A13484W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B5: GENERAL RELATIVITY & COSMOLOGY

TRINITY TERM 2017

Saturday 17 June, 9.30 am – 11.30 am

Candidates are strongly advised to use the first 10 minutes to read the whole paper before starting writing.

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. Explain what is meant by the *covariant derivative*, ${v^\beta}_{;\gamma}$, of a vector field $v^\beta(x^\alpha)$. Given a curve $x^\gamma(\lambda)$, what does $(\mathrm{d}x^\gamma/\mathrm{d}\lambda){v^\beta}_{;\gamma}$ represent? [4]

By definition, the energy–momentum tensor of a perfect fluid has components $T^{\alpha\beta} = \rho\langle v^\alpha v^\beta\rangle$, where ρ is the local mass density of the fluid and the angle brackets denote averages over the four-velocities v^α, v^β of the swarm of particles that contribute to that density. Show that $T^{\alpha\beta} = (\rho + p/c^2)u^\alpha u^\beta + g^{\alpha\beta}p$, where $g^{\alpha\beta}$ is the metric tensor, $u^\alpha \equiv \langle v^\alpha\rangle$ is the mean four-velocity of the fluid and ρ and p are scalar functions that you should define carefully. In the non-relativistic limit, what physical principles does the equation ${T^{\alpha\beta}}_{;\alpha} = 0$ encapsulate? [5]

For the case $p = 0$, show that a consequence of requiring ${T^{\alpha\beta}}_{;\alpha} = 0$ is that fluid elements follow geodesics. [6]

For a homogeneous, isotropic spacetime having scale factor $a(t)$ and curvature constant k, the Einstein equation, $R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = -\frac{8\pi G}{c^4}T^{\alpha\beta}$, reduces to the pair of equations

$$\begin{aligned}
\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a, \\
\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 &= \frac{8\pi G}{3}\rho a^2 - c^2 k.
\end{aligned}$$

Assume now that this universe has two non-interacting components that contribute to $T^{\alpha\beta}$: non-relativistic matter and radiation. Explain why the equation of state of each component can be approximated by an expression of the form $p_i = w_i\rho_i c^2$ (no summation), stating the approximate value of w_i for each (does p_i contribute significantly to non-relativistic matter?). Show that each ρ_i evolves according to

$$\frac{1}{\rho_i}\frac{\mathrm{d}\rho_i}{\mathrm{d}t} + 3(1 + w_i)\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = 0.$$

[5]

At time t_1 this universe has nonzero matter and radiation densities and is expanding. Show that there was some time $t_0 < t_1$ at which $a(t_0) = 0$. [5]

2. A light ray moves in the vicinity of a massive body, around which the line element is given in terms of coordinates (ct, r, θ, ϕ) by

$$-c^2 \mathrm{d}\tau^2 = \mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2.$$

Starting from the fact that photons follow geodesics, explain how to obtain the equations that describe the path followed by the light ray in terms of an affine parameter λ. Why may we assume that the ray is confined to the plane $\theta = \pi/2$? Show that the t and ϕ coordinates for such a ray satisfy the equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(1 - \frac{2GM}{c^2 r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right] &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right] &= 0, \end{aligned}$$

and hence identify two constants of motion. [9]

What other invariant is associated with photon geodesics? Use this other invariant to write down an expression that must be satisfied by $\mathrm{d}r/\mathrm{d}\lambda$. Show that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2\left(1 - \frac{2GMu}{c^2}\right) = C,$$

where $u = 1/r$ and C is a constant. [5]

This metric describes the spacetime outside a spherically symmetric neutron star that extends to coordinate radius $R = 4GM/c^2$. An observer hovers at fixed coordinate radius $r_0 > R$. Show that the angular diameter, α, of the neutron star measured by the observer is given by

$$\cos\tfrac{1}{2}\alpha = \left[1 + r_0^2\left(1 - \frac{2GM}{c^2 r_0}\right)\frac{1}{A^2}\right]^{-1/2},$$

where A is the value of $\mathrm{d}r/d\phi$ at $r = r_0$ for a ray that just skims the surface of the neutron star. [Hint: how is the angle between two vectors defined in terms of the metric?] Obtain an expression for A and hence simplify your equation for α. [11]

3. An almost-flat spacetime has metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$

in coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric and the perturbation $h_{\alpha\beta}$ is small, with $|h_{\alpha\beta}| \ll 1$.

