Cosmology

- General principles:

1) The observable Universe is homogeneous and isotropic at sufficiently large scales

– what scales?

– compare with gas and l_{mfp} $\Rightarrow$
hom. and isotr. for $L \gg l_{mfp}$
but not for $L \sim l_{mfp}$

– this follows from observations; may change tomorrow but only as a small correction (still important)

– limited info about the Universe as a whole (old problem of G. Bruno – is our patch unique or typical?)

2) From this conjecture (yes, backed by evidence) – known as the Cosmolog. Principle – it follows (see Weinberg's „Grav. and Cosmology") that

the metric of the Universe must be of the FRW form

$$ds^2 = -c^2 dt^2 + R^2(t)\left[\frac{dr^2}{1 - r^2/a^2} + r^2 d\Omega^2\right],$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$,

$a^2 = 0$, $a^2 < 0$ or $a^2 > 0$.

Here, $[r] = L$. One can rescale $r \to \bar{r}$, $\bar{r}^2 \equiv r^2/|a^2|$, $[\bar{r}] = 1$ but $[R] = L$. Then (we denote $\bar{r}$ by r again):

$$ds^2 = -c^2 dt^2 + R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right],$$

$k = 0, \pm 1$.

- Note: standard current notation for the scale factor is $a(t)$, not $R(t)$.

- So far this metric is an ansatz (compare with ansatz for spher.-symm. metric) - need to solve e.o.m. (i.e.

Einstein's eq) to get dynamical information => determine $R(t)$.

3) Dynamics (eom)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

$T_{\mu\nu}$ is the energy-momentum tensor of all matter / fields in the Universe, including us, poor souls. We model the „gas" of galaxies and everything else by perfect (no friction) fluid:

$$T_{\mu\nu} = P g_{\mu\nu} + (\rho c^2 + P) U_\mu U_\nu$$

with some e.o.s. $P = P(\rho)$.

=> expect 2 indep. eq. for 2 var.: $\rho(t)$ and $R(t)$.

Insert the FRW ansatz and $T_{\mu\nu}$ into eom => 3 eqs (one is not indep.)

(1) $$\dot{R}^2 + kc^2 - \frac{c^2}{3}\Lambda R^2 = \frac{8\pi G}{3}\rho R^2$$

(2) $$2R\ddot{R} + \dot{R}^2 + kc^2 - c^2\Lambda R^2 = -\frac{8\pi G}{c^2} P R^2$$

(3) $$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\frac{P}{c^2} + \rho\right) = 0$$

Eq. (3) also follows from $\nabla_\mu T^{\mu\nu} = 0$

- Exercise (optional): take the FRW ansatz metric and compute connection coeffi, Riemann tensor components etc. Show that the Kretschmann invar.

$K = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}$ is

$$K = \frac{12\left(R^2(t)\ddot{R}^2 + (k + \dot{R}^2)^2\right)}{R^4(t)}.$$

- Usually, (1) and (3) are taken as indep. equations. Friedmann eq:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

Fate of $R(t)$:

- Big Bang: $R(t_{init}) = 0$
- Big Crunch: $R(t_{finite}) = 0$
- Universe expands forever: $R(t \to \infty) \to \infty$
- Big Rip: $R(t_{finite}) \to \infty$

- EOS is assumed:

  $P = P(\rho)$, e.g. $P = \rho c^2/3$ for ultrarel. matter („radiation")

  or $P \simeq 0$ for „dust" (non-rel. matter)

Parametrised as $P = w\rho c^2$.

- $H = \dot{R}/R$ Hubble parameter

- With $P = w\rho c^2$, eq. (3) is

$$\dot{\rho} R + 3\rho\dot{R}(1+w) = 0, \text{ i.e.}$$

$$\frac{d}{dt}\left(\rho R^{3(1+w)}\right) = 0$$

$$\Rightarrow \rho = \rho_0 / R^{3(1+w)} \qquad R(t_0) = 1: \text{„now"}$$

$$\begin{cases} w = 0 \text{ („dust")}: & \rho = \rho_{0,M}/R^3 \\ w = 1/3 \text{ („rad")}: & \rho = \rho_{0,R}/R^4. \end{cases}$$

<u>Problem</u>: solve for $R(t)$, $\rho(t)$, given $P = P(\rho)$: (2 eq for 2 variables)