OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

GENERAL RELATIVITY AND COSMOLOGY

PROBLEM SET 5

(problems 1-5)

Solution notes

by

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1. **Redshift**

The Friedmann-Robertson-Walker metric for a homogeneous and isotropic Universe is given by

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1}$$

where ds is the proper time interval between two events, t is the cosmic time, k measures the spatial curvature, r, θ and ϕ are radial, polar and azimuthal coordinates, respectively.

(a) Explain what is meant by $a(t)$ and discuss its physical significance.

(b) Describe what is meant by *redshift* and how spectroscopic observations of extragalactic objects may be used to deduce their redshifts.

(c) What does the above expression become in the case of a light ray? Hence derive an integral expression for a light ray which leaves the origin at time t_{em} and reaches a comoving distance r_0 at time t_{obs}. A second ray is emitted a time dt after the first. By considering the two intervals as corresponding to successive wave crests, derive the relation

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} \equiv 1+z\,, \tag{2}$$

where z the redshift and λ_{em} and λ_{obs} are the emitted and observed wavelengths, respectively.

(d) How does the separation of galaxies today compare with the separation of galaxies when light left the galaxies we observe at redshift 1?

Solution:

Interaction of matter with (low-energy) gravity is described by the action

$$S = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left(R - 2\Lambda + \mathcal{L}_m\right) = S_{EH} + S_m\,, \tag{3}$$

where the Einstein-Hilbert action includes the cosmological constant Λ, and $\mathcal{L}_m$ is the Lagrangian containing all non-gravitational fields (i.e. the Standard Model matter fields) coupled to gravity. We know that $\mathcal{L}_m$ is a valid description of Nature up to a TeV scale (and possibly beyond). Einstein's gravity may be valid up to the (four-dimensional) Planck scale $E_P \approx 10^{19}$ GeV ($l_P = \sqrt{G\hbar/c^3} \approx 10^{-33}$ cm), where quantum gravity effects become relevant, but experimentally gravity is directly tested only up to a submillimeter scale (i.e. for distances $r \gtrsim 0.22$ mm), so it is possible that the true quantum gravity scale is less than the four-dimensional Planck scale (current experimental bounds suggest that this scale is greater than approximately 1 TeV).

The equations of motion resulting from the variational principle applied to the action (3) are

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}\,, \tag{4}$$

where the energy-momentum tensor of matter fields is defined as $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$. Observations suggest that the Universe is spatially homogeneous and isotropic on a large scale[1]. Assuming this is true for the whole Universe (this is known as the "cosmological principle" hypothesis), one finds (see e.g. S. Weinberg, "Gravitation and Cosmology") that the metric of the Universe must have the Friedmann-Robertson- Walker (FRW) form (i.e. be a metric whose hypersurfaces of constant time are maximally symmetric spaces in three dimensions):

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right), \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and k determines the scalar curvature of the three-dimensional space: $R = 6k$ (the parameter k can be positive, negative or zero, $k = 0$ corresponds to a flat three-dimensional space). If the radial coordinate r has the dimension of length, then $k \sim 1/L^2$ (since kr^2 in the metric should be dimensionless). Thus k is inversely proportional to the square of the scale of the corresponding space. In the FRW metric one can rescale variables $k \to \lambda^2 k$, $r \to r/\lambda$, $a \to \lambda a$, so that in the new metric $k = 0, \pm 1$, the radial coordinate is dimensionless, and the scale factor $a(t)$ has the dimension of length. Another form of the RW metric is

$$ds^2 = -c^2 dt^2 + a^2(t)\left(d\chi^2 + F^2(\chi) d\Omega^2\right), \tag{6}$$

where $F(\chi) = \chi, \sin\chi, \sinh\chi$ for $k = 0, +1, -1$, respectively.

The coordinates r, θ, ϕ in (1) are the so called "comoving" coordinates. (Imagine a coordinate grid on the surface of an inflating balloon, with galaxies represented by dots on the surface of the balloon - as the balloon inflates, the dots move, but the coordinate lines move with them, so each dot will have the same coordinates - if we neglect the "local" motion of a given galaxy in the gravitational potential of nearby galaxies.)

The Cosmological Principle also implies (again, see e.g. S. Weinberg, "Gravitation and Cosmology") that on average at sufficiently large scales the matter in the Universe is de-

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[1] Note that the isotropy at *each* point of a space implies homogeneity of the space, see Weinberg's book "Gravitation and Cosmology" for more details.

scribed by the energy-momentum tensor of the form:

$$T_{\mu\nu} = \frac{P(t)}{c^2}\, g_{\mu\nu} + \left(\rho(t) + \frac{P(t)}{c^2}\right) u_\mu u_\nu \tag{7}$$

which is the energy-momentum tensor of a perfect (i.e. non-viscous, non-dissipative) fluid.

