B2: Symmetry and Relativity
Problem Set 3: Forces and fields
MT 2019

1. Obtain the transformation equations for 3-force, by starting from the Lorentz transformation of energy-momentum, and then differentiating with respect to t'. [Hint: argue that the relative velocity $\mathbf{v}$ of the reference frame is constant, and use or derive an expression for dt/dt'.]

2. Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows:

   (i) Write down the solution for $\mathbf{p}$ as a function of time, taking as initial condition $\mathbf{p}(0) = \mathbf{p}_0$.

   (ii) Show that the Lorentz factor as a function of time is given by $\gamma^2 = 1 + \alpha^2$ where $\boldsymbol{\alpha} = (\mathbf{p}_0 + \mathbf{f}t)/mc$.

   (iii) You can now write down the solution for $\mathbf{v}$ as a function of time. Do so.

   (iv) Now restrict attention to the case where $\mathbf{p}_0$ is perpendicular to $\mathbf{f}$. Taking the x-direction along $\mathbf{f}$ and the y-direction along $\mathbf{p}_0$, show that the trajectory is given by

   $$\begin{aligned} x &= \frac{c}{f}(m^2c^2 + p_0^2 + f^2t^2)^{1/2} + \text{const} \\ y &= \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const} \end{aligned}$$

   where you may quote that $\int (a^2 + t^2)^{-1/2} dt = \log(t + \sqrt{a^2 + t^2})$.

   (v) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation. [N.B. the calculation as a function of proper time is best done another way, see later problems.]

3. For motion under a pure (rest mass preserving) inverse square law force $\mathbf{f} = -\alpha\mathbf{r}/r^3$, where α is a constant, derive the energy equation $\gamma mc^2 - \alpha/r = \text{constant}$.

4. Prove that the time rate of change of the angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ of a particle about an origin O is equal to the couple $\mathbf{r} \wedge \mathbf{f}$ of the applied force about O.

   If $L^{\mu\nu}$ is a particle's 4-angular momentum, and we define the 4-couple $G^{\mu\nu} \equiv X^\mu F^\nu - X^\nu F^\mu$, prove that $(d/d\tau)L^{\mu\nu} = G^{\mu\nu}$, and that the space-space part of this equation corresponds to the previous 3-vector result.

5. Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$ and ϕ such that

   $$\begin{aligned} \mathbf{B} &= \nabla \wedge \mathbf{A} \\ \mathbf{E} &= -\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \nabla\phi. \end{aligned}$$

(i) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ and ϕ.

(ii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators, and Lorentz scalars (a *manifestly covariant* form).

6. How does a 2nd rank tensor change under a Lorentz transformation? By transforming the field tensor, and interpreting the result, prove that the electromagnetic field transforms as

$$\begin{aligned}
\mathbf{E}'_{\|} &= \mathbf{E}_{\|} \\
\mathbf{B}'_{\|} &= \mathbf{B}_{\|} \\
\mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}) \\
\mathbf{B}'_{\perp} &= \gamma(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2)
\end{aligned}$$

[Hint: you may find the algebra easier if you treat $\mathbf{E}$ and $\mathbf{B}$ separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]

Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.

7. The electromagnetic field tensor $F^{\mu\nu}$ (sometimes called the Faraday tensor) is defined such that the 4-force on a charged particle is given by

$$f^{\mu} = qF^{\mu\nu}U_{\nu}.$$

By comparing this to the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

which defines the electromagnetic and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $dP^{\mu}/d\tau$), show that the components of the field tensor are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

8. Show that the field equation

$$\partial_{\lambda}F^{\lambda\nu} = -\mu_0\rho_0 U^{\nu}$$

is equivalent to

$$\partial^{\lambda}\partial_{\lambda}A^{\nu} - \partial^{\nu}(\partial_{\lambda}A^{\lambda}) = -\mu_0 J^{\nu}$$

where $J^{\nu} \equiv \rho_0 U^{\nu}$ (here ρ_0 is the proper charge density, and J^{ν} is the 4-current density). Comment.

9. Show that the following two scalar quantities are Lorentz invariant:

$$\begin{aligned} D &= B^2 - E^2/c^2 \\ \alpha &= \mathbf{B} \cdot \mathbf{E}/c. \end{aligned}$$

[Hint: for the second, introduce the dual field tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$.]
Show that if $\alpha = 0$ but $D \neq 0$ then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example such frame (by specifying its velocity relative to one in which the fields are $\mathbf{E}$, $\mathbf{B}$). Suggest a type of field for which both $\alpha = 0$ and $D = 0$.

