B5: GENERAL RELATIVITY & COSMOLOGY Saturday 18 June20169.30 am11.30 am

1. A satellite orbits a Schwarzschild black hole having metric

$$-c^2\mathrm{d}\tau^2 = -\left(1-\frac{2GM}{c^2r}\right)c^2\,\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2.$$

Explain why we may restrict our attention to motion confined to the equatorial plane $\theta = \frac{\pi}{2}$ to understand possible orbits of the satellite. Using a variational method to extremise the proper time interval $\mathrm{d}\tau$ along the satellite's orbit, or otherwise, find expressions for $\mathrm{d}t/\mathrm{d}\tau$ and $\mathrm{d}\phi/\mathrm{d}\tau$. Identify two independent constants of motion. [4]

Obtain an equation for $\mathrm{d}r/\mathrm{d}\tau$. From this show that $u = 1/r$ satisfies the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = A + Bu^2,$$

where A and B are constants that you should identify. [9]

The satellite is launched from a space station that is maintained at constant $(r, \theta, \phi) = (R, \frac{\pi}{2}, 0)$. As measured by an astronaut on the station, with what speed (that is, proper distance moved per unit proper time), v_{circ}, should the satellite be launched so that it follows a circular orbit around the black hole? [You may assume that the satellite is launched in the azimuthal direction, so that $\mathrm{d}r/\mathrm{d}\tau = \mathrm{d}\theta/\mathrm{d}\tau = 0$, but $\mathrm{d}\phi/\mathrm{d}\tau \neq 0$.] How long would the astronaut have to wait before a satellite placed on this orbit collides with the station? [8]

In an attempt to delay the collision, the astronaut decides to launch the satellite in the same direction, but with speed $v_{\mathrm{circ}} + \Delta v$, where $|\Delta v| \ll v_{\mathrm{circ}}$. Sketch the motion of the satellite, indicating the position of station and the circular orbit $r = R$. Is the station safe from a collision for all time? [4]

2. The energy–momentum tensor for a perfect fluid has components

$$T^{ab} = \left(\rho + \frac{p}{c^2}\right) u^a u^b + g^{ab} p, \qquad (1)$$

where ρ and p are the rest-frame density and pressure, respectively, u^a is the 4-velocity and g is the metric. The Einstein field equations relate T^{ab} to the curvature, quantified by the Einstein tensor G^{ab}. Starting from the assumption that G^{ab} should be proportional to T^{ab}, outline the reasoning that leads to the linear combination $G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R$, where R^{ab} is the Ricci tensor and R is the Ricci scalar. [6]

In coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$ a spacetime has metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)} \mathrm{d}(ct)^2 + \mathrm{e}^{2\Lambda(r)} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2,$$

for which the only nonzero components of the Einstein tensor turn out to be

$$\begin{aligned}
G_{00} &= -\frac{1}{r^2} \mathrm{e}^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} \left[r(1 - \mathrm{e}^{-2\Lambda}) \right], \\
G_{11} &= \frac{1}{r^2} \mathrm{e}^{2\Lambda} - \frac{1}{r^2} - \frac{2}{r} \Phi', \\
G_{22} &= -r^2 \mathrm{e}^{-2\Lambda} \left[\Phi'' + \Phi'(\Phi' - \Lambda') + \frac{1}{r}(\Phi' - \Lambda') \right], \\
G_{33} &= G_{22} \sin^2\theta,
\end{aligned}$$

where primes denote derivatives with respect to r. Assuming a perfect fluid having an energy–momentum tensor of the form (**??**), explain why the matter distribution for this Einstein tensor has velocity components $u^1 = u^2 = u^3 = 0$. Hence write down expressions for u^0 and u_0. [3]

Use the 00 component of the field equation $G_{ab} = -\frac{8\pi G}{c^4} T_{ab}$ to obtain an expression for $\Lambda(r)$ in terms of the function

