OXFORD UNIVERSITY
PHYSICS DEPARTMENT
3RD YEAR UNDERGRADUATE COURSE

GENERAL RELATIVITY
AND
COSMOLOGY

PROBLEM SET 4
(problems 1-5)

Solution notes
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1. **Closed Universe**

(a) Show that

$$a(\eta) = C\,(1-\cos\eta) \tag{1}$$

$$t(\eta) = C\,(\eta-\sin\eta) \tag{2}$$

satisfies the closed, matter dominated FRW equation and find an expression for C in terms of H_0, Ω and the scale factor today, a_0.

(b) If the parameter η that occurs there is used as a time coordinate, show that the metric takes the form

$$ds^2 = a^2(\eta)\left[-c^2 d\eta^2 + d\chi^2 + \sin^2\chi\left(d\theta^2+\sin^2\theta d\phi^2\right)\right]\,. \tag{3}$$

Solution:

(a) The FRW equation is

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}\,. \tag{4}$$

The value of ρ such that $k=0$ in the absence of a cosmological constant is known as the *critical density* ρ_c. The critical density is time-dependent: $\rho_c(t) = 3H^2/8\pi G$.

In the matter-dominated Universe, $\rho=\rho_M$, with $\rho_M a^3 = \rho_{M,0}a_0^3$. Introducing the cosmological parameters

$$\Omega_M = \frac{8\pi G\rho_{M,0}}{3H_0^2}\,, \qquad \Omega_k = -\frac{kc^2}{H_0^2}\,, \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}\,, \tag{5}$$

where $H_0 = H(t_0)$, the FRW equation can be re-written as

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_M a_0^3}{a^3} - \frac{\Omega_k}{a^2} + \Omega_\Lambda\right)\,. \tag{6}$$

In our case,

$$da = C\sin\eta d\eta \tag{7}$$

$$dt = C\,(1-\cos\eta)\,d\eta \tag{8}$$

and therefore $\dot a = da/dt = C\sin\eta/a$. Therefore,

$$\left(\frac{\dot a}{a}\right)^2 = \frac{C^2\sin^2\eta}{a^4}\,. \tag{9}$$

But $\cos\eta = 1 - a/C$, and thus $\sin^2\eta = 2a/C - a^2/C^2$. We find therefore

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2C}{a^3} - \frac{1}{a^2}\,. \qquad (10)$$

Comparing this to the FRW equation (4), we find $k = 1$ (closed Universe), $\Lambda = 0$, $C = 4\pi G\rho_{M,0}/3$. A comparison to the FRW equation in the form (6) gives $C = \Omega_M a_0^3 H_0^2/2$.

(b) Observations suggest that the Universe is spatially homogeneous and isotropic on a large scale. Assuming this is true for the whole Universe (this is known as the "cosmological principle" hypothesis), one finds (see e.g. S. Weinberg, "Gravitation and Cosmology") that the metric of the Universe must have the RW form (i.e. be a metric whose hypersurfaces of constant time are maximally symmetric spaces in three dimensions):

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \qquad (11)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and k determines the scalar curvature of the three-dimensional space: $R = 6k$ (the parameter k can be positive, negative or zero, $k = 0$ corresponds to flat three-dimensional space). If the radial coordinate r has the dimension of length, then $k \sim 1/L^2$ (since kr^2 in the metric should be dimensionless). Thus k is inversely proportional to the square of the scale of the corresponding space. In the RW metric one can rescale variables $k \to \lambda^2 k$, $r \to r/\lambda$, $a \to \lambda a$, so that in the new metric $k = 0, \pm 1$, the radial coordinate is dimensionless, and the scale factor $a(t)$ has the dimension of length. Another form of the RW metric is

$$ds^2 = -c^2dt^2 + a^2(t)\left(d\chi^2 + F^2(\chi)d\Omega^2\right), \qquad (12)$$

where $F(\chi) = \chi, \sin\chi, \sinh\chi$ for $k = 0, +1, -1$, respectively.

In our case, $dt = a d\eta$ and $k = 1$ which immediately implies that the corresponding RW metric is given by Eq. (3).

2. **Linearly Expanding Universe**

Consider a homogeneous, isotropic cosmological model described by the line element

$$ds^2 = -c^2dt^2 + \left(\frac{t}{t_*}\right)\left[dx^2 + dy^2 + dz^2\right] \tag{13}$$

where t_* is a constant.

(a) Is this model open, closed or flat?

(b) Is this a matter-dominated Universe? Explain.

(c) Assuming the Friedmann equation holds for this Universe, find $\rho(t)$.

Solution:

(a) Comparing the metric (13) to the standard RW metric (11), we find $k = 0$ (i.e. the geometry is flat): $dx^2 + dy^2 + dz^2$ is written in spherical coordinates as $dr^2 + r^2d\Omega^2$.

