B2

SYMMETRY AND RELATIVITY

EXAM PAPER

2017

SOLUTION NOTES

(PROBLEMS 1 & 2)

A13481W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B2. SYMMETRY AND RELATIVITY

TRINITY TERM 2017

Wednesday 14 June, 2.30 pm – 4.30 pm

Candidates are strongly advised to use the first 10 minutes to read the whole paper before starting writing.

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The following notation is used throughout the paper: capital bold letters (e.g. $\mathbf{U}$) indicate 4-vectors; lower case bold letters (e.g. $\mathbf{v}$) indicate 3-vectors. Note, however, that the symbols $\mathbf{E}$ and $\mathbf{B}$ in questions 2 and 3 indicate 3-vectors: the electric and magnetic fields respectively.

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1. For a particle of mass m moving along a world-line $X^\mu = X^\mu(\tau)$ in the inertial reference frame S, define the 4-velocity $\mathbf{U}$ and the 4-force $\mathbf{F}$. Show that if the scalar product of two non-zero 4-vectors $\mathbf{D}$ and $\mathbf{G}$ is zero, then one of the vectors is space-like and the other is time-like. Prove that if the particle has a 4-momentum $\mathbf{P}$ such that $\mathbf{P} \cdot \mathbf{P} = 0$, then the particle has a rest mass equal to zero. Give an example of such a particle. [5]

Define proper time and pure force. Show that if $\mathbf{F}$ is a pure force, then $\mathbf{F} \cdot \mathbf{U} = 0$, where $\mathbf{U}$ is the 4-velocity. Show that if 3-velocity $\mathbf{v}$ and 3-acceleration $\mathbf{a}$ are orthogonal to each other, i.e. $\mathbf{v} \perp \mathbf{a}$, then the 4-acceleration invariant $A^\mu A_\mu$ is given by $A^\mu A_\mu = \gamma^4 a^2$. [5]

A free particle, having rest mass M_0, is in vacuum and initially at rest in the lab frame S. It undergoes an acceleration under the action of a constant pure force $\mathbf{F} = (f_0, f_x, 0, 0)$, where

$$f_0 = \frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}.$$

Find its 4-velocity $\mathbf{U}$ as a function of time and force. Sketch the graphs of the dependence of normalised 3-velocity $\boldsymbol{\beta} = \mathbf{v}/c$ and the Lorentz factor $\gamma(t)$ of the particle as a function of time t. [5]

An electron and a positron are annihilated during a head-on collision. Before the collision they had 3-velocities $\mathbf{v}_e = (v_x, 0, 0)$ and $\mathbf{v}_p = (-v_x, 0, 0)$, respectively. After the collision, some number of photons are detected. What is the minimum number of photons that can be registered in this experiment? Explain your answer. Find the energy of each of the minimal number of photons taking into account that the rest mass of the electron and the positron is each equal to 0.511 MeV and their total kinetic energy before the collision was 1 GeV. [7]

An electron is accelerated from rest through a gap of $L = 3$ m by an electric field of strength 10 MV m^{-1} which is constant throughout the gap. Find γ and β at the other end of the gap. [3]

2. Write down the components of the 4-wave vector $\mathbf{K}$ of an electromagnetic wave [you may assume that it is a 4-vector]. Show that the phase φ of the wave is Lorentz invariant. Show that a single, isolated electron, propagating in vacuum cannot emit a photon. [5]

Two events in the laboratory frame S are characterised by the following 4-coordinates $\mathbf{D} = (ct_d, \mathbf{x}_d)$ and $\mathbf{G} = (ct_g, \mathbf{x}_g)$, where $\mathbf{x}_{d,g}$ are 3-vectors. Write down the condition for these events to be connected by a space-like interval. Can we find an inertial frame, S', where these two events are occurring simultaneously? Find the answer without drawing diagrams. [4]

Two photons of the same angular frequency ω and with 4-momenta $\mathbf{P}_1$ and $\mathbf{P}_2$ move in the lab frame S. The first photon moves along the x direction, while the second photon moves along the y direction. Find (a) the rest mass of the system as a function of ω and (b) the velocity of the centre of mass frame relative to the lab frame. [4]

In the lab frame S an electron is injected with initial 3-velocity $\mathbf{v} = (0, 0, 0)$ into a region with uniform, static and orthogonal magnetic and electric fields $\mathbf{B} = (0, 0, B)$ (magnetic field) and $\mathbf{E} = (0, E, 0)$ (electric field).