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 \, \mathrm{d}r',$$

and, from the 11 component, show that $\Phi(r)$ satisfies

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM(r) + \frac{4\pi G}{c^2} p r^3}{c^2 r^2 \left(1 - \frac{2GM(r)}{c^2 r}\right)}.$$

Assuming the energy–momentum conservation relation,

$$(\rho c^2 + p) \frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r},$$

use these results to obtain a differential equation for the pressure p in terms of $\rho(r)$ and $M(r)$. [5]

Consider a star of radius R and constant density ρ. By integrating your expression for $\mathrm{d}p/\mathrm{d}r$ subject to appropriate boundary conditions, show that the pressure at the centre of the star is given by

$$p_0 = \rho c^2 \frac{1 - \left(1 - \frac{2GM}{c^2 R}\right)^{1/2}}{3\left(1 - \frac{2GM}{c^2 R}\right)^{1/2} - 1}$$

and find an upper bound to the star's mass given its radius R. An astronomer claims to detect emission from the surface of a neutron star in which spectral lines have gravitational redshift $z = 3.1$. Comment on the plausibility of this claim. [11]

[You may assume that $\int \frac{\mathrm{d}x}{(A+3x)(A+x)} = \frac{3}{2A} \int \frac{\mathrm{d}x}{A+3x} - \frac{1}{2A} \int \frac{\mathrm{d}x}{A+x}$.]

3. In 4-dimensional (w, x, y, z) space, a 3-sphere of radius A is the 3-dimensional subvolume $w^2 + x^2 + y^2 + z^2 = A^2$. By expressing (w, x, y, z) in terms of suitable trigonometric functions, or otherwise, show that the line element $\mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ for points confined to the 3-sphere can be written as

$$\mathrm{d}s^2 = \frac{A^2}{A^2 - \bar{r}^2}\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2 \sin^2\theta \mathrm{d}\phi^2.$$

State the corresponding line elements for flat space and for hyperbolic space. Write down an expression for the surface area of the two-dimensional surface $\bar{r} = \bar{R}$ in each case, where $\bar{R}$ is a constant. [6]

Hence or otherwise explain how the Friedmann–Robertson–Walker (FRW) metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right],$$

where $k = 0$, $+1$ or -1, is consistent with the principles of homogeneity and isotropy. [4]

In a universe having an FRW metric, light emitted at time t_{E} from a galaxy having constant radial coordinate $r = R$ is received at time t_0 by an observer at $r = 0$. Derive an expression for the redshift z of the observed light in terms of the scale factors $a(t_0)$ and $a(t_{\mathrm{E}})$. [5]

The galaxy has proper luminosity L spread uniformly across a disc of proper diameter d. Viewed from $r = 0$, what is the angular diameter $\Delta\theta$ of the galaxy? How does the surface brightness of the galaxy depend on its redshift? Use your result to comment on the effectiveness or otherwise of increasing telescope diameter for studying high-redshift galaxies. [10]

[You may assume that the galaxy is observed perfectly face on. Surface brightness is energy received per unit time per unit area per unit solid angle.]

4. Compare and contrast the physical processes and conditions involved during *nucleosynthesis* and during *decoupling*, putting your discussion of these in the context of the thermal history of the Universe. [10]

What is the spectral shape and current temperature of the cosmic microwave background (CMB)? The energy density in the CMB at present is $\sim 4 \times 10^{-14}\,\mathrm{J\,m^{-3}}$. By considering the energy of a typical CMB photon derive an estimate of the number density of photons at the current time. Calculate the approximate number of photons there are for every baryon at the current epoch. Hence estimate the temperature at which decoupling took place, identifying the most significant approximations you have made. Are the electrons relativistic during decoupling? [10]

When the universe is matter dominated, how does the scale factor $a(t)$ vary with cosmic time t? Hence or otherwise show how t and the temperature T are related. Now estimate of the age of the Universe at decoupling. [5]