B2: Symmetry and Relativity
Problem Set 5: Groups and Revision
MT 2019

1. Show that the Lorentz transformations in a single spatial direction form a group.

2. Show that $e^{in\theta}$, with θ a constant, is a representation of the group of integers n under the addition operator. If $\theta = \pi/N$, how many elements does the representation have, and in what sense is it still a representation of the infinite group of integers?

3. Write down a set of 3×3 matrices to represent the permutation group on three objects, such that the action of swapping the second and third objects is the matrix

$$(D_{132})^i{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Show that this matrix representation is reducible by the following steps.

(i) Find a common eigenvector for all the $D^i{}_j$ matrices.

(ii) Write down a suitable similarity transformation matrix $S^i{}_j$, such that $(D')^i{}_j = S^i{}_m D^m{}_n (S^{-1})^n{}_j$ with the common eigenvector becoming $(1, 0, 0)$ in the new basis.

(iii) Show that the transformation matrices in the new basis take on block-diagonal form.

4. Show that the following matrix generates a rotation around the x axis

$$(J_1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

using the exponential $R(\theta) = e^{-i\theta J_1}$.

Show that the matrix

$$(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

generates a boost in the x direction with $\Lambda(\eta) = e^{-i\eta K_1}$.

Write the matrix form of the generator K_2 for infinitesimal boosts along the y axis. Multiply an infinitesimal boost along x by another along y. What does the form of the matrix indicate about whether non-aligned Lorentz transformations can form a group?

5. The canonical $j = 1$ representation of the generators of 3D rotations can be derived from following rules:

$$\begin{aligned} J_3|m\rangle &= m|m\rangle \\ J_\pm|m\rangle &= [j(j+1) - m(m \pm 1)]^{1/2}|m \pm 1\rangle \\ J_\pm &= J_1 \pm iJ_2 \end{aligned}$$

   Write down matrices representing the J_i generators using the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$.

   Verify that the generators satisfy the same Lie algebra as that of the $SO(3)$ group, *i.e.*,

$$[J_j, J_k] = i \sum_m \epsilon_{jkm} J_m.$$

   Using the appropriate spherical harmonics as basis of the $j = 1$ representation space, show that J_3 generates rotations around $\hat{\mathbf{z}}$. Similarly, show that the spherical harmonics representing a direction in the yz plane are rotated around $\hat{\mathbf{x}}$ by J_1.

6. Consider the Dirac equation of a fermion field in the presence of an electromagnetic field A^μ:

$$(i\gamma^\mu \partial_\mu + q\gamma^\mu A_\mu - m)\psi = 0$$

   where the γ^μ are 4×4 matrices with the anti-commutation relationship

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu\nu}$$

   (The important thing to keep in mind here is merely that there are 4 distinct matrices; you shouldn't need the anti-commutation relationship itself.) Apply the local gauge transformation to the fermion field

$$\psi \to \psi' = e^{iq\alpha}\psi$$

   where α is also a function of spacetime. Show that local gauge invariance can be restored by applying a simultaneous gauge transformation to the electromagnetic field,

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi.$$

   What must the relationship be between χ and α?

Past exam problems

7. (Based on B2 2014 Q3.) Write down the relationship between electric and magnetic fields and the scalar and vector potentials. The Faraday tensor is related to the four-vector potential by the expression

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

   Use this to find the expression for $F^{\mu\nu}$ in terms of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

The electromagnetic field satisfies $\partial_\lambda F^{\lambda\kappa} = -\mu_0 J^\kappa$ where J^κ is the four-current. Use this equation to obtain two of the Maxwell equations. Also, obtain the other two Maxwell equations given that $F^{\lambda\kappa}$ can be obtained from a 4-potential.

The stress-energy tensor of the electromagentic field may be written

$$T^{\mu\nu} = \varepsilon_0 c^2 (-F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}).$$

Find the top row of this tensor (*i.e.*, $T^{0\nu}$) in terms of $\mathbf{E}$ and $\mathbf{B}$ for a general field, and identify the physical quantities obtained. Find the $\nu = 0$ component of $\partial_\lambda T^{\lambda\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, and state how it is related to the current density $\mathbf{j}$.

A parallel-plate capacitor has its plates parallel to the xz plane and moves relative to the laboratory in the x direction at speed v. Let E be the electric field between the plates in the rest frame of the capacitor. Write down the Faraday tensor for this field in the rest frame, and hence obtain the stress-energy tensor, first in the rest frame and then in the laboratory. Hence find the Poynting vector of the field observed in the laboratory.

Repeat the calculation for a capacitor moving in the same way whose plates are parallel to the yz plane. In both cases describe the physical processes whereby the capacitor's stored energy is transported.

8. (Based on B2 2014 Q4.) Write down the scalar and vector potential for the field of a charged particle at rest. Hence, carefully explaining your reasoning, show that the 4-vector potential of an arbitrarily moving charged particle is given by

$$A^\mu = \frac{q}{4\pi\varepsilon_0} \frac{U^\mu/c}{(-R^\nu U_\nu)}$$

and define the quantities U^μ and R^ν involved in this expression. Make sure you justify the claim that your argument leads to a genuinely covariant result.