(b) The scale factor is $a(t) = \sqrt{t/t_*}$. Thus $\dot{a}/a = 1/2t$, $H^2 \sim 1/t^2 \sim 1/a^4$. Therefore, the model corresponds to a radiation-dominated rather than a matter-dominated Universe.

(c) Since $(\frac{\dot{a}}{a})^2 = \frac{1}{4t^2} = \frac{1}{4t_*^2a^4}$, from the FRW equation we find $\rho = 3/32\pi G t_*^2 a^4$.

3. **Hubble parameter**

Assume that the Universe is dust-dominated. Take $H_0 = 100\, km\, s^{-1}\, Mpc^{-1}$.

(a) Give a rough estimate of the age of the Universe.

(b) How far can light have traveled in this time?

(c) The microwave background radiation has been traveling towards us uninterrupted since decoupling, when the Universe was 1/1000 of its current size. Compute the value of the Hubble parameter H at the time of decoupling.

(d) How far could light have traveled in the time up to decoupling (assume the Universe was dominated by radiation until then)?

(e) Between decoupling and the present, the distance that light traveled up to the time of decoupling has been stretched by the subsequent expansion. What would its physical size be today?

(f) Assuming that the distance to the last-scattering surface is given by part (b) of this question, what angle is subtended by the distance light could have traveled before decoupling?

(g) What is the physical significance of this value?

Solution:

(a) For the dust-dominated Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2 a_0^3}{a^3}\,. \tag{14}$$

With the normalization choice $a_0 = 1$, the relevant ODE is $\dot{a} = H_0/\sqrt{a}$. The solution is given by

$$t = C + \frac{1}{H_0}\int_0^a \sqrt{x}dx\,, \tag{15}$$

where the integration constant C is set to zero by the initial condition $a(0) = 0$. The age of the Universe is given by

$$t_0 = \frac{1}{H_0}\int_0^1 \sqrt{x}dx = \frac{2}{3H_0}\,. \tag{16}$$

Note that 1 Mpc = 3.086 $\times\, 10^{22}$ m, and thus $t_0 \approx 6.52 \times 10^9$ years. Thus *in this model* the age of the Universe is 6.52 Gyr (this is shorter than the age of some stars in the Universe, and thus the model cannot be completely correct). (Note that the current standard cosmological model gives $t_0 = 13.69 \pm 0.13$ Gyr.)

(b) The light emitted at $r = r_1$ at time $t = t_1$ and received at the origin $r = 0$ at $t = t_0$ (now) travels along the radial null geodesic given by the equation $ds^2 = 0$ or $cdt = -a(t)dr/\sqrt{1-kr^2}$ (the minus sign reflects the direction of propagation: $dr/dt < 0$ in our coordinate system). We have

$$c\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}\,. \tag{17}$$

The proper distance (determined at time $t = t_0$) from the origin $r = 0$ to a comoving object at $r = r_1$ is given by

$$d(r_1, t_0) = a(t_0)\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = a(t_0)c\int_{t_1}^{t_0} \frac{dt}{a(t)}\,. \tag{18}$$

For the light emitted at $t_1 = 0$, t_0 is the age of the Universe, and so $a(t_1) = 0$, $a(t_0) = 1$. Using the FRW equation for a matter-dominated Universe, $\dot{a} = H_0/\sqrt{a}$, we find

$$d(t_0) = ca(t_0)\int_0^{t_0} \frac{dt}{a(t)} = \frac{c}{H_0}\int_0^1 \frac{da}{\sqrt{a}} = \frac{2}{H_0}c = 3ct_0\,. \tag{19}$$

With $H_0 = 100\,km\,s^{-1}\,Mpc^{-1}$ one finds $d(t_0) = 6000$ Mpc. The distance $d(t_0)$ is known as the "particle horizon".

(c) The Hubble parameter at the time $t = t_d$ of the decoupling is $H(t_d) = \dot{a}(t_d)/a(t_d)$, where $a(t_d) = a(t_0)/1000$. Using the FRW equation for the assumed matter-dominated model, $\dot{a} = H_0/\sqrt{a}$, we find $\dot{a}(t_d) = H_0\sqrt{1000}$. Thus, $H(t_d) = H_0 1000\sqrt{1000} \approx 3.16 \times 10^6\,km\,s^{-1}\,Mpc^{-1}$.

