B2: Symmetry and Relativity
Problem Set 4: Radiation
MT 2019

1. Obtain the electric field of a uniformly moving charge, as follows: place the charge at the origin of the primed frame S' and write down the field in that frame, then transform to S using the equations for the transformation of the fields (not the force transformation method) and the coordinates. Be sure to write your result in terms of coordinates in the appropriate frame. Sketch the field lines. Prove (from the transformation equations, or otherwise) that the magnetic field of a uniformly moving charge is related to its electric field by $\mathbf{B} = \mathbf{v} \wedge \mathbf{E}/c^2$.

2. A sphere of radius a in its rest frame is uniformly charged with charge density $\rho = 3q/4\pi a^3$ where q is the total charge. Find the fields due to a moving charged sphere by two methods, as follows.

   [N.B. it will be useful to let the rest frame of the sphere be S' (not S) and to let the frame in which we want the fields be S. This will help to avoid a proliferation of primes in the equations you will be writing down. Let S and S' be in the standard configuration.]

   (i) Field method: write down the electric field as a function of position in the rest frame of the sphere, for the two regions $r' < a$ and $r' \geq a$ where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. Use the field transformation equations to find the electric and magnetic fields in frame S (re-using results from previous questions where possible), making clear in what regions of space your formulae apply.

   (ii) Potential method: in the rest frame of the sphere the 3-vector potential is zero, and the scalar potential is

   $$\phi' = \frac{q}{8\pi\varepsilon_0 a}(3 - r'^2/a^2)$$

   for $r' < a$, and

   $$\phi' = \frac{q}{4\pi\varepsilon_0 r'}$$

   for $r' \geq a$.

   Form the 4-vector potential, transform it, and thus show that both ϕ and $\mathbf{A}$ are time-dependent in frame S. Hence derive the fields for a moving sphere. [Beware when taking gradients that you do not muddle $\partial/\partial x$ and $\partial/\partial x'$, etc.]

3. In a frame S a point charge first moves uniformly along the negative x-axis in the positive x direction, reaching the point $(-d, 0, 0)$ at $t = -\Delta t$, and then it is slowed down until it comes to rest at the origin at $t = 0$. Sketch the lines of electric field in S at $t = 0$, in the region $(x + d)^2 + y^2 + z^2 > (c\Delta t)^2$.

4. Give a 4-vector argument to show that the 4-vector potential of a point charge q in an arbitrary state of motion is given by

$$A^\mu = \frac{q}{4\pi\varepsilon_0} \frac{U^\mu/c}{-R_\nu U^\nu}$$

where U^μ and R^μ are suitably chosen 4-vectors which you should define in your answer.

5. The electromagnetic field of a charge in an arbitrary state of motion is given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0\kappa^3}\left(\frac{\hat{\mathbf{n}} - \mathbf{v}/c}{\gamma^2 r^2} + \frac{\hat{\mathbf{n}} \wedge [(\hat{\mathbf{n}} - \mathbf{v}/c) \wedge \mathbf{a}]}{c^2 r}\right)$$

where $\hat{\mathbf{n}} = \mathbf{r}/r$ and $\kappa = 1 - v_r/c = 1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c$, and

$$\mathbf{B} = \hat{\mathbf{n}} \wedge \mathbf{E}/c$$

where $\mathbf{r}$ is the vector from the source point to the field point, and $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration of the charge at the source event. Without detailed derivation, outline briefly how this result may be obtained. How is the source event identified?

A charged particle moves along the x axis with constant proper acceleration ("hyperbolic motion"), its worldline being given by

$$x^2 - t^2 = \alpha^2$$

in units where $c = 1$. Find the electric field at $t = 0$ at points in the plane $x = \alpha$, as follows:

(i) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$. Show that the source event is at

$$x_s = \alpha + \frac{y^2}{2\alpha}.$$

(ii) Show that the velocity and acceleration at the source event are

$$v_s = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}$$

$$a_s = \frac{\alpha^2}{x_s^3}.$$

(iii) Consider the case $\alpha = 1$, and the field point $y = 2$. Write down the values of x_s, v_s, and a_s. Draw on a diagram the field point, the source point, and the location of the charge at $t = 0$. Mark at the field point on the diagram the directions of the vectors $\hat{\mathbf{n}}$, $\mathbf{v}$, $\mathbf{a}$, and $\hat{\mathbf{n}} \wedge (\hat{\mathbf{n}} \wedge \mathbf{a})$. Hence, by applying the formula above, establish the direction of the electric field at $(t, x, y, z) = (0, 1, 2, 0)$.

(iv) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x axis such that their separation is constant, comment on the forces they exert on one another.

