B5: GENERAL RELATIVITY AND COSMOLOGYSaturday, 20 June20159.30 am11.30 am

1. Show that if you extremize the action for a test particle $S = -\int \mathrm{d}\lambda\, g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$ (where λ is the affine parameter and $g_{\alpha\beta}$ the metric), you will obtain the correct expressions for the connection coefficients for a general metric. [4]

Consider the "global rain" metric,

$$\mathrm{d}s^2 = -c^2\left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{d}\bar{t}^2 + 2\sqrt{\frac{r_\mathrm{S}}{r}}c\,\mathrm{d}\bar{t}\,\mathrm{d}r + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\ ,$$

where $(\bar{t}, r, \theta, \phi)$ are space-time coordinates and r_S is a constant. Show that the non-zero components of the inverse metric are

$$g^{\bar{t}\bar{t}} = -\frac{1}{c^2}\ ,\ g^{\bar{t}r} = \frac{1}{c}\sqrt{\frac{r_\mathrm{S}}{r}}\ ,\ g^{rr} = \left(1-\frac{r_\mathrm{S}}{r}\right)\ ,\ g^{\theta\theta} = \frac{1}{r^2}\ ,\ g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}\ .$$

Using the definition of the connection coefficients (or otherwise), show that the radial geodesic equation is

$$\begin{aligned}\ddot{r} - \frac{1}{2r_\mathrm{S}}\left(\frac{r_\mathrm{S}}{r}\right)^2\dot{r}^2 &+ \frac{c^2}{2r_\mathrm{S}}\left(1-\frac{r_\mathrm{S}}{r}\right)\left(\frac{r_\mathrm{S}}{r}\right)^2\dot{\bar{t}}^2 - \frac{c}{r_\mathrm{S}}\left(\frac{r_\mathrm{S}}{r}\right)^{\frac{5}{2}}\dot{r}\dot{\bar{t}} \\ &- \left(1-\frac{r_\mathrm{S}}{r}\right)r\dot{\theta}^2 - \left(1-\frac{r_\mathrm{S}}{r}\right)r\sin^2\theta\,\dot{\phi}^2 = 0\ .\end{aligned}$$

[10]

Compare what happens to this metric at $r = r_\mathrm{S}$ with what happens to the Schwarzschild metric in the usual coordinates. By looking only at light-like radial geodesics, explain why, if $r < r_\mathrm{S}$, photons always fall inwards [hint: show that $\mathrm{d}r/\mathrm{d}\bar{t} < 0$]. [6]

Consider a change of coordinates such that $\bar{t} = \bar{t}(r,t)$ where

$$\begin{aligned}\frac{\partial\bar{t}}{\partial t} &= 1\ , \\ \frac{\partial\bar{t}}{\partial r} &= \sqrt{\frac{r_\mathrm{S}}{r}}\left(1-\frac{r_\mathrm{S}}{r}\right)^{-1}\frac{1}{c}\ .\end{aligned}$$

Rewrite the global rain metric in terms of t and r, and show that it is equivalent to the Schwarzschild metric. [5]

2. Consider a conformally flat space-time with metric given in Cartesian coordinates by

$$\mathrm{d}s^2 = \mathrm{e}^{\frac{2\varphi}{c^2}}\, \eta_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta \ ,$$

where φ is a scalar function of space-time coordinates and $\eta_{\alpha\beta}$ is the Minkowski metric. Show that the connection coefficients take the form

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{c^2} \left[\partial_\beta \varphi\, \delta^\mu{}_\alpha + \partial_\alpha \varphi\, \delta^\mu{}_\beta - \partial^\mu \varphi\, \eta_{\alpha\beta}\right] \ ,$$

where $\partial^\mu = \eta^{\mu\nu}\, \partial_\nu$. [7]

The Ricci tensor is given by

$$R_{\nu\beta} \equiv \partial_\mu \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\mu\nu} + \Gamma^\mu{}_{\mu\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\mu} \ .$$

Assume that $\varphi/c^2 \ll 1$, and show that the Einstein tensor takes the form

$$G_{\alpha\beta} = \frac{2}{c^2} \left(\partial_\mu \partial^\mu \varphi\, \eta_{\alpha\beta} - \partial_\alpha \partial_\beta \varphi\right) \ .$$ [6]