(d) Assuming the Universe was dominated by radiation until the time of decoupling $t = t_d$ (this means the FRW equation gives $a(t) = a(t_d)\sqrt{t/t_d}$ for the time dependence of the scale factor), the distance the light could have traveled during the time from $t = 0$ and $t = t_d$ is given by

$$d(t_d) = ca(t_d)\int_0^{t_d} \frac{dt}{a(t)} = c\sqrt{t_d}\int_0^{t_d} \frac{dt}{\sqrt{t}} = 2ct_d\,. \tag{20}$$

During the later matter-dominated epoch, the scale factor evolves as $a(t) \sim t^{2/3}$, so

$$\left(\frac{t_d}{t_0}\right)^{2/3} = \frac{a(t_d)}{a(t_0)} = \frac{1}{1000}\,.$$

and $t_d = t_0(1000)^{-3/2}$. The distance is then $d(t_d) = 2ct_d = \frac{2}{3}(1000)^{-3/2}3ct_0 = \frac{2}{3}(1000)^{-3/2}d(t_0) \approx$ 0.126 Mpc.

(e) Since $d(t_d) = a(t_d)\, r$ and $d(t_{d,0}) = a(t_0)\, r$, we have $d(t_{d,0}) = \frac{a(t_0)}{a(t_d)}d(t_d) = 1000\, d(t_d) \approx$ 126 Mpc.

(f) The corresponding angle is $\theta \sim d(t_{d,0})/d(t_0) = 126/6000 \approx 0.021$ rad $\sim 1.2°$.

(e) The significance of this value is in the fact that in this model the distance traveled by light at the decoupling as seen now (i.e. $d(t_{d,0})$) is much smaller than the size of the observable Universe. The regions that could be in a causal contact with each other at the time of decoupling thus occupy a small patch (with the angular size of about one degree) on a present day sky. This poses a difficulty for the "old" cosmological models, since the observed CMB is highly uniform throughout the sky. This problem is resolved in the "new" inflationary cosmological models.

4. **Conformal time**

We can define conformal time, η, in terms of physical time, t, through $dt = ad\eta$, where a is the scale factor which is a function of t or η.

(a) Show that $\eta \propto a^{1/2}$ in a matter-dominated Universe, and $\eta \propto a$ in one dominated by radiation

(b) Consider a Universe with only matter and radiation, with equality at a_{eq}. Show that

$$\eta = \frac{2}{\sqrt{\Omega_M H_0^2}} \left(\sqrt{a + a_{eq}} - \sqrt{a_{eq}}\right)$$

(c) What is the conformal time today? And at recombination?

Solution:

(a) For the matter-dominated Universe, the solution of the FRW equation is $a(t) = (t/t_0)^{2/3}$. Direct integration gives $\eta = 3t_0^{2/3}\, t^{1/3}$. Expressing this through $a(t)$, we get $\eta = 3t_0\sqrt{a}$. Similarly, for the radiation-dominated Universe, where $a(t) = \sqrt{t/t_0}$, we find $\eta = 2\, t_0\, a$.

(b) Including matter and radiation only, the FRW equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_M}{a^3} + \frac{\Omega_R}{a^4}\right) = H_0^2\, \Omega_M \left(\frac{1}{a^3} + \frac{a_{eq}}{a^4}\right). \tag{21}$$

Since $dt = ad\eta$, we find

$$d\eta = \frac{da}{H_0\sqrt{\Omega_M(a + a_{eq})}}. \tag{22}$$

Integrating with the initial condition $a(0) = 0$ we obtain

$$\eta = \frac{2}{H_0\sqrt{\Omega_M}} \left(\sqrt{a + a_{eq}} - \sqrt{a_{eq}}\right). \tag{23}$$

(c) The conformal time today, in a matter-dominated Universe, is $\eta(t_0) = 3t_0\sqrt{a(t_0)} = 3t_0$, where $t_0 = 2/3H_0$. Thus, $\eta(t_0) = 2/H_0 \approx 20$ Gyrs. At recombination, $\eta(t_d) = 3t_0\sqrt{a(t_d)}$. Since $a(t_d) = 10^{-3}a(t_0)$, $\eta(t_d) = 3t_0 10^{-3/2} \approx 0.632$ Gyrs.

5. **Contributions to the dynamics of the Universe**

(a) Suppose the Universe contains four different contributions to the Friedmann equation, namely dust, radiation, a cosmological constant and negative curvature. What is the behaviour of each as a function of the scale factor $a(t)$?

(b) Which will dominate at early times and which will dominate at late times?

Solution:

(a) The FRW equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \,. \tag{24}$$

For dust (cold matter), the energy density ρ is essentially $Mc^2/Volume$, so $\rho \sim 1/a^3(t)$. For radiation, from e.g. the Stefan-Boltzmann law we have $\rho \sim T^4 \sim 1/a^4(t)$. The curvature term scales as $1/a^2(t)$. Finally, the cosmological constant term is independent of the scale factor.

(b) Since $a(0) = 0$, at early times the dominant contribution to the right hand side of Eq. (24) comes from radiation. At late times, the cosmological constant term dominates (provided $\Lambda \neq 0$).