Considering the following cases

a) $|E/B| > c$ and

b) $|E/B| < c$,

where c is the speed of light. In each case find the trajectory exactly in a suitably chosen frame. Qualitatively describe, discuss and sketch the trajectories in the lab frame S. [7]

For an isolated system of particles, each of which is non-interacting, consider the expression (here $c = 1$)

$$s^2 = \left(\sum W_i\right)^2 - \left(\sum \mathbf{p}_i\right)^2,$$

where W_i and p_i are the energy and 3-momentum of a particle from the isolated system of particles respectively. Show that s^2 is invariant. Assuming the isolated system of particles is a non-interacting photon gas, find s^2. [5]

3. Name three phenomena, each of which can be detected via the studies of emitted or reflected light in astrophysics, that would represent a direct test of Special Relativity. Derive an expression for the photon's frequency shift in Compton scattering. [6]

In the laboratory frame S, a plane monochromatic electromagnetic wave with angular frequency ω and wave 3-vector $\mathbf{k} = (k_x, 0, 0)$ propagates in vacuum. Derive general expressions which describe the relativistic Doppler effect i.e. the equation for the photon's frequency shift from ω to ω' between the frame S and the frame S' moving with relative velocity $\mathbf{v}$ with respect to the laboratory frame S. For the same wave as described above consider cases when the frame S' is moving with velocities: (a) $\mathbf{v} = (v_x; 0; 0)$; and (b) $\mathbf{v} = (0; v_y; 0)$, relative to the frame S and find the frequency transformations for both cases. [6]

A particle of mass M_1 is moving in the laboratory frame S towards a stationary particle M_2. The first particle has the 3-velocity $v_1 = (v_x, 0, 0)$. At some point an elastic collision takes place. Show that after the collision, the angle between the trajectory of the particle M_1 and the axis x and the angle between the trajectory of the particle M_2 and the axis x are equal only if $M_1 = M_2$. [6]

An ultra-relativistic electron propagates with constant velocity $\mathbf{v}$ along the z-axis through a periodic electromagnetic field defined in the laboratory frame S by a 3-vector potential with only one non-vanishing component $A_y = A_0 \cos(k_u z)$, where $k_u = 2\pi/d$ and d is the period of the field. Find the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the rest frame of the electron. Can the field observed in the electron's rest frame be a propagating electromagnetic wave? Explain your answer. [7]

4. Frame S' moves with a constant 3-velocity $\mathbf{v}=(v_x, 0, 0)$ relative to the lab frame S. In S, the components of the electric field and the magnetic field are, respectively: $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$. Find the electric (E'_x, E'_y, E'_z) and magnetic (B'_x, B'_y, B'_z) field components in the frame S'. [3]

Define a 4-vector potential $\mathbf{A}$ and a 4-wave vector $\mathbf{K}$. A plane, linearly polarised electromagnetic wave propagates in the z direction through vacuum. Using the 4-vector potential $\mathbf{A} = (0, 0, A_y, 0)$, where

$$A_y = -\frac{\sin\left(K^\mu \cdot X_\mu\right)}{\sqrt{\omega \cdot k}},$$

find the field strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha.$$

Define the 4-current J_ν. Write down the 4-current continuity condition in 3-vector form and 4-vector form. Using Maxwell's equations written in 3-vector form show that $\partial^\mu F_{\mu\nu} = \mu_0 J_\nu$. The Lorentz force acting on a unit volume of charge density ρ can be written as $f_\mu = J^\nu F_{\nu\mu}$. What is the physical meaning of the f_0 component of this 4-vector? [7]

Consider a unit vector $U^\mu_W = W^\mu/(m_0 c|\mathbf{s}|)$ parallel to the Pauli-Lubanski spin vector $W^\mu = (\mathbf{s} \cdot \mathbf{p}, \mathbf{s}(E/c))$. Show that U^μ_W and the 4-momentum are orthogonal to each other. [3]

A 4-current $\mathbf{J} = (0, j_x, 0, 0)$, where $|j_x| = |\rho \cdot v_0|$ flows along an infinitely long straight wire which is stationary in the lab frame S. Find the electric and magnetic fields generated in the lab frame S, in the frame moving perpendicular to the wire with velocity $\mathbf{u} = (0, v_0, 0)$, and in the frame co-moving with the electrons, i.e. having 3-velocity $\mathbf{u} = (v_0, 0, 0)$ in S.