Rewrite the metric into spherical coordinates, (t, r, θ, ϕ), and assuming that φ is a function of r only, show that for an equatorial orbit the geodesic equations for t and ϕ take the form

$$\mathrm{e}^{\frac{2\varphi}{c^2}}\, \dot{t} = d \ \text{ and } \ \mathrm{e}^{\frac{2\varphi}{c^2}}\, r^2 \dot{\phi} = \ell \ ,$$

where $\dot{f} \equiv \mathrm{d}f/\mathrm{d}\lambda$ for any function $f(\lambda)$ (λ is the affine parameter), and d and ℓ are integration constants. Write down an expression for the null condition for photons in this metric. [5]

Assume that $\varphi = -GM/r$, where M is a constant and r is the distance from the origin. Show that there is no light deflection around $r = 0$. [7]

3. The Riemann tensor is defined to be

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_{\alpha}\Gamma^{\mu}{}_{\beta\nu} - \partial_{\beta}\Gamma^{\mu}{}_{\alpha\nu} + \Gamma^{\mu}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\nu\beta} - \Gamma^{\mu}{}_{\epsilon\beta}\Gamma^{\epsilon}{}_{\nu\alpha} \ ,$$

where $\Gamma^{\mu}{}_{\beta\nu}$ is the connection coefficient tensor. Show that

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})V^{\mu} = R^{\mu}{}_{\nu\alpha\beta}V^{\nu} \tag{1}$$

for any contravariant vector V^{μ}. [7]

A "Killing" vector, U^{μ}, satisfies the condition $\nabla_{\mu}U_{\nu} + \nabla_{\nu}U_{\mu} = 0$. We define the commutator between two vectors to be

$$W^{\mu} \equiv [U, V]^{\mu} = U^{\nu}\nabla_{\nu}V^{\mu} - V^{\nu}\nabla_{\nu}U^{\mu} \ .$$

Using equation 1 (and the symmetry of the Riemann tensor, $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$), show that the commutator of two Killing vectors is also a Killing vector. [8]

Consider a tensor $T_{\mu\nu} = \nabla_{\mu}\nabla_{\nu}\phi - g_{\mu\nu}\nabla^{\sigma}\nabla_{\sigma}\,\phi$, where ϕ is a scalar function of the space-time coordinates. Show that

$$\nabla^{\mu}T_{\mu\nu} = R_{\nu}{}^{\sigma}\,\nabla_{\sigma}\phi \ .$$ [6]

Assume now that $\nabla^{\mu}T_{\mu\nu} = 0$. If k^{ν} is a Killing vector, show that

$$\nabla^{\mu}(T_{\mu\nu}k^{\nu}) = 0 \ .$$ [4]

4. Consider an inflationary universe that undergoes three phases of expansion: an initial inflationary phase in which the pressure, P, and the density, ρ, satisfy $P = -\rho c^2$ up until the scale factor $a = a_1$, followed by a radiation phase in which $P = \frac{1}{3}\rho c^2$ up until the scale factor $a = a_2$, followed by a matter phase in which $P = 0$ up until the scale factor $a = 1$. Find an expression for the Hubble rate and deceleration rate, as a function of the scale factor a, in each of these regimes (neglecting all other non-dominant components of the energy density). Solve the Friedman-Robertson-Walker (FRW) equation in each one of the three phases. [7]

Explain why the expansion rate in such a universe is slower when $a < a_1$ than in a universe where there is *no* initial inflationary phase (i.e. a universe where there is no period of inflation for $a < a_1$ and that has exactly the same expansion rate as a function of the scale factor for $a > a_1$). [Hint: assume continuity in the Hubble rate at $a = a_1$ for the inflationary universe]. [5]

Assume that the initial scale factor of the universe is $a_{\rm in}$. Find an expression for the age of the universe in terms of an integral over the Hubble rate. Taking the limit $a_{\rm in} \to 0$, show that the inflationary universe must be older than the non-inflationary universe. [6]

Find an expression for the particle horizon in each one of the phases of the inflationary universe. Comparing the physical size of a length scale of fixed comoving size with the particle horizon, explain the qualitative difference between what happens for $a < a_1$ and $a > a_1$. [7]