OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

SYMMETRY AND RELATIVITY

TUTORIAL I

Tensors. Problem Set 1.

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I. A BRIEF COMMENT ABOUT THE LITERATURE

The titles recommended by the lecturer are [1] and [2].

When reading about tensors, one should remember that many books discuss them twice - once in the context of Special Relativity and then in full generality. For example, in Weinberg's book [3] tensors appear in Chapter 2 and then in Chapter 4 (and similarly in Landau-Lifshitz [4]). In this course, we shall not be making such a distinction and always treat tensors in the most general way, unless explicitly stated otherwise, because this is how they appear in various branches of physics (not only in SR & GR).

A very useful collection of problems (with solutions!) in Special and General Relativity (tensors appear in Chapter 3) is [5].

Useful books where tensors and other structures are introduced rigorously but in the language accessible to physicists are [6], [7], [8], [9].

II. TENSORS AND TENSOR ALGEBRA

In this course, we will be dealing mostly with 3-dimensional Euclidean space and 4-dimensional Minkowski space. These are examples of the so called "metric spaces", i.e. spaces, equipped with a machinery (metric) to measure distances between points. To be more precise, one has to define the notion of "space" first. This is done in topology and differential geometry courses (one starts with sets, then introduces topology to have a sense of continuity, then gradually adds other structures, including a metric). This is important to know for a physicist, since at small (Planckian, i.e. $l \sim l_P = \sqrt{G\hbar/c^3} \sim 10^{-33} cm$) distances some of these structures may not be adequate (e.g. Riemannian geometry may have to be replaced by a more general construction, reducing at $l \gg l_P$ to the "standard" one).

Coordinates are introduced to quantify a space and objects associated with it. One can introduce many coordinate systems for the same space, e.g. Cartesian, spherical or cylindrical coordinates for the 3-dimensional Euclidean space $\mathbb{R}^3$, or Cartesian or polar for $\mathbb{R}^2$. (See Morse and Feshbach "Methods of Theoretical Physics", Chapter 5, for a list of some useful coordinate systems.)

Suppose we have two coordinate systems in n-dimensional space M^n (not necessarily Euclidean): $x^i = (x^1, x^2, \ldots x^n)$ and $x'^i = (x'^1, x'^2, \ldots x'^n)$. Here $i = 1, 2, \ldots n$. Each point $p \in M^n$ is characterised by the set of coordinates (either x^i or x'^i), and there is one-to-one correspondence[1] $x'^i = x'^i(x)$ between the two descriptions at a point p provided the determinant of the *Jacobi matrix*

$$J^i_j = \frac{\partial x'^i}{\partial x^j} \tag{1}$$

known as the *Jacobian* does not vanish at this point: $J = \det J^i_j \neq 0$. Points where $J = 0$ are known as coordinate singularities (they are singularities associated with a given coordinate system, not the space itself).

Excercise: Compute J^i_j and J for the sets of Cartesian and polar coordinates in $\mathbb{R}^2$ and Cartesian and spherical coordinates in $\mathbb{R}^3$. Identify the coordinate singularities.

Now consider vectors on M^n, e.g. the velocity vector of a point moving in

[1] All transformations $x'^i = x'^i(x)$ are assumed to be smooth, e.g. of C^∞ class. The important class of discrete transformations (such as $x^i \to -x^i$), including parity inversion ($x \to -x$, $y \to -y$, $z \to -z$) and time reversal ($t \to -t$), are considered separately.

M^n. Vectors are specified by their components $a^i(x) = (a^1(x), a^2(x), \dots a^n(x))$ at each point $x \in M^n$. Consider the gradient of a function f in the direction of a^i:

$$a^i \nabla_i f = a^i(x) \frac{\partial f}{\partial x^i} , \tag{2}$$

where summation over repeated indices is assumed (this is known as "Einstein summation convention"). What happens to this expression if we write it in the new coordinates $x'^i = x'^i(x)$? We have

$$a^i(x) \frac{\partial f}{\partial x^i} = a^i \left(x(x') \right) \frac{\partial f}{\partial x'^j} \frac{\partial x'^j}{\partial x^i} = a'^j(x') \frac{\partial f}{\partial x'^j} , \tag{3}$$

where

$$\boxed{a'^j(x') = \frac{\partial x'^j}{\partial x^i} a^i \left(x(x') \right)} \tag{4}$$

is the law of transformations of vectors (old name: contravariant vectors), or, more precisely, vector's components, under the coordinate transformation $x'^i = x'^i(x)$. Eq. (4) appears naturally: indeed, we could have started in x' coordinates, writing the gradient of the function as on the RHS of Eq. (3) (its functional form should not depend on the choice of coordinates). In fact, we can define a vector in a way independent of the choice of coordinates by

$$v = a^i(x) \frac{\partial}{\partial x^i} , \tag{5}$$

where $\partial/\partial x^1, \partial/\partial x^2 \dots \partial/\partial x^n$ can be thought as the basis in the linear vector space, similar to the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $V = a^1\mathbf{i} + a^2\mathbf{j} + a^3\mathbf{k}$. Sometimes, the notation $\mathbf{e_i} \equiv \partial/\partial x^i$ is used. Then

$$v = a^i(x) \frac{\partial}{\partial x^i} = a^i(x)\, \mathbf{e_i} \tag{6}$$

are contravariant vectors (or just vectors) with components $a^i(x)$. More precisely, they are *vector fields*, since a^i are not constant but depend on x.