Consider a second identical wire having the same 4-current and separated from the first one by the distance d. Is it possible to find a frame such that the forces between the wires can be described as purely electrical? Prove the statement. [6]

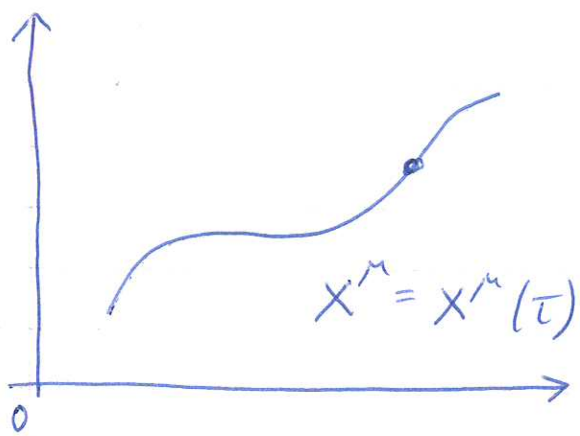
During the head-on collision between a photon defined by a 4-wave vector $\mathbf{K} = (\omega/c, k_x, 0, 0)$ and an ultra-relativistic particle of rest mass M_0 and total energy W an inverse Compton scattering was observed. Show that the maximum energy, which the photon can gain during this process, can be estimated as

$$E_{ph}{}^{max} = 4\left(\frac{W}{M_0 \cdot c^2}\right)^2 \cdot E_{ph},$$

where E_{ph} is the initial energy of the photon. [6]

SR Exam 2017 (Solution notes)

Problem 1



1) 4-velocity:

$u^\mu = dx^\mu / d\tau$

Since $d\tau = dt/\gamma \Rightarrow$

$u^\mu = (\gamma c, \gamma \bar{v})$

$\gamma = (1-\beta^2)^{-1/2}$, $\beta = |\bar{v}|/c$

2) 4-force: $f^\mu = dp^\mu / d\tau$, where

$p^\mu = m u^\mu = (\gamma m c, \gamma m \bar{v}) = \left(\frac{\mathcal{E}}{c}, \bar{p}\right)$.

We have $f^\mu = \left(\frac{\gamma}{c}\frac{d\mathcal{E}}{dt}, \gamma \bar{f}\right)$, where $\bar{f} = \frac{d\bar{p}}{dt}$.

3) $D \cdot G = 0$ $\quad D = (D^0, \bar{D})$ $\quad G = (G^0, \bar{G})$.

<u>Note</u>: this question is formulated incorrectly. However, one can show that if D or G is time-like, say $|G^0| > |\bar{G}|$, then the other is space-like: $D \cdot G = 0 \Rightarrow \frac{|D^0|}{|\bar{D}|} = \frac{|\bar{G}|}{|G^0|}\cos\varphi$ $< 1 \Rightarrow |\bar{D}| > |D^0|$, i.e. D is space-like.

(Note: the converse is not necessarily true.)

4) $P \cdot P = 0 \Rightarrow -\gamma^2 m^2 c^2 + \gamma^2 m^2 \bar{v}^2 =$
$= -m^2\gamma^2 c^2 (1 - \bar{v}^2/c^2) = -m^2 c^2 = 0$

$\Rightarrow m = 0$. Photon is an example of such a particle.

5) Proper time is the time in the ref frame where an observer is at rest. Pure force is the force acting on the body of mass m in case $\dot{m} = 0$.

Since $f^\mu = dp^\mu / d\tau$, we can write

$$f^\mu = \left(\frac{\gamma}{c} \frac{d\mathcal{E}}{dt}, \gamma \bar{f} \right), \text{ where } \bar{f} = \frac{d\bar{p}}{dt},$$

$$\mathcal{E} = \gamma m c^2.$$

We have $f^\mu u_\mu = 0$: $f^\mu u_\mu$ is a scalar, i.e. an invariant that can be computed in any frame. In the ~~body's~~ own frame,

$$u^\mu = (c, \bar{0}), \quad f^\mu = (\dot{m}c, \bar{f}) \Rightarrow$$

$$f^\mu u_\mu = \dot{m} c^2 = 0 \text{ for pure force.}$$

So, $f^\mu u_\mu = 0$ for pure ~~frame~~ force in any frame. In general, $f^\mu u_\mu = -\gamma^2 \dot{\mathcal{E}} + \gamma^2 \bar{f} \cdot \bar{v}$

So, $f^\mu U_\mu = 0$ implies $\dot{\mathcal{E}} = \bar{f} \cdot \bar{v}$. 3

6) Compute the invariant $A^\mu A_\mu$, where A^μ is 4-acceleration, $A^\mu = dU^\mu / d\tau$.