With the FRW ansatz (5) and the energy momentum tensor as in Eq. (7), the Einstein's equations (4) give

$$\dot{a}^2 + kc^2 - \frac{c^2}{3}\Lambda a^2 = \frac{8\pi G}{3}\, \rho a^2 \,, \tag{8}$$

$$2a\ddot{a} + \dot{a}^2 + kc^2 - c^2\Lambda a^2 = -\frac{8\pi G}{c^2} P(t) a^2 \,. \tag{9}$$

The energy-momentum tensor is covariantly conserved: $T^{\mu\nu}{}_{;\nu} = 0$ (note that this is compatible with the Einstein equations (4) since the covariant derivative of the LHS of (4) vanishes). With Eqs. (5), (7), the conservation equation gives

$$\dot{P} a^3 = \frac{d}{dt}\left[a^3 \left(P + \rho\, c^2\right)\right] \tag{10}$$

which can be rewritten as

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\frac{P}{c^2} + \rho\right) = 0\,. \tag{11}$$

We can also arrive at Eq. (11) by taking the derivative of Eq. (8) and combining the result with Eq. (9). See e.g. L.Ryder, "Introduction to General Relativity", Chapter 10.7. This means that Eqs. (8), (9) and (11) are not independent. Normally, Eqs. (8) and (11) are used as fundamental equations

$$\dot{a}^2 + kc^2 - \frac{c^2}{3}\Lambda a^2 = \frac{8\pi G}{3}\, \rho a^2 \,, \tag{12}$$

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\frac{P}{c^2} + \rho\right) = 0 \tag{13}$$

for $a(t)$, $\rho(t)$, $P(t)$. Supplemented by the equation of state $P = P(\rho)$ (for example, for radiation we have $P = \rho c^2/3$) this is a system of two coupled non-linear ODE for two variables.

(a) The scale factor $a(t)$ describes the expansion of the Universe and measures the physical (proper) distances (distances traveled by e.g. light) in the Universe. Recall that r, θ, ϕ in the FRW metric are comoving coordinates, i.e. their values do not change while the Universe expands. At a given moment of time $t = t_*$, the proper distance between "us" at $r = 0$ and an object at the radial coordinate r is given by

$$d(t_*) = a(t_*) \int_0^r \frac{dx}{\sqrt{1-kx^2}} \,.$$

(b) The redshift z is defined as $z = (\lambda_{obs} - \lambda_{em})/\lambda_{em}$, where λ_{obs} and λ_{em} are the wavelengths of the observed and emitted light. The observed spectroscopic line emitted by a moving object will not in general coincide with the "canonical" one, i.e. the one emitted by the same source at rest with respect to an observer. A generic motion of a source results in the Doppler shift of spectroscopic lines. The cosmological redshift z refers exclusively to the motion due to the expansion of the Universe (i.e. ignoring all "local" motion of galaxies, stars and so on which in principle contributes to the Doppler effect). The cosmological redshift is related to the scale factor of the Universe $a(t)$.

(c) To relate z with a, recall that the geodesic equation for light traveling to us along the radial direction from the object at $r = r_1$ is

$$ds^2 = -c^2 dt^2 + a^2 \frac{dr^2}{1-kr^2} = 0 \,. \tag{14}$$

This can be written as $cdt = -adr/\sqrt{1-kr^2}$, where the minus sign reflects the fact that the light is moving towards the origin $r = 0$ and its radial velocity component dr/dt is negative in the associated coordinate system. If the light was emitted at $t = t_1$ at $r = r_1$ and is observed at $t = t_0$ at $r = 0$, then

$$c \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dx}{\sqrt{1-kx^2}} \,. \tag{15}$$

For a second pulse, emitted at $t = t_1 + \delta t_1$ at the same comoving location $r = r_1$ and observed at $t = t_0 + \delta t_0$ at $r = 0$ we have

$$c \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dx}{\sqrt{1-kx^2}} \,. \tag{16}$$

This implies

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} - \int_{t_1}^{t_0} \frac{dt}{a(t)} = 0 \,. \tag{17}$$

Re-arranging the limits of integration carefully, we find

$$\int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{t_1}^{t_1+\delta t_1} \frac{dt}{a(t)} \,. \tag{18}$$