6. The far field due to an elementary wire segment dz carrying oscillating current I is given by

$$dE = \frac{I \sin\theta \, dz}{2\varepsilon_0 cr \;\; \lambda} \cos(kr - \omega t).$$

Compare and contrast the case of a short antenna and the *half-wave dipole antenna*. Roughly estimate E in the far field for each case by proposing a suitable model for the distribution of current $I(z)$ in the antenna. What happens (qualitatively) for still longer antennae?

7. Show that the space-space part of the energy-momentum tensor

$$T^{\mu\nu} = \varepsilon_0 c^2 \left(-F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right)$$

is

$$\sigma^{ij} = \frac{1}{2}\varepsilon_0 (E^k E_k + c^2 B^k B_k)\delta^{ij} - \varepsilon_0 (E^i E^j + c^2 B^i B^j)$$

(Greek indices run over space and time, and Latin indices over space only.)

Use the stress-energy tensor $T^{\mu\nu}$ to find the forces exerted by the magnetic field inside a long cylindrical solenoid of radius 3 cm and field 1 tesla. *Mu-metal* is an alloy of high magnetic permeability that can be used to provide shielding against magnetic fields. If a piece of mu-metal is placed against the end of a solenoid, it "confines" the magnetic field to the interior of the solenoid. By interpreting the stress-energy tensor for the field on each side of the mu-metal sheet, discover whether the latter is attracted or repelled by the solenoid, and find the net force.

8. Write down the stress-energy tensor and the 4-wave vector for an electromagnetic plane wave propagating in the x direction.

Such a wave is observed in two frames in standard configuration. Show that the values of radiation pressure P, momentum density g, energy density u, and frequency ν in the two frames satisfy

$$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2}$$

(Optional: can you prove this for any relative motion of the frame? [Hint: write $T^{\mu\nu}$ in terms of K^μ]).

A confused student proposes that these quantities should transform like ν'/ν rather than ν'^2/ν^2, on the grounds that energy-momentum $N^\mu = (uc, \mathbf{N})$ is a 4-vector and so should transform in the same way as the wave vector. What is wrong with this argument?

Additional questions

9. An isolated parallel plate capacitor has charge $\pm Q$ on the plates. It is initially at rest in the laboratory frame. Assuming the capacitor's proper dimensions are fixed, what uniform motion should be given to the capacitor in order to increase the electric field between the plates? Does this result in an increased force of attraction between the plates?

10. A current-carrying wire is electrically neutral in its rest frame S. The wire has cross-sectional area A and a current I flows uniformly through this cross-section. Write down the 4-vector current density in the rest frame of the wire. Obtain the 4-vector current density in a rest frame S' moving with non-zero velocity v parallel to the wire. Hence show that in this frame there is a non-zero charge density in the wire. Does this imply that charge is not Lorentz invariant? Explain. Find the electric field in S' produced by the charge density of the wire, in the region outside the wire, and show that it is the same as the field obtained by transformation of the magnetic field in frame S.

Retarded potentials and radiative emission

11. Retarded potential:

    (i) Write down the solution to Poisson's equation for the case of a point charge q.

    (ii) In electrostatics, how is the electric potential at a point in space obtained if the charge distribution is known?

    (iii) Now consider the wave equation

    $$\partial_\mu \partial^\mu \phi = -\frac{\rho}{\varepsilon_0}.$$

    Show that the spherical wave form $\phi = \kappa g(t - r/c)/r$ (where $\kappa \equiv 1/4\pi\varepsilon_0$) is a solution of the wave equation for $r \neq 0$ if ρ is zero everywhere except at the origin.

    (iv) We would like to show that this is a solution also as $r \to 0$, if the charge density ρ is concentrated at a point at the origin. Using your knowledge of Poisson's equation, or otherwise, show that this is true as long as $g(t) = \int \rho(t) dV$.

    (v) Hence write down the solution to the wave equation for a given arbitrary time-dependent distribution of charge.

    (vi) Why is this called a *retarded* solution?

Field energy and momentum

12. In this problem we will compare power loss in electron accelerators.

    (i) The electric field in a linear accelerator is 10^6 V/m. Find the power emitted by an electron travelling down the accelerator. Express your results in eV per metre assuming the electrons travel at close to the speed of light. (You may quote Larmor's formula for emitted power.)

    (ii) A magnetic field of 1 tesla is used to maintain electrons in their orbits around a synchrotron of radius 10 m. Show that the electron energy is approximately 3 GeV. Find the radiative energy loss per revolution.

    (iii) What is the main reason why the loss rate is so much higher in the synchrotron case, compared with the linear accelerator?