Since $d\tau = dt/\gamma$, we have

$$A^\mu = \left(c \frac{d\gamma}{d\tau}, \frac{d}{d\tau} (\gamma \bar{v}) \right) =$$

$$= \left(c\gamma\dot{\gamma}, \gamma (\gamma \dot{\bar{v}}) \right), \text{ where } \dot{\gamma} = d\gamma/dt.$$

But
$$\dot{\gamma} = \frac{d}{dt} \left[\left(1 - \bar{v}^2/c^2\right)^{-1/2} \right] =$$

$$= \frac{v^i}{c^2} \left(1 - \frac{\bar{v}^2}{c^2}\right)^{-3/2} \dot{v}^i = \frac{\bar{v} \cdot \bar{a}}{c^2} \gamma^3.$$

Therefore, $A^\mu = \left(\gamma^4 \frac{\bar{v} \cdot \bar{a}}{c}, \gamma^2 \bar{a} + \gamma^4 \bar{v} \frac{\bar{v} \cdot \bar{a}}{c^2} \right)$.

For $\bar{v} \perp \bar{a}$, this expression is simpler:

$$A^\mu = (0, \gamma^2 \bar{a})$$

Therefore, $A^\mu A_\mu = \gamma^4 \bar{a}^2$.

7) $f^\mu = (f_0, f_x, 0, 0)$

The equation of motion is

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$$\frac{dp^\mu}{d\tau} = f^\mu, \quad p^\mu = M_0 u^\mu$$

$$f^\mu = \left(\frac{\gamma}{c}\dot{\mathcal{E}}, \gamma \bar{f}\right) \qquad \dot{M}_0 = 0$$

We have then $M_0 \dfrac{du^\mu}{d\tau} = f^\mu \Rightarrow$

$$M_0 \gamma \frac{du^i}{dt} = \gamma f^i \Rightarrow \dot{u}^i = f^i / M_0 .$$

$$\begin{cases} \dot{u}_x = f_x / M_0 = \text{const} \\ \dot{u}_y = 0 \\ \dot{u}_z = 0 \end{cases}$$

$$\Rightarrow u_x(t) = \frac{f_x}{M_0} t + u_x(0) = f_x t / M_0 ,$$

since the particle is initially at rest.

Also, $u_y = 0$, $u_z = 0$.

So, $u^\mu = (\gamma c, \dfrac{f_x t}{M_0}, 0, 0)$. To find γ, recall that $u_i = \gamma v_i \Rightarrow$

$$u_x = f_x t / M_0 = \gamma v_x \Rightarrow$$

$\gamma^2 \dfrac{v_x^2}{c^2} = f_x^2 t^2 / M_0^2 c^2$. Since $v_y = 0$, $v_z = 0$,

