f = \kappa\Delta x$, where κ is the spring constant and Δx is the spring extension, both observed in frame S. Using your results, or otherwise, determine the spring constant κ' as measured by an observer in frame S' when the spring is (a) aligned with and (b) orthogonal to the direction of motion. Comment on your result. [5]

Suppose the proper force is provided by electrostatic repulsion of two point charges attached to the ends of the spring. Sketch the field lines around a point charge both at rest and moving at a relativistic speed in the x-direction. Explain how your sketch relates to your results concerning the behaviour of the ideal spring. [6]

8. Define the terms *four-force* and *four-velocity*. If the four-force F^a on a particle carrying a proper charge q moving with four-velocity U_b is given by

$$F^a = q\mathcal{F}^{ab}U_b \quad ,$$

where $\mathcal{F}^{ab}$ is a second-rank tensor, show that a necessary and sufficient condition that the force does not affect the rest-mass of the particle is that the tensor $\mathcal{F}^{ab}$ must be anti-symmetric. [6]

The Faraday tensor, $\mathcal{F}^{ab}$, is given by:

$$\mathcal{F}^{ab} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Show that, if $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ are the electric and magnetic fields experienced by a particle in a given reference frame, and $\mathbf{E}' = (E'_x, E'_y, E'_z)$ and $\mathbf{B}' = (B'_x, B'_y, B'_z)$ these fields in a reference frame moving in the positive x-direction at speed v relative to the first, then

$$\begin{aligned} E'_x &= E_x \\ E'_y &= a_1E_y + a_2B_x + a_3B_z \\ B'_x &= B_x \\ B'_y &= a_1B_y + \frac{a_2E_x - a_3E_z}{c^2} \end{aligned}$$

and find the constants a_1, a_2 and a_3. [9]

Show that $\mathcal{F}_{ab}\mathcal{F}^{ab}$ is proportional to $B^2 - E^2/c^2$ and explain why you would expect this quantity to be Lorenz invariant. [4]

A space-ship of rest-mass m_0, travelling with a velocity of βc in the x-direction, acquires a charge q from interstellar dust before entering a uniform interstellar magnetic field oriented in the z-direction, $\mathbf{B} = (0, 0, B)$. Derive an expression for electric and magnetic field in the rest-frame of the rocket immediately after entering the field. Hence, or otherwise, find the proper three-force on and proper acceleration of the rocket immediately after entering the field. [6]

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5. $a = a_0 / \gamma^3$

$-c^2 d\tau^2 = -c^2 dt^2 + dx'^2 \Rightarrow d\tau = dt/\gamma$

$\tanh \rho = \beta = v/c$

$a^\mu = \frac{du^\mu}{d\tau}, \quad u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{v})$

$a^\mu = (c\gamma\dot{\gamma}, \gamma\dot{\gamma}\vec{v} + \gamma^2 \dot{\vec{v}}) = (\gamma^4 \frac{\vec{v}\cdot\vec{a}}{c},$

$\gamma^4 \frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} + \gamma^2 \vec{a})$, since

$\dot{\gamma} = \frac{\vec{v}\cdot\vec{a}}{c^2}\gamma^3$

In particle's rest frame $a^\mu = (0, \vec{a}_0)$

$a^\mu a_\mu = \text{invar} \Rightarrow$

$a_0^2 = -\left(\gamma^4 \frac{\vec{v}\cdot\vec{a}}{c}\right)^2 + \left(\vec{a} + \gamma^2 \frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v}\right)^2 \gamma^2$

$\Rightarrow$ for $\vec{v} \perp \vec{a}$: $a_0 = \gamma^2 a$

for $\vec{v} \parallel \vec{a}$, $a_0 = \gamma^3 a$

$\tanh \rho = v/c$

$$\frac{1}{\cosh^2\rho}\frac{d\rho}{d\tau} = \frac{\gamma}{c} a$$

$$1 - th^2\rho = \frac{1}{ch^2\rho} = 1 - \beta^2 \Rightarrow \frac{d\rho}{d\tau} = \gamma^3\frac{a}{c} = \frac{a_0}{c}$$