Additional questions

10. Twin paradox:

 (i) Evaluate the acceleration due to gravity at the Earth's surface (9.8 m/s^2) in units of years and light-years.

 (ii) In the twin paradox, the travelling twin leaves Earth on board a spaceship undergoing motion at constant proper acceleration of 9.8 m/s^2. After 5 years of proper time for the spaceship, the direction of the rockets are reversed so that the spaceship accelerates towards Earth for 10 proper years. The rockets are then reversed again to allow the spaceship to slow and come to rest on Earth after a further 5 years of spaceship proper time. How much does the travelling twin age? How much does the stay-at-home twin age?

Motion under a given force

11. Consider a particle moving in a straight line with speed v, rapidity ρ, and proper acceleration a_0. Prove that $d\rho/d\tau = a_0/c$. [Hint: use the fact that colinear rapidities are additive.]

12. The axis of a cylinder lies along the x' axis. The cylinder has no translational motion in S', but it rotates about its axis with angular speed ω'. When observed in S the cylinder travels and rotates.

 (i) Prove that in S at any instant the cylinder is twisted, with a twist per unit length $\gamma\omega' v/c^2$. [Hint: consider the rotation of the flat surfaces at the two ends of the cylinder; a line painted on either surface rotates like the hand of a clock.]

 (ii) Is the cylinder in mechanical equilibrium? Comment on whether or not you expect there to be internal shear forces in the cylinder in frame S.

13. A "photon rocket" propels itself by emitting photons in the rearwards direction. The rocket is initially at rest with mass m. Show that when the rest mass has fallen to αm the speed (as observed in the original rest frame) is given by

$$\frac{v}{c} = \frac{1 - \alpha^2}{1 + \alpha^2}.$$

[Hint: don't bother with equations of motion, use conservation of momentum.]

It is desired to reach a speed giving a Lorentz factor of 10. What value of α is required? Supposing the rocket cannot pick up fuel en route, what proportion of its initial mass must be devoted to fuel if it is to make a journey in which it first accelerates to $\gamma = 10$, then decelerates to rest at the destination (the destination being a star with negligible relative speed to the sun)?

14. A rocket propels itself by giving portions of its mass m a constant velocity $\mathbf{u}$ relative to its instantaneous rest frame. Let S' be the frame in which the rocket is at rest at time t. Show that, if v' is the speed of the rocket in S', then to first order in dv',

$$(-dm)u = mdv'.$$

Hence prove that, when the rocket attains a speed v relative to its initial rest frame, the ratio of final to initial rest mass of the rocket is

$$\frac{m_f}{m_i} = \left(\frac{1 - v/c}{1 + v/c}\right)^{c/2u}.$$

Note that the least expenditure of mass occurs when $u = c$, *i.e.*, the "photon rocket".

Prove that if the rocket moves with constant proper acceleration a_0 for a proper time τ, then $m_f/m_i = \exp(-a_0\tau/u)$.

15. Show that the motion of a particle in a uniform magnetic field is in general helical, with the period for a cycle independent of the initial direction of the velocity. [Hint: what can you learn from $\mathbf{f} \cdot \mathbf{v}$?]

16. A particle moves hyperbolically with proper acceleration a_0, starting from rest at $t = 0$. At $t = 0$ a photon is emitted towards the particle from a distance c^2/a_0 behind it. Prove that in the instantaneous rest frames of the particle, the distance to the photon is always c^2/a_0.

Electromagnetic field tensor

17. Assuming the relation of fields $\mathbf{E}$ and $\mathbf{B}$ to potentials ϕ and $\mathbf{A}$, show that the field tensor can be written

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

(Note, the right hand side here is the 4-vector equivalent of a curl operation.) [Hint: use cyclic permutation to avoid unnecessary repetition.] Now write down $\partial^\lambda F^{\mu\nu}$ in terms of ∂ operators and A^μ. By keeping track of the sequence of indices, show that

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0.$$

(In an axiomatic approach, one could argue in the opposite direction, asserting the above as an axiom and then deriving the relation of fields to potentials.)

18. If $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ are two different electromagnetic fields, show that $\mathbf{E}\cdot\mathbf{E}' - c^2\mathbf{B}\cdot\mathbf{B}'$ and $\mathbf{E}\cdot\mathbf{B}' + \mathbf{B}\cdot\mathbf{E}'$ are invariants.