$$\frac{v_x^2}{c^2} = 1 - 1/\gamma^2$$

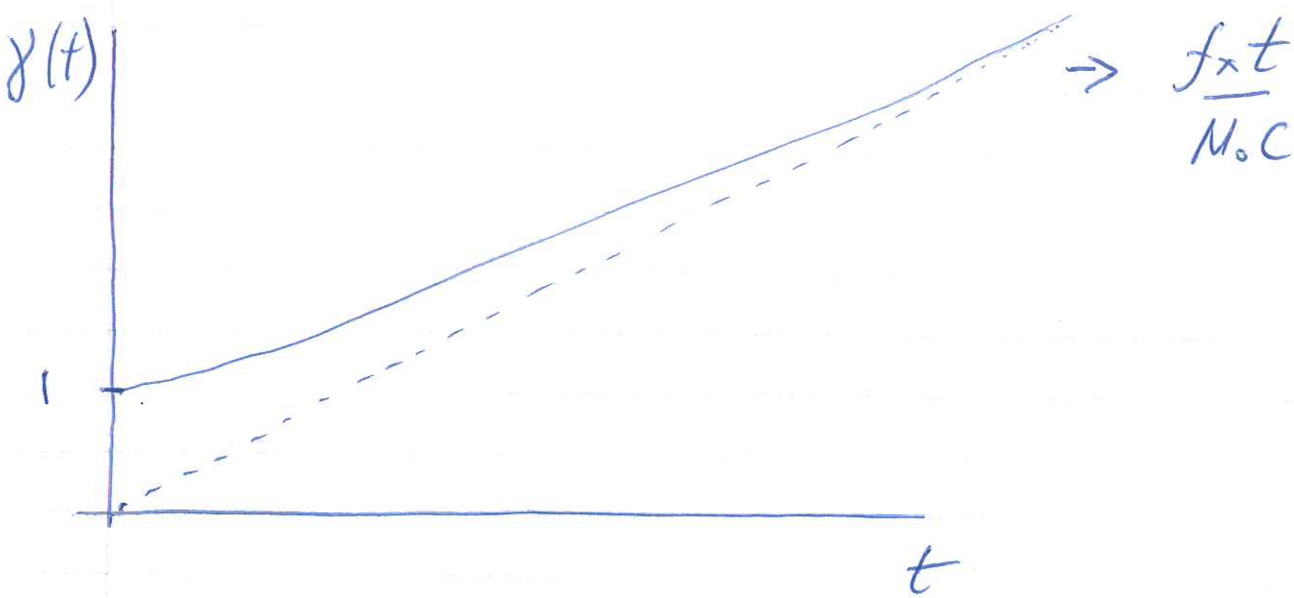
(5)

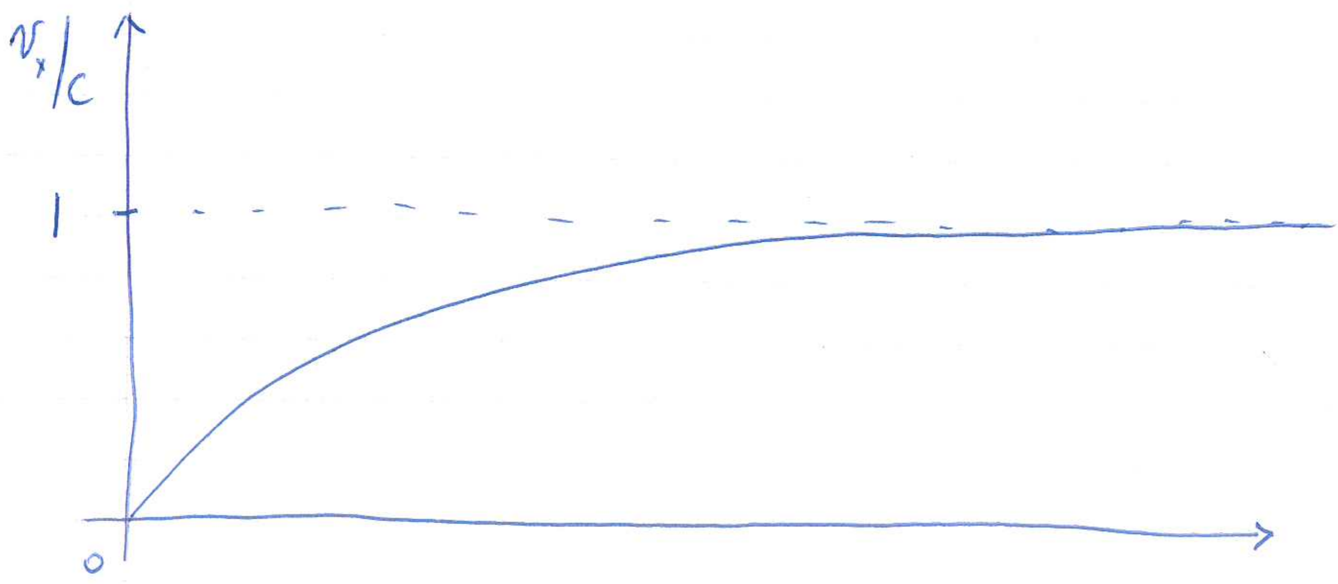
Therefore, $\gamma^2 = 1 + f_x^2 t^2 / M_0^2 c^2$

$$\gamma(t) = \sqrt{1 + f_x^2 t^2 / M_0^2 c^2}$$

$$\bar{\beta} = \bar{v}/c = (v_x/c,\ 0,\ 0).$$

$$\frac{v_x}{c} = \frac{f_x t}{M_0 c \sqrt{1 + f_x^2 t^2 / M_0^2 c^2}}$$





8) $e^+ + e^-$

$\bar{v}_e = (v_x, 0, 0)$ $\bar{v}_p = (-v_x, 0, 0)$

Note $m_e = m_p = m$.