Taking δt_1 and δt_0 to be the periods of the emitted (observed) light wave, $\delta t_1 = T_1 = 1/\nu_{em}$ and $\delta t_0 = T_0 = 1/\nu_{obs}$, and taking into account the fact that during the time intervals δt_1 and δt_0 the scale factor $a(t)$ is essentially constant, we find

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

which implies

$$\frac{\nu_{obs}}{\nu_{em}} = \frac{a(t_{em})}{a(t_{obs})}\,. \tag{19}$$

Since $\lambda\nu = c$, for the redshift z we find

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} - 1\,. \tag{20}$$

If the light is observed "now" at $t_{obs} = t_0$, the standard normalization $a(t_0) = 1$ simplifies the relation (20) even further: $z + 1 = 1/a(t_{em})$.

(d) The distance to an object located at $r = r_*$ is given now (i.e. at $t = t_0$) by

$$d(t_0) = a(t_0) \int_0^{r_*} \frac{dx}{\sqrt{1 - kx^2}}\,.$$

The distance to the same object at $t = t_{em}$ was

$$d(t_{em}) = a(t_{em}) \int_0^{r_*} \frac{dx}{\sqrt{1 - kx^2}} = a(t_{em}) \frac{d(t_0)}{a(t_0)} = \frac{1}{2}\, d(t_0)\,,$$

since $a(t_0)/a(t_{em}) = 1 + z = 2$. Thus, $d(t_0) = 2\, d(t_{em})$, i.e. the separation today is twice the separation at $t = t_{em}$.

2. **Horizons**

(a) Describe the concept of our past and future light-cone. Explain the meaning of the terms *particle horizon distance*, *event horizon distance* and *world-line*, and discuss the difference between time-like and space-like locations.

(b) Show that in an Einstein-de Sitter Universe in which the scale-factor $a(t)$ at time t follows $a(t) \propto t^{2/3}$, the particle horizon is at $3ct$ and the event horizon is at infinity.

(c) Suppose that the scale factor were given by $a(t) \propto \exp{(mt)}$, where m is a positive constant. Show that the event horizon is finite and that the particle horizon grows exponentially when $t \gg 1/m$.

(d) Explain how such behaviour of the particle horizon might be useful in explaining observations of the cosmic microwave background.

Solution:

(a) A world-line is a trajectory of a point-like object moving in a space-time. A world-line of a stationary object is parallel to the time axis. For a moving object, the slope of its world-line w.r.t. time axis can be less or equal to c. At each fixed space-time location, the world-lines with the slope equal to c form a cone - a future light-cone in the direction of increasing time, and a past light-cone in the direction of decreasing time.

Recall that the Equivalence Principle (which is an experimentally testable physical assumption) implies that at each space-time point P there exists a coordinate system such that the metric at P reduces to Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ and the Christoffel symbols vanish (note that for space-times obeying the Equivalence Principle, the torsion tensor then necessarily vanishes and the Christoffel symbols are symmetric in their lower indexes, $\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\sigma\nu}$). Using a coordinate transformation $x' = x'(x)$ and tensor transformation rules, one can show explicitly that at a given point P it is possible to reduce $g_{\mu\nu}(x)$ to Minkowski metric and make the Christoffel symbols (i.e. the metric's first derivatives) vanish. However, this coordinate freedom is not sufficient to make the *second derivatives* of the metric vanish - thus if the Riemann tensor at P is non-zero, it remains non-zero after the coordinate transformation - the spacetime is *curved*. (Note in passing that it is possible to make the Christoffels vanish along a given world-line.) Thus, at each space-time point P one can choose a *local inertial frame*. Thus in an infinitesimal neighborhood of P, GR reduces to Special Relativity, and one can introduce the usual concepts of a light cone as well as time-like, space-like and null infinitesimal separations between points according to $ds^2 < 0$, $ds^2 > 0$ or $ds^2 = 0$, where $ds^2 = g_{\mu\nu}(x_P)dx^{\mu}dx^{\nu}$.

Particle horizon of a given observer includes all points the observer can see at present (all points emitting light or other signals since the beginning of time).