Similarly, consider the differential of a function, $df = b_i(x)dx^i$. Under $x'^i = x'^i(x)$, we have

$$df = b_i(x)dx^i = b_i\left(x(x')\right)\frac{\partial x^i}{\partial x'^j}dx'^j = b'_j(x')dx'^j\,, \tag{7}$$

where

$$\boxed{b'_j(x') = \frac{\partial x^i}{\partial x'^j}\,b_i\left(x(x')\right)} \tag{8}$$

is the law of transformation of covectors or differential forms (old name: covariant vectors) under the coordinate transformation $x'^i = x'^i(x)$. In fact, we can define a covariant vector in a way independent of the choice of coordinates by

$$v^* = b_i(x)dx^i\,, \tag{9}$$

where $dx^1, dx^2 ... dx^n$ can be thought as the basis in the linear vector space, i.e. notation $\mathbf{e^i} \equiv dx^i$ is used. Then

$$v^* = b_i(x)dx^i = b_i(x)\,\mathbf{e^i} \tag{10}$$

are the covariant vectors with componets $b_i(x)$. More precisely, they are *covector fields*, since b_i are not constant but depend on x.

Denoting the space of all vectors by V and all covectors by V^*, we see that there us a natural map $V \otimes V^* \to \mathbf{R}$ (in principle, other fields such as $\mathbf{C}$ can be used as well, but some care should be exercised then, especially in the case of curved spaces) given by

$$v^*(v) = b_j dx^j\left(a^i\frac{\partial}{\partial x^i}\right) = a^i b_i \in \mathbf{R}\,. \tag{11}$$

We can think of generalising these constructions to objects with more than one index. For example,

$$w^* = c_{ij}(x)dx^i dx^j \tag{12}$$

is an obvious generalisation of (9). A more highbrow notation is

$$w^* = c_{ij}(x)dx^i \otimes dx^j \tag{13}$$

but it is really the same thing. An operation

$$w^*(v) = c_{ij}(x)dx^i \otimes dx^j \left(a^k \frac{\partial}{\partial x^k}\right) = c_{ij}a^j dx^i \in V^* \tag{14}$$

is a map $V^* \otimes V^* \otimes V \to V^*$ which can be seen as linear operators (matrices) acting on vectors.

Obviously, we can add more components

$$w^* = p_{ijk}(x)dx^i \otimes dx^j \otimes dx^k \,, \tag{15}$$

and so on. For vectors we have,

$$w = h^{ijk}(x)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} \,, \tag{16}$$

and we can have mixed objects as well, such as

$$t = s^{ij}_{k}(x)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k \,, \tag{17}$$

A generic tensor (more precisely - tensor field, since components depend on x) then is an object

$$T = T^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x)\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \cdots \otimes dx^{j_q} \,, \tag{18}$$

whose components $T^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x)$ transform under a continuous ${x'}^i = {x'}^i(x)$ such that each upper index transforms as in (4) and each lower index - as in (8), i.e.

$$\boxed{T'^{\,i_1 i_2 \dots i_p}_{\;j_1 j_2 \dots j_q}(x') = \frac{\partial x'^{\,i_1}}{\partial x^{k_1}} \frac{\partial x'^{\,i_2}}{\partial x^{k_2}} \cdots \frac{\partial x'^{\,i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial x'^{\,j_1}} \frac{\partial x^{l_2}}{\partial x'^{\,j_2}} \cdots \frac{\partial x^{l_q}}{\partial x'^{\,j_q}} \, T^{k_1 k_2 \dots k_p}_{l_1 l_2 \dots l_q}\left(x(x')\right)} \tag{19}$$

Tensors are often called rank (p, q)-tensors, specifying the number of upper (contravariant) and lower (covariant) components. In N-dimensional space, a generic rank (p, q)-tensor has N^{p+q} components.

The simplest example of the transformation law (19) is a transformation of a scalar $\varphi(x)$ (under continuous $x'^{\,i} = x'^{\,i}(x)$):

$$\varphi'(x') = \varphi(x) \,. \tag{20}$$

Note: if, in addition to the property (20) *under a continuous transformation,* $\varphi(x)$ *changes sign under a parity transformation* $x^i \to x'^i = -x^i$*, it is called a pseudoscalar. If the sign remains the same it is sometimes called a true scalar. The same terminology applies to higher tensors, e.g. we have pseudovectors etc.*

Another important example of a $(0, 2)$ tensor is the metric $g_{ij}(x)$:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^{\,\mu}} \frac{\partial x^\sigma}{\partial x'^{\,\nu}} \, g_{\rho\sigma}\left(x(x')\right) \tag{21}$$

Note: since rank 2 tensors are represented by matrices, the transformation (21) *can be written as*

$$G'(x') = S^T G(x) S \,, \tag{22}$$

where

$$S^\rho_{\;\mu}(x) = \frac{\partial x^\rho}{\partial x'^{\,\mu}} \,. \tag{23}$$

In special relativity, the matrix Λ *representing Lorentz transformations* $x' = \Lambda x$ *is independent of* x *and is given by*

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

whereas

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

where $\beta = v/c$ *and* $\gamma = 1/\sqrt{1-\beta^2}$*. Thus,* $S = \Lambda^{-1}$*. Now,* G *is the Minkowski metric tensor (normally denoted by* η*)*

$$G = \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

One can easily check that $\eta' = S^T \eta S = \eta$*, i.e. the Minkowski metric is invariant under the Lorentz transformations.*

Important note: NOT EVERY OBJECT WITH INDICES IS A TENSOR. A canonical example here is the connection coefficient $\Gamma^i_{jk}(x)$ of GR which is not a tensor (see e.g. [3]). To check whether an object with indices is a tensor, one has to check the transformation law (19) explicitly or use some simple facts of tensor algebra:

- A linear combination of (p,q)-tensors tensors is a (p,q)-tensor
- A contraction of tensors is a tensor

  If S^{ijk} and $T_{ijlm}$are tensors, so is

$$S^{ijk}T_{ijlm} = U^k_{lm}\,,$$

  where summation over repeated ("dummy") indices is assumed. An important contraction is

$$T = T^i_i$$

  known as "trace" (or "Spur" in German), with the notation tr (or Sp).

