B2: Symmetry and Relativity

Problem Set 5

Problem 1

We need to check conditions that transf. have to satisfy to form a group.

One-dim. Lorentz boosts are of the form

$$g(\beta) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix},$$

where $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$.

A group G is a set of elements equipped with a product operation such that for $g_1, g_2 \in G$ we have $g_1 \cdot g_2 \in G$ and group axioms are satisfied

1) $\forall g_1, g_2, g_3 \in G$: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ (associativity of product)

2) Existence of identity element $e \in G$

$\exists e \in G \Rightarrow \forall g \in G \quad e \cdot g = g$ and

$g \cdot e = g$

3) Existence of inverse:

$\forall g \in G \quad \exists g^{-1} \in G$ such that

$g \cdot g^{-1} = e, \quad g^{-1} g = e$.

In our case, the product is matrix multiplication. We have

$$g_1 \cdot g_2 = \begin{pmatrix} \gamma_1 & -\beta_1 \gamma_1 \\ -\beta_1 \gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_2 & -\beta_2 \gamma_2 \\ -\beta_2 \gamma_2 & \gamma_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma_1 \gamma_2 (1 + \beta_1 \beta_2) & -\gamma_1 \gamma_2 (\beta_1 + \beta_2) \\ -\gamma_1 \gamma_2 (\beta_1 + \beta_2) & \gamma_1 \gamma_2 (1 + \beta_1 \beta_2) \end{pmatrix} =$$

$$= \begin{pmatrix} \bar{\gamma} & -\bar{\beta} \bar{\gamma} \\ -\bar{\beta} \bar{\gamma} & \bar{\gamma} \end{pmatrix}, \quad \text{where} \quad \bar{\gamma} = \frac{1 + \beta_1 \beta_2}{\sqrt{1 - \beta_1^2} \sqrt{1 - \beta_2^2}}$$

$$\bar{\beta} = (\beta_1 + \beta_2) / (1 + \beta_1 \beta_2)$$

It is easy to check that $\bar{\gamma} = \dfrac{1}{\sqrt{1-\bar{\beta}^2}}$,

so $g_1 \cdot g_2$ is again a Lorentz boost.

Axioms 1-3 are standard properties of matrices. Identity element corresponds to $\beta = 0$.

Note that Lorentz boosts can be viewed as pseudorotations preserving the metric

$$ds^2 = -c^2 dt^2 + dx^2,$$

pretty much like rotations

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

preserve Euclidean metric

$$ds^2 = dx^2 + dy^2.$$

Then

$$g = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \Rightarrow \begin{matrix} x = x(t', x') \\ t = t(t', x') \end{matrix}$$

and one can check that

$$g_1 \cdot g_2 = \begin{pmatrix} \cosh(y_1+y_2) & \sinh(y_1+y_2) \\ \sinh(y_1+y_2) & \cosh(y_1+y_2) \end{pmatrix},$$

so it is again a boost with $y = y_1 + y_2$.
The parameter y satisfies $\cosh y = \gamma$
and is called rapidity.

Problem 3

The matrix $\mathcal{D}_{132}$ representing the action $123 \to 132$ is

$$\mathcal{D}_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ Indeed,

$$\mathcal{D}_{132} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$ The other elements are:

$$\mathcal{D}_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{D}_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathcal{D}_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Finally, $\mathcal{D}_{123} = e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

A reducible rep. is the one of the form

$$g = \begin{pmatrix} R_1 & & 0 \\ & R_2 & \\ 0 & & \ddots \end{pmatrix}$$ - i.e. the one which does not mix coordinates of subspaces.

In $\mathbb{R}^3$, where $\mathcal{D}$s are acting, there are invar. vectors - they are of the form $\bar{v} = (x, x, x)$, e.g. $(1, 1, 1)$ - they are mapped into themselves by all $\mathcal{D}$s.

We can choose a basis in which

$\bar{v}' = (1, 0, 0) = S\bar{v}$.

For example, $S = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$

does this.

Then in this basis $\mathcal{D}' = S\mathcal{D}S^{-1}$.