Using the equation for the radial light-like geodesic, one finds for the light emitted at r at time t and received at $r = 0$ at $t = t_0 > t$

$$c \int_{t}^{t_0} \frac{dt}{a(t)} = - \int_{r}^{0} \frac{dx}{\sqrt{1-kx^2}} . \tag{21}$$

The maximal value of r corresponds to $t = 0$:

$$c \int_{0}^{t_0} \frac{dt}{a(t)} = \int_{0}^{r_{max}} \frac{dx}{\sqrt{1-kx^2}} . \tag{22}$$

The distance to the location $r = r_{max}$ at $t = t_0$ (now) is given by

$$d_H = d(t_0) = a(t_0) \int_0^{r_{max}} \frac{dx}{\sqrt{1 - kx^2}} = c\, a(t_0) \int_0^{t_0} \frac{dt}{a(t)} \, . \tag{23}$$

This is known as the *particle horizon distance.* Alternatively, the distance d_H is the maximal distance a hypothetical signal emitted at $t = 0$ at our location $r = 0$ can reach at $t = t_0$. Indeed, for the outward traveling light ray,

$$c \int_0^{t_0} \frac{dt}{a(t)} = \int_0^{r_{max}} \frac{dx}{\sqrt{1 - kx^2}} \tag{24}$$

and then the distance to r_{max} now (at $t = t_0$) is given by d_H as in Eq. (23).

Event horizon includes all points (events) in the Universe at time t, the signal from which will be able to reach us during the remaining life-time of the Universe.

In a sense, an event horizon is a concept complementary to the concept of a particle horizon. Both terms were introduced in 1956 by W.Rindler (see e.g. S.Weinberg, "Cosmology", Section 1.13).

Again, imagine an observer sitting at $r = 0$. If this observer was sending the light signals starting from $t = 0$ (here we ignore all issues related to the decoupling of matter and radiation and so one - it is an idealized situation), then the *farthest* proper distance such signals could travel during the time from $t = 0$ to $t = t_0$ (now) is given by d_H: this is the particle or object horizon distance (alternatively and symmetrically, this is the maximum proper distance we can look into the Universe now). But now suppose the observer emits a signal at time $t = t_*$ and wonders how far (measured at $t = t_*$) this signal will propagate during the remaining life-time of the Universe (i.e. from $t = t_*$ until $t = T_{fin}$ which can be infinite or finite, depending on the model). In other words, what part of the Universe (considered at $t = t_*$) will ever be accessible for observations. The answer is given by the equations

$$c \int_{t_*}^{T_{fin}} \frac{dt}{a(t)} = \int_0^{r_{max}} \frac{dx}{\sqrt{1 - kx^2}} \tag{25}$$

and

$$d_{EH} = d(t_*) = a(t_*) \int_0^{r_{max}} \frac{dx}{\sqrt{1 - kx^2}} = c\, a(t_*) \int_{t_*}^{T_{fin}} \frac{dt}{a(t)} \, . \tag{26}$$

The distance d_{EH} is known as the *event horizon distance.*

Consider the proper distance between two objects (e.g. galaxies) fixed at their comoving coordinates, say $r = 0$ and r. In a spatially flat Universe ($k = 0$), this distance at time t is given by $d(t) = a(t)r$. The rate of separation, $\dot{d}$, is then

$$\dot{d} = \frac{\dot{a}}{a} ar = H(t)\, d(t)\,.$$

So $\dot{d}/c = Hd/c > 1$ if $d > c/H$. This implies that the objects separated by the distance exceeding the Hubble distance, are receding from each other at a rate exceeding the speed of light (this does not contradict special relativity since the motion is due to a cosmological expansion and is not related to a local change of coordinates of the objects or a signal propagation).

For the matter-dominated Universe whose scale factor is given by $a(t) = (t/t_0)^{2/3}$, where $t_0 = 2/3H_0$ is the age of the Universe in this model, we find

$$d_H = c\, t_0^{2/3} \int\limits_0^{t_0} \frac{dt}{t^{2/3}} = 3\, c\, t_0 = \frac{2}{H_0}\, c \tag{27}$$

and

$$d_{EH} = c\, a(t_*) t_0^{2/3} \int\limits_{t_*}^{\infty} \frac{dt}{t^{2/3}} \to \infty\,. \tag{28}$$

Note that the Hubble constant H_0 (more precisely, its inverse, $1/H_0$) effectively sets the time scale (”Hubble time”) and the length scale (”Hubble length”) of the observable Universe. The Hubble constant is usually written as $H_0 = 100\, h$ km/s Mpc, where the current (2010) value of h is 0.72(3).