Tensor product: For two tensors, A and B, one can define a tensor product $S = A \otimes B$. For example, if $A = A^i$ and $B = B^j$ are vectors, $S^{ij} = A^i B^j$. E.g. in $d = 2$ we have

$$S = \begin{pmatrix} A^1B^1 & A^1B^2 \\ A^2B^1 & A^2B^2 \end{pmatrix}.$$

This can be extended to tensors of any rank. Note that the tensor product operation is generically not commutative ($A \otimes B \neq B \otimes A$) but associative.

A. Special tensors

- The Kronecker tensor δ^i_j (the identity matrix) is a $(1,1)$-rank tensor. It is the same in all coordinate systems,

$$\delta'^i_j = \delta^i_j\,,$$

as can be seen from the transformation law of tensors. Note that lowering or raising indices of δ^i_j we get the metric tensor or its inverse,

$$g_{ij}\delta^j_k = g_{ik}\,, \qquad g^{ij}\delta^k_j = g^{ik}\,, \qquad g^{ij}g_{jk} = \delta^i_k\,.$$

In this sense, the notations δ_{ij} and δ^{ij} only make sense in Euclidean space, where the metric itself is a unit matrix, $g_{ij} = \delta_{ij}$.

• The Levi-Civita absolutely antisymmetric (pseudo) tensor. In $\mathbf{R}^3$, we had a useful object ε_{ijk}, where $\varepsilon_{123} = +1$ and any interchange of indices changes the sign. In $4d$ Minkowski space, we define a similar object with $\varepsilon_{0123} = -\varepsilon_{1023} = \cdots = +1$. Note that $\varepsilon^{0123} = -1$. (Also note that some authors define $\varepsilon_{0123} = -1$.)

In general curvilinear coordinates, one can introduce a generalisation of this object (written here in 4 dimensions)

$$\epsilon_{ijkl}(x) = \sqrt{|g(x)|}\varepsilon_{ijkl}\,,$$

where $g = det\, g_{ij}$ is the determinant of a metric tensor. Such an object is a covariant tensor. The corresponding contravariant tensor is

$$\epsilon^{ijkl}(x) = \frac{1}{\sqrt{|g(x)|}}\,\varepsilon^{ijkl}\,,$$

whereas ε^{ijkl} is known as *tensor density* (more details can be found e.g. in [3] or in the exercises in [5]).

B. Vector components in curvilinear coordinates

It is helpful to consider a number of standard examples familiar from earlier studies, such as the orthogonal curvilinear coordinates (polar, cylindrical, spherical) in $\mathbb{R}^3$. It is important to emphasize that these are coordinates in flat space (the criterium for this is simple - all components of the Riemann curvature tensor for a given metric are zero and thus there exists a coordinate transformation bringing the metric into the form $ds^2 = dx^2 + dy^2 + dz^2$).

Nevertheless, such coordinates exhibit non-trivial features. For example, the connection coefficients or Christoffel symbols for them are non-zero (they are known as flat connections since the curvature tensor remains zero), and therefore the covariant derivative is non-trivial, and so on.

For a cylindrical coordinate system, we have (here $x^\mu = (x, y, z)$ are Cartesian coordinates, and $x'^\mu = (r, \phi, z)$):

$$x = r\cos\phi \tag{27}$$

$$y = r\sin\phi \tag{28}$$

$$z = z \tag{29}$$

The metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The inverse metric is given by

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

To represent a vector in the new system, usually a set of basis unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$, similar to the Cartesian unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, is introduced, so that

$$\vec{A} = \hat{A}_x\,\hat{\mathbf{i}} + \hat{A}_y\,\hat{\mathbf{j}} + \hat{A}_z\,\hat{\mathbf{k}} = \hat{A}_r\,\hat{\mathbf{r}} + \hat{A}_\phi\,\hat{\boldsymbol{\phi}} + \hat{A}_z\,\hat{\mathbf{z}}\,. \tag{30}$$

Vector components with hats such as $\hat{A}_r$ were introduced to distinguish them from the contravariant and covariant componets A^i and A_i used earlier. We shall explain the difference shortly.

More generally, one can write

$$d\vec{r} = \mathbf{e}_i dx^i = \mathbf{e}'_k dx'^k \,, \tag{31}$$

with

$$\mathbf{e}'_k = \frac{\partial x^i}{\partial x'^k}\mathbf{e}_i \,. \tag{32}$$

For the cylindrical cordinates, with $\mathbf{e}_1 = \mathbf{e}_x \equiv \hat{\mathbf{i}}$, $\mathbf{e}_2 = \mathbf{e}_y \equiv \hat{\mathbf{j}}$, $\mathbf{e}_3 = \mathbf{e}_z \equiv \hat{\mathbf{k}}$ and $\mathbf{e}'_1 = \mathbf{e}_r$, $\mathbf{e}'_2 = \mathbf{e}_\phi$, , $\mathbf{e}'_3 = \mathbf{e}_z$, we have from Eq. (32)

$$\mathbf{e}_r = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \,, \tag{33}$$

$$\mathbf{e}_\phi = -r\sin\phi\,\hat{\mathbf{i}} + r\cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \,, \tag{34}$$

$$\mathbf{e}_z = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \,. \tag{35}$$

This can also be written as

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = \cos\phi\,\frac{\partial}{\partial x} + \sin\phi\,\frac{\partial}{\partial y} + 0\,\frac{\partial}{\partial z} \,, \tag{36}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} = -r\sin\phi\,\frac{\partial}{\partial x} + r\cos\phi\,\frac{\partial}{\partial y} + 0\,\frac{\partial}{\partial z} \,, \tag{37}$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = 0\,\frac{\partial}{\partial x} + 0\,\frac{\partial}{\partial y} + 1\,\frac{\partial}{\partial z} \,, \tag{38}$$

which is an explicit form of the identification $\mathbf{e}_i = \partial/\partial x^i$.