We find

$$\mathcal{D}'_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad \mathcal{D}'_{321} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}'_{213} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{D}'_{231} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\mathcal{D}'_{312} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{D}'_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The block-diagonal form $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ is obvious.

The 2-dim irrep. is called the standard repr. of S_3 (in general: $n-1$ irrep of S_n).

Problem 4

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$R(\theta) = e^{-i\theta J_1} = \hat{\mathbb{1}} - i\theta J_1 - \frac{\theta^2}{2!} J_1^2 + \cdots$$

$$J_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Similarly, $\Lambda(\eta) = e^{-i\eta K_1}$ with

$$K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ and

$$K_1^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ is written as

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is $x'^\mu = x'^\mu(x)$: Lorentz boost.

(Note: inverse ~~boost~~ $x^\mu = x^\mu(x')$ does not have minuses in front of $\sinh\eta$.)

Boost along y:

$$K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Two infinitesimal boosts will give

$$(\mathbb{1} - i\eta_1 K_1)(\mathbb{1} - i\eta_2 K_2) =$$

$$= \begin{pmatrix} 1 & -\eta_1 & -\eta_2 & 0 \\ -\eta_1 & 1 & \eta_1\eta_2 & 0 \\ -\eta_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

– ~~boost~~ and rotation.

Problem 5

$$J_3 |m\rangle = m|m\rangle$$

$$J_\pm |m\rangle = \left[j(j+1) - m(m\pm 1)\right]^{1/2} |m\pm 1\rangle$$

$$J_\pm = J_1 \pm i J_2$$

Basis $|1\rangle$, $|0\rangle$, $|-1\rangle$:

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then: $J_1 = \frac{1}{2}(J_+ + J_-)$

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

We can verify explicitly that

$$[J_i\, J_k] = i\varepsilon_{ik\ell} J_\ell$$

Lie algebra of SO(3) (and SU(2))

$$R_3(\varphi) = e^{-i\varphi J_3} = \begin{pmatrix} e^{-i\varphi} & & \\ & 1 & \\ & & e^{i\varphi} \end{pmatrix}$$

Since $Y_{11} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi},$$

$R_3(\varphi)$ generates a rot. by φ around z-axis.

Similarly for $R_1(\varphi)$ rotating a vector in the yz plane (i.e. at $\phi = \pi/2$

$$\Rightarrow \left(i\sqrt{\frac{3}{8\pi}} \sin\theta,\ \sqrt{\frac{3}{4\pi}} \cos\theta,\ i\sqrt{\frac{3}{8\pi}} \sin\theta \right)$$

$$\Rightarrow \left(i\sqrt{\frac{3}{8\pi}} \sin(\theta+\varphi),\ \sqrt{\frac{3}{4\pi}} \cos(\theta+\varphi),\ i\sqrt{\frac{3}{8\pi}} \sin(\theta+\varphi) \right).$$

Problem 6 The Dirac eq

$$\left(i\gamma^\mu \partial_\mu + q\gamma^\mu A_\mu - m\right)\psi = 0$$

in an external $U(1)$ field

$$\psi \to \psi' = e^{iq\alpha}\psi \quad , \quad \alpha = \alpha(x).$$

$$(i\gamma^\mu \partial_\mu + q\gamma^\mu A'_\mu - m)\psi' =$$

$$= e^{iq\alpha}(i\gamma^\mu \partial_\mu + q\gamma^\mu A'_\mu - m)\psi -$$

$$-q e^{iq\alpha}\gamma^\mu(\partial_\mu \alpha)\psi =$$

$$= e^{iq\alpha}(i\gamma^\mu \partial_\mu + q\gamma^\mu A_\mu - m)\psi +$$

$$+ e^{iq\alpha} q\gamma^\mu(\partial_\mu \chi)\psi - e^{iq\alpha} q\gamma^\mu(\partial_\mu \alpha)\psi$$

$$= 0 \quad \Rightarrow \quad A'_\mu = A_\mu + \partial_\mu \chi \text{ with}$$

$\chi = \alpha$ leaves the eq. invariant.