The exponential scale factor $a(t) \sim \exp{(mt)}$ arises in models with dominating cosmological constant. Setting all Ω_i except Ω_Λ to zero in the FRW equation, we find $a(t) = a(t_f) e^{m(t-t_f)}$, where $m = c\sqrt{\Lambda/3} = H_0$, and t_f is the time when the inflation (exponential expansion) ends. With $t_f = t_0$, we find

$$d_H = c\, e^{mt_0} \int\limits_{t_i}^{t_0} e^{-mt}\, dt = \frac{c}{m} \left(e^{\mathcal{N}} - 1\right)\,, \tag{29}$$

$$d_{EH} = c\, e^{mt_*} \int\limits_{t_*}^{\infty} e^{-mt}\, dt = \frac{c}{m}\,. \tag{30}$$

where t_i is the time when the inflation starts, $\mathcal{N} = m(t_f - t_i) = m(t_0 - t_i)$ is known as the number of *e-foldings* during the inflation. It is clear that the event horizon distance is finite, and the particle horizon distance grows exponentially for $\mathcal{N} \gg 1$.

(d) The cosmic microwave background (CMB) radiation is highly isotropic, with temperature differences between different directions in the sky of the order of $\delta T/T \sim 10^{-5}$. These fluctuations in CMB photons energies is due to density fluctuations at the time of photon emission (more dense regions had gravitational potentials that redshifted emitted photons stronger than less dense regions). The fact that the density fluctuations are so small implies that the regions emitting CMB were in causal contact with each other at some point, so that different regions could influence each other. As explained in Problem 3 of the Problem Set 4, the particle horizon distance at the time of radiation decoupling as seen now is of the order of 100 Mpc, whereas the particle horizon now in the "old" matter-dominated models is much larger, about 6000 Mpc. Thus, within these models, the region in the sky (now) whose interior was causally connected at the time of CMB release subtends an angle of about $\sim 1°$. It is difficult to explain why the CMB is so highly isotropic beyond such a region.

Indeed, assuming the Universe was dominated by radiation until the time of decoupling $t = t_d$ (this means the Friedmann equation gives $a(t) = a(t_d)\sqrt{t/t_d}$ for the time dependence of the scale factor), the distance the light could have traveled during the time from $t = 0$ and $t = t_d$ is given by

$$d(t_d) = c\, a(t_d) \int_0^{t_d} \frac{dt}{a(t)} = c\sqrt{t_d} \int_0^{t_d} \frac{dt}{\sqrt{t}} = 2\, c\, t_d\,. \tag{31}$$

By now, this distance has been stretched by the subsequent cosmological expansion (during the matter-dominated epoch) up to

$$d(t_{d,0}) = \frac{a(t_0)}{a(t_d)}\, d(t_d) = 2\, c\, t_d \left(\frac{t_0}{t_d}\right)^{2/3}.$$

This can be compared to this model's particle horizon at $t = t_0$ (now), $d_H = 3\, c\, t_0$:

$$\frac{d(t_{d,0})}{d_H} = \frac{2}{3}\left(\frac{t_d}{t_0}\right)^{1/3} \sim (1 + z_d)^{-1/2} \ll 1\,.$$

More precisely, one should be calculating the ratio of $d(t_{d,0})$ and the so called *angular diameter distance* d_A defined so that the proper distance $d(t_{d,0})$ perpendicular to the line of sight would be seen today to subtend a (small) angle $\theta = d(t_{d,0})/d_A$. This distance is the proper distance to the decoupling surface now

$$d_A = c\,a(t_0) \int\limits_{t_d}^{t_0} \frac{dt}{a(t)} = 3\,c\,t_0 \left[1 - \left(\frac{t_d}{t_0}\right)^{1/3}\right] \approx 3\;c\;t_0 \sim d_H$$

since $t_d/t_0 \ll 1$. So $\theta = d(t_{d,0})/d_A \sim (1+z_d)^{-1/2} \ll 1$.

A similar calculation with the scale factor $a(t) = e^{m(t-t_0)}$ shows that[2]

$$d(t_d) = c\,a(t_d) \int\limits_{t_i}^{t_d} \frac{dt}{a(t)} = \frac{c}{m}\left(e^{m(t_d-t_i)} - 1\right) = \frac{c}{m}\left(e^{\mathcal{N}} - 1\right),$$

whereas $d_A \sim 1/H_0$ (assuming $\Omega_\Lambda \neq 0$, $\Omega_M \neq 0$ only). It is therefore clear that in this case $d(t_{d,0})/d_A \sim z\exp(\mathcal{N}) \gg 1$, i.e. the patches causally connected at the time of radiation decoupling are now spead throughout the observable sky. Thus, the inflationary expansion can be invoked to explain the CMB isotropy.