Note that the basis vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ are not normalised to unity, e.g. $|\mathbf{e}_\phi| = \sqrt{\mathbf{e}_\phi \cdot \mathbf{e}_\phi} = r$. One can introduce the normalised vectors

$$\hat{\mathbf{r}} = \frac{\mathbf{e}_r}{|\mathbf{e}_r|} \,, \qquad \hat{\boldsymbol{\phi}} = \frac{\mathbf{e}_\phi}{|\mathbf{e}_\phi|} \,, \qquad \hat{\mathbf{z}} = \frac{\mathbf{e}_z}{|\mathbf{e}_z|} \,,$$

whose Cartesian coordinates are $\hat{\mathbf{r}} = (\cos\phi, \sin\phi, 0)$, $\hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi, 0)$, $\hat{\mathbf{z}} = (0, 0, 1)$. In general, $\hat{\mathbf{e}}_i = \mathbf{e}_i/|\mathbf{e}_i|$, where $|\mathbf{e}_i|^2 = \mathbf{e}_i \cdot \mathbf{e}_i = g_{\alpha\beta}\mathbf{e}^\alpha_{(i)}\mathbf{e}^\beta_{(i)}$. Clearly, components of a vector $\vec{A}$ will be different in these two bases,

$$\vec{A} = A^i\mathbf{e}_i = A^i|\mathbf{e}_i|\hat{\mathbf{e}}_i = A^1\,|\mathbf{e}_r|\hat{\mathbf{r}} + A^2\,|\mathbf{e}_\phi|\hat{\boldsymbol{\phi}} + A^3\,|\mathbf{e}_z|\hat{\mathbf{z}} \,. \tag{39}$$

Raising the indices with the metric g^{ij}, we get

$$\mathbf{e}^r = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}\,, \tag{40}$$

$$\mathbf{e}^\phi = -\frac{1}{r}\sin\phi\,\hat{\mathbf{i}} + \frac{1}{r}\cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}\,, \tag{41}$$

$$\mathbf{e}^z = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\,. \tag{42}$$

Note that $\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij}$, $\mathbf{e}^i \cdot \mathbf{e}^j = g^{ij}$, and $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$. We can introduce $\hat{\mathbf{e}}^i = \mathbf{e}^i/|\mathbf{e}^i|$. For diagonal metrics, $\hat{\mathbf{e}}^i = g^{ij}\mathbf{e}_j/|g^{ij}\mathbf{e}_j| = \hat{\mathbf{e}}_j\,\mathrm{sgn}(g^{ij})$. In particular, $\hat{\mathbf{e}}_r = \hat{\mathbf{e}}^r$, $\hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}^\phi$, $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}^z$. This is convenient, since in the expansion of a vector

$$\vec{A} = A^i\mathbf{e}_i = A^i|\mathbf{e}_i|\hat{\mathbf{e}}_i = \hat{A}^i\hat{\mathbf{e}}_i = A_i\mathbf{e}^i = A_i|\mathbf{e}^i|\hat{\mathbf{e}}^i = \hat{A}_i\hat{\mathbf{e}}^i \tag{43}$$

we have $\hat{A}^i = \hat{A}_i$, i.e. there is no difference between components with upper and lower indices in this basis. For details, see [**?**] and [**?**]. This basis is typically used when dealing with orthogonal curvilinear coordinates in $\mathbb{R}^3$.

We note that for a generic curved space-time with the metric tensor $g_{\mu\nu}(x)$ the standard bases $\partial/\partial x^i$ (for vectors) and dx^i (for covectors) are used, and, respectively, one has the standard contravariant and covariant components A^i and A_i.

C. Differential operators

In curved space (and even in flat space when using curvilinear coordinates) one has to generalise various differential operations accordingly. This is fully considered in GR courses but we mention some operations here. When differentiating tensors, ordinary derivatives should be replaced with covariant derivatives. For example, acting on vectors, the covariant derivative is

$$\nabla_i A^j = \partial_i A^j + \Gamma^j_{ik}A^k\,,$$

where Γ^j_{ik} are Christoffel symbols (coefficients of the metric connection). We also have

$$\nabla_i A_j = \partial_i A_j - \Gamma^k_{ij} A_k \, ,$$

Some operations can be easily generalised any dimension, for example, the divergence $\nabla_i A^i$, whereas others, such as curl, may be dimension-specific and are replaced in other dimensions by more general constructions.

The curl of a vector $\vec{A}$ in $3d$ can be written as

$$\left(\mathrm{curl}\vec{A}\right)^i = \epsilon^{ijk} \nabla_j A_k \, ,$$

where the tensor ϵ^{ijk} is defined as

$$\epsilon^{ijk}(x) = \frac{1}{\sqrt{|g(x)|}} \, \varepsilon^{ijk} \, ,$$

whereas ε^{ijk} is the permutation coefficient, with $\varepsilon^{123} = 1$. For example, in spherical coordinates in $\mathbb{R}^3$, $g = r^4 \sin^2\theta$ and e.g. the r-component of a curl is given by

$$\left(\mathrm{curl}\vec{A}\right)^r = \frac{\varepsilon^{r\theta\phi}}{r^2 \sin\theta} \left(\nabla_\theta A_\phi - \nabla_\phi A_\theta\right) \, ,$$

where

$$\nabla_\theta A_\phi = \partial_\theta A_\phi - \Gamma^k_{\theta\phi} A_k \, ,$$

and

$$\nabla_\phi A_\theta = \partial_\phi A_\theta - \Gamma^k_{\phi\theta} A_k \, .$$