A particle of charge q is moving at constant, non-relativistic speed around a circle in the xy plane, such that its position is given by $(x, y, z) = (a \cos \omega t_s, a \sin \omega t_s, 0)$ at any given time t_s. It is desired to obtain the electric field $\mathbf{E}$ at points on the z axis. Let $\mathbf{v}_s$ be the velocity of the particle at the source time. Find an expression for A^μ in terms of q, a, z, $\mathbf{v}_s$, and fundamental constants. Hence obtain E_z.

To find the other components of $\mathbf{E}$, the gradient of the scalar potential ϕ is required. Consider a field event at $(\Delta x, 0, z)$ at time t, and obtain the dependence of ϕ on Δx to first order. Hence show that the x component of the electric field at $(0, 0, z)$ is given by

$$E_x = \frac{qa}{4\pi\varepsilon_0 c^2 \sqrt{a^2 + z^2}} \left(\left(\omega^2 - \frac{c^2}{a^2 + z^2} \right) \cos \omega t_s + \frac{\omega c}{\sqrt{a^2 + z^2}} \sin \omega t_s \right).$$

9. (Based on B2 2017 Q1.) For a particle of mass m moving along a worldline $X^\mu = X^\mu(\tau)$ in the inertial reference frame S, define the 4-velocity U^μ and the 4-force F^μ. Show that if the scalar product of two non-zero 4-vectors D^μ and G^μ is zero, and D^μ is time-like, then G^μ must be space-like. Prove that if the particle has a 4-momentum P^μ

such that $P^\mu P_\mu = 0$, then the particle has a rest mass equal to zero. Give an example of such a particle.

Define proper time and pure force. Show that if F^μ is a pure force, then $F^\mu U_\mu = 0$, where U^μ is the 4-velocity. Show that if 3-velocity $\mathbf{v}$ and 3-acceleration $\mathbf{a}$ are orthogonal to each other, *i.e.*, $\mathbf{v} \perp \mathbf{a}$, then the 4-acceleration invariant $A^\mu A_\mu$ is given by $A^\mu A_\mu = \gamma^4 a^2$.

A free particle, having rest mass M_0, is in vacuum and initially at rest in the lab frame S. It undergoes an acceleration under the action of a constant pure force with $\mathbf{f} = (f, 0, 0)$. Find its 4-velocity U^μ as a function of time and force. Sketch the graphs of the dependence of normalised 3-velocity $\boldsymbol{\beta} = \mathbf{v}/c$ and the Lorentz factor $\gamma(t)$ of the particle as a function of time t.

An electron and a positron are annihilated during a head-on collision. Before the collision they had 3-velocities $\mathbf{v}_e = (v_x, 0, 0)$ and $\mathbf{v}_p = (-v_x, 0, 0)$, respectively. After the collision, some number of photons are detected. What is the minimum number of photons that can be registered in this experiment? Explain your answer. Find the energy of each of the minimal number of photons taking into account that the rest mass of the electron and the positron is each equal to 0.511 MeV and their total kinetic energy before the collisions was 1 GeV.

An electron is accelerated from rest through a gap of $L = 3$ m by an electric field of strength 10 MV m^{-1} which is constant throughout the gap. Find γ and β at the other end of the gap.

10. (Based on part of B1 2004 Q3.) For an isolated system of particles, let

$$s^2 = \left(\sum E_i\right)^2 - \left(\sum \mathbf{p}_i c\right)^2$$

where the sums are taken over the particles in the system at some given time. What is s for a single particle of mass m?

In the laboratory frame a particle of mass m and momentum p_m is incident on a particle of mass M, at rest. Find an expression for the total available energy in the centre-of-mass frame.

Show that the momentum of the particle of mass m in the centre-of-mass frame is given by $p'_m = Mc^2 p_m / s$.

11. (Based on B1 2006 Q8.) It is proposed to generate a pure beam of either electron neutrinos or electron antineutrinos by accelerating ions of unstable nuclei to relativistic speeds and then allowing them to decay in a long straight section of the accelerator.

An unstable ion of rest mass M decays after it has been accelerated to total energy E and Lorentz factor $\gamma = E/Mc^2$ and emits a neutrino of energy E_ν at an angle of θ to the beam direction. (i) Derive an expression for the neutrino's energy E_ν^* in the rest frame of the ion in terms of E_ν, θ, and the velocity of the ion βc. (ii) Show that in the rest frame of the ion, the neutrino's path is inclined to the beam direction by the angle θ^* that satisfies

$$\cos\theta^* = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}.$$

Ions are accelerated to $\gamma = 100$ and decay in the straight section of the accelerator. A cylindrical detector that is coaxial with the beam and has radius $r = 30$ m, is placed $D = 300$ km downstream. Show that the angle between the beam direction and that of a neutrino which will hit the outer edge of the detector, measured in the rest frame of the ion, is approximately given by

$$\cos\theta^* = \frac{1 - \gamma^2\theta^2}{1 + \gamma^2\theta^2 - \theta^2/2}$$

where $\theta \simeq r/D$. Given that the emission of neutrinos is isotropic in the ion rest frame, find the fraction of the neutrinos that pass through the detector.

Show that in the ion rest frame the detector subtends an angle $2\theta_r^*$ at the ion at the emission event, where $\theta_r^* = \tan^{-1}(\gamma r/D)$. Why does θ_r^* differ from θ^*?