3. **The size of the Universe**

Assume the Universe today is flat with both matter and a cosmological constant but no radiation.

(a) Compute the horizon of the Universe as a function of Ω_M and sketch it (you will need a computer or calculator to do this).

(b) What is the current horizon size for a Universe with $\Omega_M = 1/3$ and $h = 1/\sqrt{2}$?

(c) What is the mass contained within the current horizon in solar masses? If all objects were $10^{13}h^{-1}M_\odot$ in mass, how many are in the observable Universe?

Solution:

(a) With only Ω_M and Ω_Λ different from zero (and thus $\Omega_M + \Omega_\Lambda = 1$), the FRW equation can be written as

$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\frac{\Omega_M}{a^3} + \Omega_\Lambda\right).$$

The horizon distance is

$$d_H(\Omega_M) = c\,a(t_0) \int\limits_0^{t_0} \frac{dt}{a(t)} = \frac{c}{H_0} \int\limits_0^1 \frac{da}{\sqrt{(1-\Omega_M)a^4 + \Omega_M a}}, \tag{32}$$

[2] To simplify the calculation, we identified t_d with t_f.

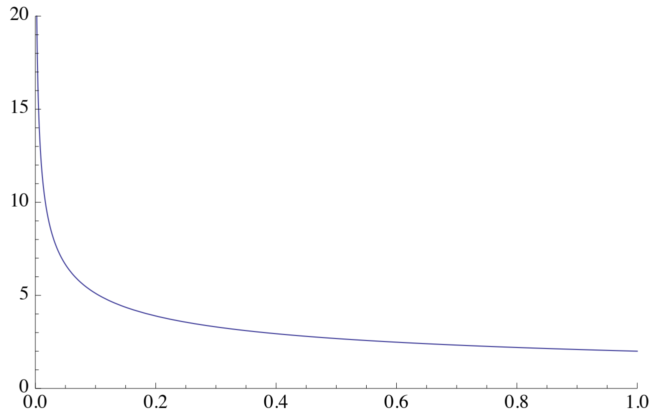

FIG. 1. The function $d_H H_0/c$ vs Ω_M.

where we used the FRW equation to express dt through da. The integral in (32) can be computed analytically and expressed through the Gauss hypergeometric function

$$d_H(\Omega_M) = \frac{2c}{H_0\sqrt{\Omega_M}}\,{}_2F_1\left(\frac{1}{6},\frac{1}{2};\frac{7}{6};1-1/\Omega_M\right)\,. \tag{33}$$

The function $d_H(\Omega_M)H_0/c$ is shown in Fig. (1).

(b) With $\Omega_M = 1/3$, we have $X = 2\,{}_2F_1\left(\frac{1}{6},\frac{1}{2};\frac{7}{6};1-1/\Omega_M\right)/\sqrt{\Omega_M} \approx 3.16636$. Then, remembering that $H_0 = 100h$ km/s Mpc, and with $h = 1/\sqrt{2}$, we find $d_H = cX/H_0 \approx 13.4$ Gpc.

(c) The critical density is $\rho_{M,0} \approx 2.775\,10^{11}h^2\,M_\odot$ Mpc^{-3}. The total mass is thus $M = 4\pi\rho_{M,0}d_H^3/3 \approx 1.4\times 10^{24}M_\odot$. This corresponds to $\sim 2\times 10^{11}$ objects of mass $10^{13}h^{-1}M_\odot$.

4. **The Big Bang and the acceleration of the Universe**

(a) Give an account of the observational evidence for the hot Big Bang model of the Universe.

(b) The Friedmann and fluid equations respectively are given by ($\Lambda = 0$)

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{34}$$

and

$$\dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{P}{c^2}\right) = 0\,, \tag{35}$$

where a is the scale factor, ρ is the density and P is the pressure, $\dot a$ and $\dot\rho$ are the derivatives of these quantities with respect to time. Use these equations to derive the acceleration equation for the Universe.

(c) Hence demonstrate that if the Universe is homogeneous and the strong energy condition $\rho c^2 + 3P > 0$ holds, then the Universe must have undergone a Big Bang.

Solution:

(a) The observational evidence for the hot Big Bang scenario includes:

1) The observed expansion of the Universe. Since $T(t) \sim 1/a(t)$, a smaller scale factor in the past implies the Universe was hotter.

2) The observed highly uniform Cosmic Microwave Background (CMB) radiation with a black body (Planckian) spectrum (intensity vs frequency dependence at a given temperature). One can show (see e.g. L. Ryder, "Introduction to General Relativity", Section 10.8) that the black body nature of the radiation is preserved under the space expansion with the temperature time dependence given by $T(t) \sim 1/a(t)$.