Note: $$\tanh(\alpha_1 \pm \alpha_2) = \frac{\tanh\alpha_1 \pm \tanh\alpha_2}{1 \pm th\alpha_1\, th\alpha_2}$$

$\Rightarrow$ add. veloc. in SR: $v_{TOT} = \ldots$

but $\rho_{TOT} = \rho_1 + \rho_2$

a) Rocket: conserv. of momentum

$$\boxed{\frac{1\, dM/c^2}{c} = M\, dV}$$

$\frac{\mathcal{E}}{c} = |\vec{k}|$ for photons

$(M - dM)\, dV \simeq M\, dV$

$\frac{dM}{d\tau} = -\alpha M \Rightarrow$

$$\alpha M\, d\tau\, c = M\, dV$$

$\Rightarrow a_0 = \frac{dV}{d\tau} = \alpha c$ – motion with const. <u>proper</u> acceleration.

In launch pad's frame:

$$a = a_0/\gamma^3$$

$$\gamma^3 dv = a_0\, dt = \alpha c\, dt$$

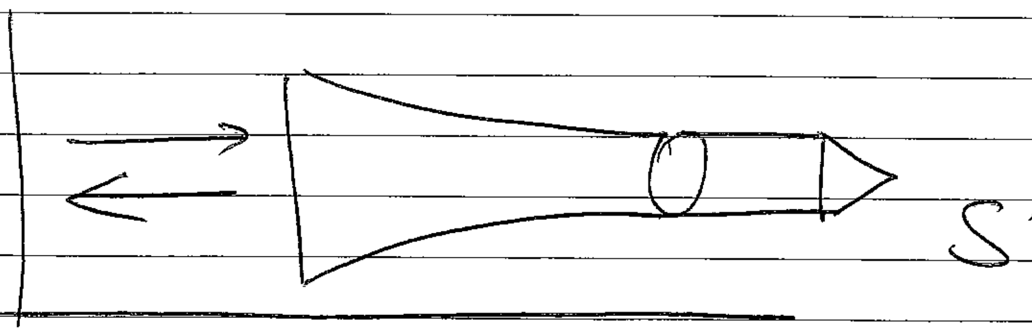
Use $\gamma^3 dv = d(\gamma v)$

$$d(\gamma v) = \alpha c\, dt$$

$$\Rightarrow \gamma v = \alpha c t$$

solve for v/c $\Rightarrow$ $\boxed{v/c = \dfrac{\alpha t}{\sqrt{1+\alpha^2 t^2}}}$

b)



S

In S: photons $\left(\frac{\varepsilon}{c}, \vec{k}\right)$,

with $\frac{\varepsilon}{c} = |\vec{k}|$

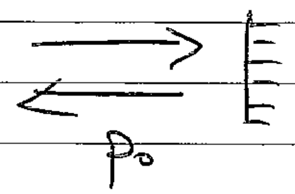
But $\varepsilon/c = \dfrac{\alpha M_0 c^2 dt}{c}$ $\Rightarrow$

in S $\quad k^\mu = (\underbrace{\alpha M_0 c\,dt}_{k^0}, \underbrace{\alpha M_0 c\,dt}_{k^1}, 0, 0)$

$\Rightarrow$ in S': $\quad k'^\mu = (k'^0, k'^1, \dots)$

$$k'^0 = \gamma(k^0 - \beta k^1) = \gamma(1-\beta)\,\alpha M_0 c\,dt = k'^1.$$

Now, in S':



$\Delta p = 2|p_0|$

$$\Rightarrow \boxed{2\gamma(1-\beta)\,\alpha M_0 c\,dt = M_0\,dv}$$

$d\tau = dt/\gamma$

$$\Rightarrow 2\gamma^2(1-\beta)\,\alpha c = \frac{dv}{d\tau} = a_0$$

$$\Rightarrow a_0 = \frac{2\alpha c}{1+\beta} \qquad \text{Note } a_0^{min} = \alpha c = a_0^{(a)})$$

$\Rightarrow$ second ship is faster.