Since for metric connection $\Gamma^k_{\theta\phi} = \Gamma^k_{\phi\theta}$, and $\varepsilon^{r\theta\phi} = 1$, we have

$$\left(\mathrm{curl}\vec{A}\right)^r = \frac{1}{r^2 \sin\theta} \left(\partial_\theta A_\phi - \partial_\phi A_\theta\right) \, .$$

Remembering our discussion of different bases for curvilinear coordinates in flat space, we note that $A_\phi = r \sin\theta \hat{A}_\phi$ and $A_\theta = r\hat{A}_\theta$. Correspondingly, we have

$$\left(\mathrm{curl}\vec{A}\right)^r = \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta\hat{A}_\phi) - \partial_\phi\hat{A}_\theta\right] .$$

Typically, this is the expression that appears in the standard literature such as ref. [2]. Finally, we note that one can also write the coordinate-free expression for curl in $3d$ space as

$$\mathrm{curl}\vec{A} = \star\, d\vec{A}\,,$$

where $\star$ denotes the Hodge dual operator.

III. PROBLEM SET I: SOLUTIONS

Problem 1

If we have two successive transformations from $u^i = u^i(x^1, x^2, \dots x^N)$ *to* $v^i = v^i(y^1, y^2, \dots y^N)$*, and from* v^i *to* $w^i = w^i(z^1, z^2, \dots z^N)$*, with* $i = 1, 2, \dots N$*,*

$$v^i = \frac{\partial y^i}{\partial x^j} u^j \,, \tag{44}$$

and

$$w^i = \frac{\partial z^i}{\partial y^j} v^j \,, \tag{45}$$

show that we can perform the transformation in one step via

$$w^i = \frac{\partial z^i}{\partial x^j} u^j \,. \tag{46}$$

Solution: What we have here is the change of coordinates

$$x \to y = y(x) \to z = z(y) \,.$$

This change induces the change of the components of a vector (rank $(1,0)$ tensor)

$$u^i = u^i(x) \to v^i = v^i(y) \to w^i = w^i(z)$$

according to the standard rules of transformation of a tensor, here given explicitly by Eqs. (44) and (45). Combining Eqs. (44) and (45), we can write (note that we can use any letter to label dummy indices, and freely change it as long as it doesn't coincide with the one already used in the expression)

$$w^i = \frac{\partial z^i}{\partial y^k} \frac{\partial y^k}{\partial x^j} u^j = \frac{\partial z^i}{\partial x^j} u^j \,, \tag{47}$$

where the right hand side is a consequence of the rule of differentiating a composite function $z = z(y(x))$:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\,.$$

Problem 2

If A^{ij}_k is a mixed tensor, B^{ij}_k is another tensor of the same kind, and α and β are scalar invariants, show that $\alpha A^{ij}_k + \beta B^{ij}_k$ is yet another tensor of the same kind.

Solution: This is a particular case of a tensor algebra property: a linear combination of tensors is a tensor. To prove it, we only need to use the definition of a tensor (19). Specifically, the $(2,1)$-rank tensors A^{ij}_k and B^{ij}_k transform under $x \to x' = x'(x)$ as

$$A'^{ij}_k(x') = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\, A^{pq}_r\,(x(x'))\ , \tag{48}$$

$$B'^{ij}_k(x') = \frac{\partial x'^i}{\partial x^s}\frac{\partial x'^j}{\partial x^t}\frac{\partial x^u}{\partial x'^k}\, B^{st}_u\,(x(x'))\ , \tag{49}$$

How does tensor $C^{ij}_k = \alpha A^{ij}_k + \beta B^{ij}_k$ transform? Well,

$$\begin{aligned} C'^{ij}_k(x') &= \alpha' A'^{ij}_k + \beta' B'^{ij}_k = \alpha\frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\, A^{pq}_r\,(x(x')) + \beta\frac{\partial x'^i}{\partial x^s}\frac{\partial x'^j}{\partial x^t}\frac{\partial x^u}{\partial x'^k}\, B^{st}_u\,(x(x')) \\ &= \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\,\left(\alpha A^{pq}_r\,(x(x')) + \beta B^{pq}_r\,(x(x'))\right) = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\, C^{pq}_r\,(x(x'))\ , \end{aligned}$$

since the scalars transform as $\alpha'(x') = \alpha(x)$, $\beta'(x') = \beta(x)$, and the dummy indices can be changed at will. Thus, C^{ij}_k transforms as a $(2,1)$-rank tensor.

Problem 3

If A^i_j are the components of a mixed tensor, show that A^i_i transforms as a scalar.

Solution: Under $x \to x' = x'(x)$, the tensor A^i_j itself transforms as

$$A'^{\,i}_j(x') = \frac{\partial x'^{\,i}}{\partial x^p}\frac{\partial x^q}{\partial x'^{\,j}}\, A^p_q\,(x(x'))\ . \tag{50}$$

The contracted tensor A^i_i therefore transforms as

$$A'^{\,i}_i(x') = \frac{\partial x'^{\,i}}{\partial x^p}\frac{\partial x^q}{\partial x'^{\,i}}\, A^p_q\,(x(x'))\ . \tag{51}$$

But

$$\frac{\partial x^q}{\partial x'^{\,i}}\frac{\partial x'^{\,i}}{\partial x^p} = \frac{\partial x^q}{\partial x^p} = \delta^q_p\,.$$

Therefore,

$$A'^{\,i}_i(x') = \ A^p_p\,(x(x')) = \ A^i_i\,(x(x'))\ . \tag{52}$$

This means that $A^i_i(x)$ is a scalar.

Note: For any tensor, full contraction of upper indices with lower ones is a scalar. More generally, a contraction of a pair of indices (one upper and one lower) lowers the rank of the tensor from (p, q) to $(p - 1, q - 1)$.