3) The primordial nucleosynthesis: during a relatively brief (minutes) period of time in the hot and dense primordial Universe fusion reactions resulted in light elements (hydrogen, helium, lithium) being formed. Their relative abundancies can be predicted and compared to the observed abundancies of these elements, with excellent agreement.

(b) Taking the time derivative of the equation (34) and combining the result with the expression for $\dot{\rho}$ from the equation (35) we get ($\Lambda = 0$):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) . \qquad (36)$$

Note that with $\Lambda \neq 0$ we would get from (12)-(13)

$$\frac{\ddot{a}}{a} = \frac{c^2\Lambda}{3} - \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) .$$

The Big Bang singularity corresponds to $a = 0$. When the strong energy condition $\rho c^2 + 3P > 0$ holds, equation (36) implies $\ddot{a} < 0$ at all times. Since $a \geq 0$, this means $a(t)$ should have crossed the time axis in the past resulting in $a = 0$.

5. **Recombination and the Surface of Last Scattering**

(a) What is the "surface of last scattering"? Would the same surface be seen by any other observer in a different galaxy?

(b) Estimate the radius of the surface of last scattering using the age of the Universe. Why might this underestimate the true value?

(c) The present number density of electrons in the Universe is the same as that of protons, namely about $0.2\,\mathrm{m}^{-3}$. Consider a time long before decoupling when the Universe was 10^4

years old and then the scale factor was one millionth of its present value[3]. Estimate the number density of electrons at that time and comment on whether the electrons would be relativistic or non-relativistic then.

(d) Given that the mean free path of photons through an electron gas of number density n_e is $d \sim 1/n_e\sigma$, where the Thompson scattering cross-section $\sigma_e = 6.7 \cdot 10^{-29}\,\mathrm{m}^2$, determine the mean free path for photons when the scale factor was one millionth of its present value.

(e) From the mean free path, calculate the typical time between interactions between the photons and electrons.

(d) Compare the interaction time with the age of the Universe at that time. What is the significance of this comparison?

Solution:

(a) Photons interact with electrons and other charged particles (”scatter” on them) with a certain cross-section which can be calculated in Quantum Electrodynamics (QED). For photons energies much less than the mass of an electron m_e, this cross-section reduces to the classical Thompson cross-section

$$\sigma_e = \frac{8\pi\alpha^2}{3m_e^2} \approx 6.7 \cdot 10^{-29}\,\mathrm{m}^2,$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$ is the fine structure constant. The mean free path of photons moving through an electron gas of number density n_e is given by $l_{mfp} \sim 1/n_e\sigma_e$.

When the Universe was sufficiently hot, the photons were energetic enough to ionize hydrogen atoms ($\gamma + H \to e^- + p^+$). Thus the density of free electrons was high at that time, and the mean free path of photons was correspondingly short. As the Universe further expanded, it cooled down (recall that $T(t) \sim 1/a(t)$). One may expect that the ionization processes stopped when $k_B T$ was of the order of eV (ionization energy scale). A detailed calculation gives $k_B T_d \sim 0.26$ eV, or $T_d \sim 3000$ K. At this time ($t \sim t_d$), the density n_e drops and the photons mean free path increases enormously: photons are now free to travel across the Universe. These photons form the CMB we observe today (of course, after $t \sim t_d$, the photon gas continued to cool down as the Universe expanded). The imaginary spherical surface (with us in the centre) where the CMB (observed by us now at $t = t_0$) originated at

[3] There is a typo in the formulation of the problem: the size of one millionth of the present value corresponds to the Universe being approximately 1 year old. The age of 10^4 years corresponds (approximately) to the size of ten thousandth. We use the size of one millionth and the age of one year in this problem.

$t = t_d$ is known as the surface of last scattering. The distance to this surface is given by

$$d = c\, a(t_0) \int_{t_d}^{t_0} \frac{dt}{a(t)} \,. \tag{37}$$

Clearly, a hypothetical observer in a different galaxy would have his/her/its/? own surface of last scattering defined in the same way.