6. $c = 1$, $(+ - - -)$

CMF: $P = (\mathcal{E}_0', \vec{0}) = (m_0, \vec{0})$

$p_1 = (\mathcal{E}_1', \vec{p}\,')$

$p_2 = (\mathcal{E}_2', -\vec{p}\,')$

$P = p_1 + p_2$ 4-vect.

$\mathcal{E}_1'^2 - \vec{p}\,'^2 = m_1^2$ but $\mathcal{E}_1' + \mathcal{E}_2' = m_0$

$\Rightarrow (m_0 - \mathcal{E}_2')^2 - \vec{p}\,'^2 = m_1^2$

$$m_0^2 - 2m_0\mathcal{E}_2' + \underbrace{\mathcal{E}_2'^2 - \vec{p}\,'^2}_{} = m_1^2$$

$$m_0^2 - 2m_0\mathcal{E}_2' + m_2^2 = m_1^2$$

$$\Rightarrow \boxed{\mathcal{E}_2' = \frac{m_0^2 + m_2^2 - m_1^2}{2m_0}}$$

$$\mathcal{E}_1' = m_0 - \mathcal{E}_2' = \frac{2m_0^2 - m_0^2 - m_2^2 + m_1^2}{2m_0} =$$

$$= \frac{m_0^2 + m_1^2 - m_2^2}{2m_0}$$

6

p': $p' = \left(\mathcal{E}_2'^2 - m_2^2 \right)^{1/2} = |\vec{p}_1'| = |\vec{p}_2'|$

$$p' = \frac{1}{2m_0} \left(\left(m_0^2 - m_2^2 - m_1^2 \right)^2 - 4 m_0^2 m_2^2 \right)^{1/2}$$

Now, Lab: $E_0 = \gamma m_0 \Rightarrow \gamma = \frac{E_0}{m_0}$

$\Rightarrow$ γ, β known.

$$\beta = \left(1 - m_0^2 / E_0^2 \right)^{1/2}.$$

Transf. to Lab from CMF:

a) $p_y' = 0$ $\quad \mathcal{E}_2 = \gamma \left(\mathcal{E}_2' + \beta p_2' \right)$

$$p_2' = \mathcal{E}_2' \left(1 - \frac{m_2^2}{\mathcal{E}_2'^2} \right)^{1/2} \approx \mathcal{E}_2' \left(1 - \frac{m_2^2}{2\mathcal{E}_2'^2} + \dots \right)$$

b) $p_x' = 0$ $\quad \mathcal{E}_2 = \gamma \mathcal{E}_2'$

p_2'

$$\tan \Theta = p_y / p_x = p_2' / \gamma (\beta \mathcal{E}_2')$$

$$\Theta = \arctan \frac{p_2'}{\gamma \beta \mathcal{E}_2'}$$

K^+: $\gamma = \frac{E_0}{m_{K^+}c^2} = \frac{8 \cdot 10^3}{493.7} \simeq 16.2$

$\beta \simeq 0.998$

With $m_\nu = 0$ $\mathcal{E}'_\nu = cp'_\nu$

$$\mathcal{E}'_\nu = \frac{m_K - m_\mu}{2m_K} \simeq 236 \text{ MeV}$$

$$p'_\nu = \left(\mathcal{E}'^2_\nu - m_\nu\right)^{1/2} = \mathcal{E}_\nu$$

$$\mathcal{E}_\nu = \gamma \mathcal{E}'_\nu (1+\beta) \simeq 7626.3 \text{ MeV}.$$

$\theta' = \pi/2$: $\mathcal{E}_\nu = \gamma \mathcal{E}'_\nu \simeq 3816.79 \text{ MeV}$

$$\theta = \arctan \frac{1}{\gamma\beta} \simeq 0.062 \text{ rad}$$

For $m_\nu = 0.2$ MeV

$$\mathcal{E}'_\nu = \frac{m_K + m_\nu - m_\mu}{2m_K} \simeq 235.5 \text{ MeV}$$

$\theta = 0.0617$ rad

— no change.