Problem 4

Assuming x and y transform as the components of a Euclidean vector, determine which of the following matrices are tensors:

$$A^{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}, \quad C^{ij} = \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix}.$$

Solution: To make this question more clear, introduce notations for coordinates $x^1 = x$, $x^2 = y$ and components of a vector $v^i = (v^1, v^2) = (v_x, v_y)$. In this particular case v^i is a vector with components $v_x = x$ and $v_y = y$. So, for example,

$$A^{ij} = \begin{pmatrix} v_x^2 & v_x v_y \\ v_x v_y & v_y^2 \end{pmatrix},$$

and so on. Under the change of coordinates

$$x \to x' = x'(x, y), \qquad y \to y' = y'(x, y)$$

the components v^i transform as

$$v'^{\,i}(x') = \frac{\partial x'^{\,i}}{\partial x^p}\, v^p\,(x(x'))\,. \tag{53}$$

Explicitly, we have

$$v'_x(x') = \frac{\partial x'}{\partial x}\, v_x\,(x(x')) + \frac{\partial x'}{\partial y}\, v_y\,(x(x'))\,, \tag{54}$$

$$v'_y(x') = \frac{\partial y'}{\partial x}\, v_x\,(x(x')) + \frac{\partial y'}{\partial y}\, v_y\,(x(x'))\,. \tag{55}$$

If A^{ij} is a tensor, its components should transform as

$$A'^{\,ij}(x') = \frac{\partial x'^{\,i}}{\partial x^p}\frac{\partial x'^{\,j}}{\partial x^q}\, A^{pq}\,(x(x'))\,. \tag{56}$$

For example, we should have

$$
\begin{aligned}
A'^{11}(x') &= \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x}\,A^{1\,1}\,(x(x')) + 2\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}\,A^{1\,2}\,(x(x')) + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial y}\,A^{2\,2}\,(x(x')) \\
&= \left(\frac{\partial x'}{\partial x}\right)^2 v_x^2 + 2\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}\,v_x v_y + \left(\frac{\partial x'}{\partial y}\right)^2 v_y^2\,. \qquad (57)
\end{aligned}
$$

Does this hold? We know that $A'^{11}(x') = v_x'^2$, and we know that v_x' transforms as in Eq. (54). Then $v_x'^2$ transforms exactly as Eq. (57), i.e. exactly as expected from a tensor component. Similar checks for the remaining 3 components show that A^{ij} is indeed a tensor. Similarly, one can check that B^{ij} and C^{ij} are not tensors.

Problem 5

Show that if the components of a contravariant vector vanish in one coordinate system, they will vanish in all coordinate systems. What can be said of two contravariant vectors whose components are equal in one coordinate system?

Solution: A contravariant vector A^i transforms under $x \to x' = x'(x)$ as

$$A'^i(x') = \frac{\partial x'^i}{\partial x^j} A^j \left(x(x')\right) . \tag{58}$$

If all components $A^j(x)$ vanish, then all components $A'^i(x')$ vanish as well.

Note: this fact is of fundamental importance and explains why tensors are used widely in physics. Indeed, fundamental physical laws can written as tensor identities: $S^\nu = 0$, $T^{\mu\nu} = 0$ *etc. For example, Maxwell equations are written in the covariant form as* $S^\nu \equiv \partial_\mu F^{\mu\nu} - \mu_0 J^\nu = 0$, *Einstein equations in vacuum as* $R^{\mu\nu} = 0$ *etc. Clearly, the laws should not depend on the coordinate system used to write them, i.e. we should have* $S'^\nu(x') = 0$ *etc. This is guaranteed by the tensorial transformation property such as the one in Eq.* (58)*. If all components of a tensor vanish in some coordinate system, they vanish in all coordinate systems.*

Problem 6

Let A_{ij} be a skew-symmetric tensor with $A_{ij} = -A_{ji}$, and S_{ij} a symmetric tensor with $S_{ij} = S_{ji}$. Show that the symmetry properties are preserved in coordinate transformations. Also show that the quantities with raised indices, A^{ij} and S^{ij}, possess the same properties. From this, show that $A^{ij}S_{ij} = 0$ and $A_{ij}S^{ij} = 0$.

Solution:

• We need to show that $A_{ij}(x) = -A_{ji}(x)$ implies $A'_{ij}(x') = -A'_{ji}(x')$. The transformation properties of A_{ij} are

$$A'_{ij}(x') = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} A_{mn}\left(x(x')\right) . \tag{59}$$

Since m and n are dummy indices, we can write

$$\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} A_{mn}\left(x(x')\right) = \frac{\partial x^n}{\partial x'^i}\frac{\partial x^m}{\partial x'^j} A_{nm}\left(x(x')\right) = -\frac{\partial x^m}{\partial x'^j}\frac{\partial x^n}{\partial x'^i} A_{mn}\left(x(x')\right) = -A'_{ji}(x') . \tag{60}$$

For the symmetric tensor, exactly the same argument works (except for the minus sign).

• For the same tensors with upper indices, we have

$$S'^{ij}(x') = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q} S^{pq}\left(x(x')\right) = \frac{\partial x'^i}{\partial x^q}\frac{\partial x'^j}{\partial x^p} S^{qp}\left(x(x')\right) = \frac{\partial x'^j}{\partial x^p}\frac{\partial x'^i}{\partial x^q} S^{pq}\left(x(x')\right) = S'^{ji}(x \tag{61}$$

and similarly for $A^{ij}(x) = -A^{ji}(x)$.

• Finally, using the fact that $A^{ij}S_{ij}$ is a scalar (all indices are contracted) and symmetry propeties of A^{ij} and S_{ij}, we find

$$A^{ij}S_{ij} = A^{ji}S_{ji} = -A^{ij}S_{ij} \tag{62}$$

which implies $A^{ij}S_{ij} = 0$. Similarly, $A_{ij}S^{ij} = 0$.