(b) Using Eq. (37) and the Friedmann equation, we can find the distance to the surface of last scattering:

$$d = \frac{c\, a(t_0)}{H_0} \int_{a(t_d)}^{a(t_0)} \frac{dx}{\sqrt{\Omega_\Lambda\, x^4 + \Omega_k\, x^2 + \Omega_M\, x + \Omega_R}} \,. \tag{38}$$

Since $a(t_0)T_0 = a(t_d)T_d$, where $T_d \sim 3000$ K, $T_0 \sim 2.7$K, and $a(t_0) = 1$, we find $a(t_d) \sim 10^{-3}$. Moreover, since $1 + z_d = a(t_0)/a(t_d)$, we find that the surface of last scattering is located at redshift $z_d \sim 1110$. The current (Particle Data Group edition 2010 (PDG - 2010)) values of various contributions to the effective energy density of the Universe are

$$\Omega_M \approx 0.26\,, \qquad \Omega_\Lambda \approx 0.74\,, \qquad \Omega_R \approx 4.8 \cdot 10^{-5}\,, \qquad \Omega_k \sim 0\,. \tag{39}$$

Using these values, the integral in (38) can be computed numerically:

$$I(\Omega_M, \Omega_\Lambda, \Omega_R) = \int_{10^{-3}}^{1} \frac{dx}{\sqrt{\Omega_\Lambda\, x^4 + \Omega_k\, x^2 + \Omega_M\, x + \Omega_R}} \approx 3.30\,. \tag{40}$$

Note that replacing the lower limit of integration with zero does not influence the result appreciably:

$$I_0(\Omega_M, \Omega_\Lambda, \Omega_R) = \int_{0}^{1} \frac{dx}{\sqrt{\Omega_\Lambda\, x^4 + \Omega_k\, x^2 + \Omega_M\, x + \Omega_R}} \approx 3.34\,. \tag{41}$$

The distance to the surface of last scattering is

$$d = \frac{c}{H_0}\, I(\Omega_M, \Omega_\Lambda, \Omega_R) \approx 1.28 \cdot 10^{26}\, \text{m} \times I(\Omega_M, \Omega_\Lambda, \Omega_R) \approx 13.6\, \text{Gpc} \tag{42}$$

which is essentially the same as the horizon distance, $d_H = c\, I_0(\Omega_M, \Omega_\Lambda, \Omega_R)/H_0 \approx 13.8$ Gpc. (We use $H_0 = 100\, h$ km/s Mpc with $h \approx 0.72$ as quoted in PDG-2010.)

From the Friedmann equation, the age of the Universe can be found to be

$$T = \frac{1}{H_0} \int_{0}^{1} \frac{x\, dx}{\sqrt{\Omega_\Lambda\, x^4 + \Omega_k\, x^2 + \Omega_M\, x + \Omega_R}} \approx 13.6\, \text{Gyr}\,.$$

Estimating d as $d \sim cT$ gives

$$d \sim \frac{c}{H_0} \int_0^1 \frac{x\,dx}{\sqrt{\Omega_\Lambda\,x^4 + \Omega_k\,x^2 + \Omega_M\,x + \Omega_R}} \approx 4220\,\mathrm{Mpc} \tag{43}$$

which is more than three times smaller than the actual value of ~ 13.6 Gpc found in (42). Mathematically, this is due to the extra power of x in (43) which makes the value of the integral smaller in the interval $[0, 1]$. Physically, such an estimate ignores the expansion of the Universe in the expression for a photon's geodesic (which leads to integrating dt instead of $dt/a(t)$) and of course is bound to underestimate the true value of the distance.

(c) For matter, we have $n_{e,0}a_0^3 \sim n_e a^3$, and therefore

$$n_e \sim n_{e,0}\,\frac{a_0^3}{a^3} \sim n_{e,0}\ \cdot 10^{18} \sim 2 \cdot 10^{17}\,\mathrm{m}^{-3}\,.$$

Before decoupling, the temperature of the electrons was the same as the temperature of photons, since they were in thermal equilibrium. For radiation, $T_0\,a_0 \sim T\,a$, so

$$T \sim T_0\,\frac{a_0}{a} \sim 10^6\,T_0 \sim 2.7 \cdot 10^6\,\mathrm{K} \sim 233\,\mathrm{eV}/k_B\,,$$

since $1\,\mathrm{eV} \approx 11604\,\mathrm{K}\,k_B$. The electrons are non-relativistic since $233\,\mathrm{eV} \ll m_e c^2 \sim 0.5$ MeV.

(d) The mean free path of photons is $l_{mfp} \sim 1/n_e\sigma_e \sim 7.5 \cdot 10^{10}$ m.

(e) The corresponding mean free time is $\tau \sim l_{mfp}/c \sim 250$ s.

(f) The mean free time is much less than the age of the Universe at that time (~ 1 year). This implies the system was in thermal equilibrium.