7. Part 1 : as in textbooks

Proper force: $\frac{dp^\mu}{d\tau} = f^\mu$

$p^\mu = mu^\mu = m\frac{dx^\mu}{d\tau}$, "Pure force": $\frac{dm}{d\tau} = 0$.

See HW solution notes of 2011 M Term.

$f^\mu = \left(\frac{1}{c}\dot{\mathcal{E}}, \vec{f}\right)$

$\Rightarrow \gamma f_x' = \gamma f_x \Rightarrow f_\parallel' = f_\parallel$

$\gamma f_y' = f_y \Rightarrow f_\perp' = f_\perp/\gamma$

Spring: $f = k\Delta x$ Hook

a) $k' = \frac{f'}{\Delta x'} = \frac{f}{\Delta x/\gamma} = \gamma k$

b) $k' = \frac{f'}{\Delta x'} = \frac{f/\gamma}{\Delta x} = k/\gamma$

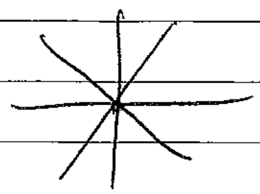
For charges: $f = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4l^2}$

a) $f' = f$ $k' = \gamma k$, $\Delta x' = \Delta x/\gamma$

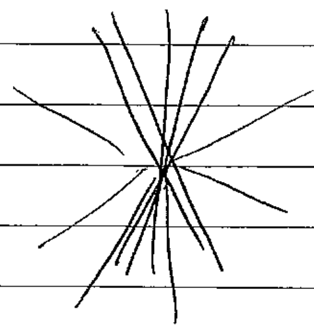
8) $\Delta x' = \Delta x \qquad k' = k/\gamma$

$f' = f/\gamma$

Recall that $\vec{E}, \vec{B}$ transform (and Δx as well). Static charge



$\rightarrow$



moving charge

field strength reduced in $\parallel$ dir. But ΔX contracted.

8. $$\frac{dp^\mu}{d\tau} = f^\mu$$

$$\frac{dx^\mu}{d\tau} = u^\mu \implies \frac{dp^\mu}{d\tau} = f^\mu = \frac{d(mU^\mu)}{d\tau} =$$

$= m \frac{dU^\mu}{d\tau}$ for the „pure" force.

$$m \frac{du^\mu}{d\tau} = qF^{\mu\nu} U_\nu$$

$$m U_\mu \frac{dU^\mu}{d\tau} = q F^{\mu\nu} U_\nu U_\mu$$

$\overset{\shortparallel}{0}$ since $\frac{d}{d\tau}(U_\mu U^\mu) = \frac{d}{d\tau} c^2 = 0$

$$\implies F^{\mu\nu} U_\mu U_\nu \equiv 0$$

but $F^{\mu\nu} U_\mu U_\nu = F^{\nu\mu} u_\nu u_\mu = F^{\nu\mu} u_\mu u_\nu$

$\equiv 0$ iff $F^{\mu\nu} = -F^{\nu\mu}$.

$F' = \Lambda F \Lambda^T \implies$ transf. law for $\vec{E}, \vec{B}$.

$F_{ab} F^{ab} = 2\left(\vec{B}^2 - \frac{\vec{E}^2}{c^2}\right)$ as discussed in HW sol. notes and tutorials.

$F^{ab} F_{ab}$ = scalar = Lor-invar.

$$\left.\begin{array}{l}\vec{B} = (0,\ 0,\ B) \\ \vec{E} = (0,0,0)\end{array}\right\} \quad E_y' = -\gamma v B_z$$

$$\vec{f} = -\gamma v q B\ (0,\ 1,\ 0)$$

$$\Rightarrow a_0 = \gamma q v B / m_0$$

$$B_z' = \gamma B_z = \gamma B.$$