Note: the last property is widely used in practical calculations to simplify expressions. Note also that any $(0,2)$ *tensor* T_{ij} *can be decomposed into symmetric and antisymmetric parts:*

$$T_{ij} = T_{ij}^{sym} + T_{ij}^{asym} \,,$$

where

$$T_{ij}^{sym} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) \,, \qquad T_{ij}^{asym} = \frac{1}{2}\left(T_{ij} - T_{ji}\right) \,.$$

Note: symmetry properties reduce the number of independent components of a tensor. For example, a generic tensor T_{ij} *in* N*-dimensional space has* N^2 *componets. How many components do* T_{ij}^{sym} *and* T_{ij}^{asym} *have?*

Problem 7

Let $C^{kl} = A^{ijk} B^l_{ij}$ be a rank-2 contravariant tensor given by contracting the N^3 functions A^{ijk} with the tensor B^l_{mn}, which is symmetric in the mn indices but otherwise arbitrary, i.e. $B^l_{mn} = B^l_{nm}$. Show that $A^{ijk} + A^{jik}$ is a rank-3 contravariant tensor. Give reasons why the same is not true for A^{ijk} and A^{jik} separately.

Solution: We need to establish transformation properties of the object A^{ijk} under $x \to x'(x)$. Since C^{kl} is a tensor, we can write

$$C'^{kl}(x') = \frac{\partial x'^k}{\partial x^m}\frac{\partial x'^l}{\partial x^n} C^{mn}(x(x')) = \frac{\partial x'^k}{\partial x^m}\frac{\partial x'^l}{\partial x^p} A^{qrm} B^p_{qr}(x(x')) \,. \tag{63}$$

On the other hand,

$$C'^{kl}(x') = A'^{ijk} B'^l_{ij} = A'^{ijk} \frac{\partial x'^l}{\partial x^p}\frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j} B^p_{qr}(x(x')) \,. \tag{64}$$

Subtracting (63) from (64), we have

$$\left(A'^{ijk} \frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j} - \frac{\partial x'^k}{\partial x^m} A^{qrm} \right) \frac{\partial x'^l}{\partial x^p} B^p_{qr}(x(x')) = 0 \,. \tag{65}$$

In (65), k and l are free (uncontracted) indices. We can simplify this expression by multiplying it by e.g. $\partial x^s / \partial x'^l$ and summing over l. We get

$$\left(A'^{ijk} \frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j} - \frac{\partial x'^k}{\partial x^m} A^{qrm} \right) B^s_{qr}(x(x')) \equiv P^{kqr} B^s_{qr} = 0 \,. \tag{66}$$

Components of B^s_{qr} are not independent: $B^s_{qr} = B^s_{rq}$. The sum in (66) is thus of the form

$$P^{k11} B^s_{11} + \left(P^{k12} + P^{k21}\right) B^s_{12} + \cdots = 0 \,. \tag{67}$$

For *generic independent* B^s_{qr} (with $q \le r$) this implies $P^{krq} + P^{kqr} = 0$ (this may not be true individually for P^{krq}), i.e.

$$\left(A'^{ijk} + A'^{jik} \right) \frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j} - \frac{\partial x'^k}{\partial x^m} \left(A^{qrm} + A^{rqm} \right) = 0 \,. \tag{68}$$

Thus, the sum $A^{ijk} + A^{jik}$ transforms as a rank $(3, 0)$ tensor:

$$\left(A'^{ijk} + A'^{jik}\right) = \frac{\partial x'^i}{\partial x^q} \frac{\partial x'^j}{\partial x^r} \frac{\partial x'^k}{\partial x^m} \left(A^{qrm} + A^{rqm}\right) . \tag{69}$$

Problem 8

In this problem, we consider a transformation from Cartesian to polar coordinate systems in two Euclidean dimensions. Let $x^1 = x$ *and* $x^2 = y$ *for the Cartesian system and* $x'^1 = r$ *and* $x'^2 = \theta$ *for the polar, with the transformations*

$$x^1 = x = r\cos\theta = x'^1 \cos x'^2 \,, \tag{70}$$

$$x^2 = y = r\sin\theta = x'^1 \sin x'^2 \,. \tag{71}$$

The metric for the Cartesian system is $g_{ij} = \delta_{ij}$. *Derive the metric tensor* $g'_{ij}(x')$ *for the polar coordinate system, its reciprocal* g'^{ij}, *and the covariant polar coordinates* x'_1 *and* x'_2 *in terms of* r *and* θ. *Why might it not be appropriate to calculate a length from the origin to a point specified by finite values of* r *and* θ *using these covariant components?*

Show that the components of the metrics g_{ij} *and* $g'_{ij}(x')$ *do not change under rotations of the coordinate system through a fixed angle* α *around the origin.*

Solution: A metric is a $(0,2)$-rank tensor (field) used to determine infinitesimal distances via

$$ds^2 = g_{ij}(x) dx^i \otimes dx^j \,. \tag{72}$$

Here we have $g_{ij} = \delta_{ij}$, and so

$$ds^2 = dx^2 + dy^2 \tag{73}$$

(the symbol $\otimes$ is usually omitted). In matrix form:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Under $x \to x' = x'(x)$ the metric transforms as

$$g'_{ij}(x') = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}\, g_{mn}\left(x(x')\right) \,. \tag{74}$$