Let $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, where $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$. The Einstein field equation becomes

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}_{\mu\nu} - \partial_\nu\partial_\lambda\bar{h}^\lambda_\mu - \partial_\mu\partial_\lambda\bar{h}^\lambda_\nu + \eta_{\mu\nu}\partial_\lambda\partial_\rho\bar{h}^{\lambda\rho} = 0$$

in the absence of matter and omitting terms of order $|\bar{h}_{\alpha\beta}|^2$. Consider a change of coordinates from x^α to $x'^\alpha = x^\alpha + \xi^\alpha$, in which the functions ξ^α are comparable in size to the $h_{\alpha\beta}$. What are the components $h'_{\alpha\beta}$ of the tensor $h_{\alpha\beta}$ after this transformation? Show that the ξ^α can be chosen to make $\partial_\rho\bar{h}'^{\lambda\rho} = 0$ and that, with this choice, the field equation becomes the wave equation,

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}'_{\mu\nu} = 0.$$

The partial derivatives in this equation are with respect to the "old" coordinates x^α, not the "new" x'^α. Comment on the significance of this. [10]

A solution to this wave equation is $\bar{h}'^{\alpha\beta} = A^{\alpha\beta}\exp[\mathrm{i}k_\gamma x^\gamma]$. State a pair of simple conditions that must be satisfied by k^γ and $A^{\alpha\beta}$. [2]

A set of test particles is arranged into a circular ring $x^2+y^2 = a^2$. They experience a wave $\bar{h}'_{\alpha\beta} = A_{\alpha\beta}\exp[\mathrm{i}k(z-ct)]$ with polarization tensor $A_{\alpha\beta} = \mathrm{diag}(0, A, -A, 0)$. Using the geodesic equations, show that the particles' (x, y, z) coordinates are unaffected by the wave. Does this mean that the particles do not move? Explain what someone sitting at the centre of the ring would observe. [8]

Two black holes, each of mass $30\,M_\odot$, are on a circular orbit of radius 10 km around their common centre of mass. Without detailed calculation, estimate the Schwarzschild radius of each black hole and hence, or otherwise, construct an order-of-magnitude estimate of $\bar{h}'_{11}$ near the black holes. An observer 500 Mpc away from the black holes sets up a ring of test particles of radius 4 km. Estimate the amplitude of the displacement of the ring caused by the black holes. [5]

$[M_\odot \sim 10^{30}$ kg; 1 pc $\sim 10^{16.5}$ m.]

4. Assume that the present number density of electrons in the Universe is the same as that of protons, namely about $0.2\,\mathrm{m}^{-3}$. Consider a time when the Universe was 10^4 years old and the scale factor was one millionth of its present value. Estimate the number density of electrons at that time and comment on whether the electrons would be relativistic or non-relativistic then. Explain your reasoning. [8]

Using the Thompson scattering cross-section ($\sigma_\mathrm{e} = 6.7 \times 10^{-29}\mathrm{m}^2$) determine the mean free path for photons when the scale factor was one millionth of its present value. [6]

From the mean free path, calculate the typical time between interactions among the photons and electrons. Compare the interaction time with the age of the Universe at that time. What is the physical significance of this comparison? [5]

By redshift $z \sim 10^3$ the Universe had cooled sufficiently that photons could stream freely. Why is there a critical temperature below which photons can stream freely? Estimate the particle horizon distance at redshift $z \sim 10^3$ and the angular scale that that corresponds to at the present. Why does this pose a puzzle in the context of the hot big bang model? [6]