Conservation of 4-momentum implies

$$p_e^\mu + p_p^\mu = K^\mu_{TOT},$$

where K^μ_{TOT} is the total 4-mom. of photons. This cannot be 1 photon, since in that case in CMF $\bar{p}_e + \bar{p}_p = 0$ (by definition of CMF) $\Rightarrow$ $\bar{K}_{TOT} = 0$ $\Rightarrow$ $\bar{K} = 0$ for a single photon $\Rightarrow$ $\mathcal{E}_\gamma = 0$, since $\mathcal{E}_\gamma = |\bar{K}|c$. Two photons are fine, no problems with conserv. laws.

$\Rightarrow$ min number is 2: $e^+ + e^- \to 2\gamma$.

For $N=2$: $p_e + p_p = K_1 + K_2$

In CMF: $|\bar{K}_1| = |\bar{K}_2| \Rightarrow \mathcal{E}_{\gamma 1} = \mathcal{E}_{\gamma 2} = \mathcal{E}_\gamma$,

since $\mathcal{E}_{\gamma 1} = \hbar\omega_1$, $|\bar{K}_1| = \omega_1/c$.

Since $\mathcal{E}_e = \sqrt{p_e^2c^2 + m_e^2c^4}$ and $|\bar{p}_e| = |\bar{p}_p|$ in CMF, $\mathcal{E}_e = \mathcal{E}_p$.
So, conservation law implies $2\mathcal{E}_e = 2\mathcal{E}_\gamma$ $\Rightarrow$ $\mathcal{E}_\gamma = \mathcal{E}_e$. Now, $\mathcal{E}_e = \gamma mc^2 =$
$= mc^2 + K$, where $mc^2 \simeq 0.5\,MeV$ and $K = 0.5\,GeV \gg 0.5\,MeV \Rightarrow$
$\mathcal{E}_\gamma \approx 0.5\,GeV$.

9) $L = 3m$ $|\bar{E}| = 10\,MV/m$. $f_x = |e\bar{E}|$
Since $v_x(t) = \dfrac{eEt}{m\sqrt{1+e^2E^2t^2/m^2c^2}}$,

we can find $x(t)$ by integrating over t (the integral is computed via change of variables $\xi = \sqrt{1+e^2E^2t^2/m^2c^2}$).
We find

$$X(t) = \frac{mc^2}{eE}\left[\sqrt{1+\frac{e^2E^2t^2}{m^2c^2}} - 1\right],$$ where

we imposed init. cond. $X(0) = 0$.

So, $L = \frac{mc^2}{eE}\left[\sqrt{} - 1\right]$ at $t = t_*$ (8)

when $x(t_*) = L$. But recall that

$$\gamma(t) = \sqrt{1 + e^2E^2t^2/m^2c^2} \Rightarrow$$

$$\Rightarrow \gamma(t_*) = \frac{LeE}{mc^2} + 1.$$

With numbers given, $\gamma \simeq 54$.

$$\beta = \sqrt{1 - 1/\gamma^2} \simeq 0.999831.$$

Alternatively, the Hamiltonian (total energy) of a particle is $\mathcal{E} = \gamma mc^2 + e\Phi$, where $\Phi = -Ex$ in this case.

We have at $t = 0$: $\mathcal{E}_0 = mc^2$

at $t = t_*$: $\mathcal{E} = \gamma mc^2 - eEL$

$$\Rightarrow mc^2 = \gamma mc^2 - eEL \Rightarrow$$

$$\gamma(t_*) = 1 + eEL/mc^2$$ as before.

Problem 2

- $K^\mu = \left(\frac{\omega}{c}, \bar{k}\right)$ (Here $\hbar = 1$.)