Explicitly,

$$g'_{11}(x') = \frac{\partial x^1}{\partial x'^1}\frac{\partial x^1}{\partial x'^1} + \frac{\partial x^2}{\partial x'^1}\frac{\partial x^2}{\partial x'^1} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 \,, \tag{75}$$

$$g'_{12}(x') = \frac{\partial x^1}{\partial x'^1}\frac{\partial x^1}{\partial x'^2} + \frac{\partial x^2}{\partial x'^1}\frac{\partial x^2}{\partial x'^2} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} \,, \tag{76}$$

$$g'_{21}(x') = \frac{\partial x^1}{\partial x'^2}\frac{\partial x^1}{\partial x'^1} + \frac{\partial x^2}{\partial x'^2}\frac{\partial x^2}{\partial x'^1} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} \,, \tag{77}$$

$$g'_{22}(x') = \frac{\partial x^1}{\partial x'^2}\frac{\partial x^1}{\partial x'^2} + \frac{\partial x^2}{\partial x'^2}\frac{\partial x^2}{\partial x'^2} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 \,, \tag{78}$$

resulting in

$$g'_{ij}(x') = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \,,$$

so the line element (often also called the metric) is

$$ds^2 = dr^2 + r^2 d\theta^2 \,. \tag{79}$$

Note that a more sraightforward way of transforming the metric is just substituting $x = r\cos\theta$ and $y = r\sin\theta$ into $ds^2 = dx^2 + dy^2$ (we have $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy = \sin\theta dr + r\cos\theta d\theta$, so the result (79) follows immediately).

We now introduce a fundamental definition of the inverse metric:

$$\boxed{\text{The inverse metric } g^{ij} \text{ is a } (2,0)\text{-rank tensor obeying } g^{ik}g_{kj} = \delta^i_j \,.} \tag{80}$$

In other words, this is just a matrix inverse to g_{ij}. In our particular case,

$$g'^{ij}(x') = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \,,$$

which is singular at $r = 0$. This is an example of a *coordinate singularity*: the space itself is not singular (as can be seen by choosing a different cordinate system, e.g. a Cartesian one, where the origin is an ordinary point). The same type of singularity is exhibited by black hole horizons.

The metric and its inverse can be used for raising and lowering indices of tensors, e.g.

$$x'_1 = g_{1j}(x')x'^j \,, \qquad x'_2 = g_{2j}(x')x'^j \,,$$

Explicitly, we find

$$x'_1 = g_{11}(x')x'^1 + g_{12}(x')x'^2 = r \,, \qquad x'_2 = g_{21}(x')x'^1 + g_{22}(x')x'^2 = r^2\theta \,,$$

The objects x'_1 and x'_2 are not related to distances (in Cartesian coordinates, $x_1 = x^1 = x$ and $x_2 = x^2 = y$, but this is not true in general). The distance (infinitesimal) is determined by Eq. (81) which can be written as

$$ds^2 = g_{ij}(x)dx^i \otimes dx^j = g'_{ij}(x')dx'^i \otimes dx'^j = dx'_i dx'^i = dr^2 + r^2 d\theta^2 \,. \qquad (81)$$

Computing finite distances, areas, volumes will in general require integration with an invariant measure $\sqrt{|g(x)|}dx$, where $g(x)$ is the determinant of the metric tensor. For polar coordinates, $\sqrt{|g(x)|} = r$, so the measure of integration (in two dimensions) is the familiar $rdrd\theta$ (in Cartesian coordinates, it is $dxdy$). More precisely, integrating a function $f(x)$ over a region Ω in an n-dimensional space with metric $g_{ij}(x)$ is done via

$$I = \int\limits_{\Omega} \sqrt{|g(x)|} f(x) d^n x \,.$$

In particular, the volume of Ω corresponds to $f(x) = 1$:

$$V = \int\limits_{\Omega} \sqrt{|g(x)|} d^n x \,.$$

For example, in $d = 2$, the "volume" (area) of the circle of radius R is given by

$$V_2 = \int_0^R \int_0^{2\pi} r dr d\theta = \pi R^2 \, .$$

How to compute the length of a curve? First of all, a curve is given by an equation (e.g. in polar coordinates in $d = 2$ it is $r = f(\theta)$). The Archimedean spiral, for example, is described by the equation $r = a + b\theta$. These equations describe an embedding of our curve (or other subspace) into the ambient space. We have $dr = f'(\theta)d\theta$. The *induced metric* on the cirve is given by substituting this dr into the ambient metric $ds^2 = dr^2 + r^2 d\theta^2$ giving $ds^2_{induced} = (f'^2 + f^2)d\theta^2$. The length is then

$$L = \int_\Omega \sqrt{|g_{induced}(x)|}dx = \int_{\theta_i}^{\theta_f} \sqrt{f'^2 + f^2}d\theta \, .$$

For the Archimedean spiral (for simplicity, we can set $a = 0$ and integrate from $\theta = 0$ to $\theta = \theta_{max}$) we have

$$\begin{aligned} L &= b\int_0^{\theta_{max}} \sqrt{1+\theta^2}d\theta = \frac{b}{2}\left(\theta_{max}\sqrt{1+\theta_{max}^2} + \text{arcsinh}\, \theta_{max}\right) \\ &= \frac{b}{2}\left[\theta_{max}\sqrt{1+\theta_{max}^2} + \ln\left(\theta_{max} + \sqrt{1+\theta_{max}^2}\right)\right] \, . \end{aligned} \tag{82}$$

A rotation by the angle α around the origin in e.g. counterclockwise direction is described by $z' = e^{i\alpha}z$, where $z = x + iy$. This can be written in the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Computing dx' and dy', one can see that the line element $ds^2 = (dx')^2 + (dy')^2 = dx^2 + dy^2$, and so the metric tensor g_{ij} is not affected by this transformation. The same can be also seen in polar coordinates, where $r' = r$ and $\theta' = \theta + \alpha$, so that $dr' = dr$ and $d\theta' = d\theta$, and thus $ds^2 = dr'^2 + r'^2 d\theta'^2 = dr^2 + r^2 d\theta^2$.

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