13. Various identities—mostly needs good organisation to attack the algebra:

(i) Show from Maxwell's equations, or otherwise, that

$$\rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial \mathbf{g}}{\partial t} + \varepsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\nabla \wedge \mathbf{E}) \wedge \mathbf{E} + c^2(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}] \qquad (1)$$

where $\mathbf{g} = \mathbf{N}/c^2$ and $\mathbf{N}$ is the Poynting vector.
We wish to show that this can be written

$$\rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial \mathbf{g}}{\partial t} - \mathbf{s} \qquad (2)$$

such that $s^i = \partial_j \sigma^{ij}$ with

$$\sigma^{ij} = \frac{1}{2}\varepsilon_0(E^k E_k + c^2 B^k B_k)\delta^{ij} - \varepsilon_0(E^i E^j + c^2 B^i B^j).$$

To this end, take the following steps (or use another method if you prefer):

(ii) Show that the x component of $(\nabla \wedge \mathbf{E}) \wedge \mathbf{E}$ is

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) E_z - \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) E_y.$$

(iii) Evaluate just the electric field contribution to s^1, the x component of $\mathbf{s}$.

(iv) Confirm that you have matched the x component of the electric field part of the square bracket in equation 1. Explain why the magnetic part also follows.

(v) Is it necessary to calculate explicitly the other components?

(vi) Multiplying equation 2 by a small volume δV, we have

$$\delta V \rho\mathbf{E} + \delta V \mathbf{j} \wedge \mathbf{B} = -\delta V \frac{\partial \mathbf{g}}{\partial t} - \delta V \mathbf{s}. \qquad (3)$$

Explain the physical significance of the terms in this equation (and thus justify all this hard work!).

14. A pair of parallel particle beams separated by a distance d have the same uniform charge per unit length λ. In the laboratory frame, a magnetic field is applied with a direction and strength just sufficient to overcome the repulsion between the beams, so that they both propagate in a straight line at constant speed v. Find the size B of this magnetic field, by both of the following methods:

(i) Do the whole calculation in the laboratory frame.

(ii) Start with a calculation of the force exerted by either beam on a particle in the other, in the rest frame of the beams. Transform this force to the laboratory frame and hence deduce the required B field in that frame. What form does the externally applied field take in the rest frame of the beams?

15. Consider the general problem of motion of a particle of charge q in a uniform constant electromagnetic field. The equation of motion is

$$qF^{\mu}{}_{\nu}U^{\nu} = m\frac{dU^{\mu}}{d\tau}$$

For a uniform constant field, $F^{\mu\nu}$ is independent of space and time, and therefore this matrix equation is precisely the same as the one obtained in a classical normal modes problem, and can be solved by the same methods.

(i) Propose a solution $U^{\mu} = U_0^{\mu}\exp(\Gamma\tau)$ and thus convert the equation into an eigenvalue equation with eigenvalues $\lambda = m\Gamma/q$. (Note that $F^{\mu}{}_{\nu}$ is not symmetric, so the right-eigenvectors are not the same as the left-eigenvectors. We only need the right-eigenvectors here.)

(ii) Without loss of generality, we can take the z-axis along $\mathbf{B}$, and $\mathbf{E}$ in the xz plane. Show that the eigenvalues are

$$\lambda^2 = -\frac{D}{2} \pm \sqrt{D^2/4 + \alpha^2}$$

where $D = |\mathbf{B}|^2 - |\mathbf{E}|^2/c^2$ and $\alpha = \mathbf{B}\cdot\mathbf{E}/c$.

(iii) Consider the case $\alpha = 0$. What does this tell us about the fields? Interpret the solution corresponding to a zero eigenvalue [Hint: Lorentz force].

(iv) Find $U^{\mu}(\tau)$ for a particle initially at rest in a uniform purely electric field.

(v) Show that the right eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

may be written

$$\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Hence find $U^{\mu}(\tau)$ for a particle moving in a plane perpendicular to a uniform purely magnetic field.

16. In a frame S there is a uniform electric field E along the y direction and a uniform magnetic field $B = 5E/3c$ along the z direction. A particle of mass m and charge q is released from rest on the x axis. What time elapses before it returns to the x axis?

17. In synchrotron radiation, in which direction is most of the energy emitted in the rest frame of the accelerating charge? Describe qualitatively the pattern of the radiation field in the rest frame of the centre of the synchrotron apparatus.

18. Why is it not feasible for mobile phones to use radio waves?

Energy-momentum tensor

19. In a certain frame S_0 having 4-velocity U^μ, a 2nd-rank tensor $T^{\mu\nu}$ has but one non-zero component, $T^{00} = c^2$. Find the components of T in the general frame S, in which $U^\mu = \gamma_u(c, \mathbf{u})$.

20. Prove that for the electromagnetic stress-energy tensor, $T^\lambda{}_\lambda = 0$.

21. Prove that for the electromagnetic stress-energy tensor, $T^\mu{}_\lambda T^\lambda{}_\nu = (I\varepsilon_0/2)^2 \delta^\mu_\nu$ where $I^2 = (|\mathbf{E}|^2 - c^2|\mathbf{B}|^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2 c^2$. [Hint: start by establishing the identity in a particular frame, such as one in which $\mathbf{E}$ is parallel to $\mathbf{B}$.]