$K^2 = K^\mu K_\mu = 0$

- The phase $-i\omega t + i\bar{k}\bar{x}$ in the solution of Maxwell's eqs $\sim e^{-i\omega t + ikx}$ can be written as $iK^\mu X_\mu$: this is a scalar = Lor.-invar.

- We have $p = p' + K$, where p, p' are 4-vectors of the electron before/after the emission.

$$\underbrace{p^2}_{m^2c^2} = \underbrace{p'^2}_{m^2c^2} + 2p'k + \underbrace{k^2}_{=0} \quad \Rightarrow \underline{p'k = 0}$$

With $p' = \left(\frac{\varepsilon'}{c}, \bar{p}'\right)$, we have

$$-\frac{\varepsilon'}{c}\frac{\omega}{c} + \bar{p}' \cdot \bar{k} = 0, \quad |\bar{k}| = \omega/c$$

or $\frac{\varepsilon'}{c} = |\bar{p}'| \cos\varphi$

$$\Rightarrow \cos\varphi = \sqrt{1 + \frac{m^2c^4}{p^2c^2}} > 1 \text{ (impossible).}$$

- $D = (ct_d, \bar{x}_d)$

  $G = (ct_g, \bar{x}_g)$

The interval squared $s^2 = -c^2(t_d - t_g)^2 + (\bar{x}_d - \bar{x}_g)^2$

Space-like s : $(\bar{x}_d - \bar{x}_g)^2 > c^2(t_d - t_g)^2$

or $s^2 > 0$.

In a frame where two events are simulton.,

$D - G = (0, \Delta x')$, where

$$\Delta x' = \gamma(\Delta x - \beta \Delta x^0)$$

$$\Delta x'^0 = \gamma(\Delta x^0 - \beta \Delta x) = 0$$

$\Rightarrow |\beta| = |\Delta x^0| / |\Delta x| < 1$ since s is space-like. So, Lor. transf. to such a frame is possible.

- $P_1 = \left(\frac{\omega}{c}, K_x, 0, 0\right)$ $\quad |K_x| = \frac{\omega}{c}$

  $P_2 = \left(\frac{\omega}{c}, 0, K_y, 0\right)$ $\quad |K_y| = \omega/c$

$P_{TOT} = P_1 + P_2 = \left(\frac{2\omega}{c}, K_x, K_y, 0\right)$.

$$P^2_{TOT} = -M^2c^2$$

$$P^2_{TOT} = -\frac{4\omega^2}{c^2} + k_x^2 + k_y^2 = -\frac{2\omega^2}{c^2}$$

$$\Rightarrow M^2 = 2\omega^2/c^4 \Rightarrow M = \sqrt{2}\,\omega/c^2$$

CMF: $\bar{p}_1' + \bar{p}_2' = 0 \qquad P'_{TOT} = \left(\frac{\varepsilon'}{c}, \bar{0}\right)$

$$P'^2_{TOT} = -\frac{\varepsilon'^2}{c^2} = P^2_{TOT} = -2\omega^2/c^2$$

$$\Rightarrow \varepsilon' = \sqrt{2}\,\omega.$$

So: $\begin{cases} \text{Lab frame } P_{TOT} = \left(\frac{2\omega}{c}, k_x, k_y, 0\right). \\ \text{CMF } P'_{TOT} = \left(\frac{\sqrt{2}\omega}{c}, 0, 0, 0\right). \end{cases}$

Lor. transf: $P^0_{TOT} = \gamma\left(P'^0_{TOT} + \underbrace{\bar{P}'_{TOT}}_{\substack{\| \\ 0 \text{ in CMF}}} \cdot \bar{\beta}\right)$

$$\Rightarrow \boxed{\gamma = P^0_{TOT} / P'^0_{TOT} = \sqrt{2}}$$

$$\Rightarrow \beta = 1/\sqrt{2} \Rightarrow |\bar{v}| = \frac{\sqrt{2}}{2}c.$$

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$$S'': \quad E''_{\parallel} = 0 \quad E''_{\perp} = \gamma\left(E + \frac{\bar{V} \times \bar{B}}{c}\right) = 0$$

$$B''_{\parallel} = 0 \quad B''_{\perp} = \frac{1}{\gamma} B = \sqrt{\frac{B^2 - E^2}{B^2}}\, B,$$

$$V = c\,\frac{\bar{E} \times \bar{B}}{B^2}.$$ In S'': circular motion.

In the lab: spiral motion.