B2: Symmetry and Relativity
Problem Set 1: Vectors and Tensors
MT 2020

Einstein summation convention is assumed, with indices taking values from 1 to N.

1. If we have two successive transformations from $u^i = u^i(x^1, x^2, \cdots x^N)$ to $v^i = v^i(y^1, y^2, \cdots y^N)$, and from v^i to $w^i = w^i(z^1, z^2, \cdots z^N)$, with $i = 1, 2, \cdots N$,
$$v^i = \frac{\partial y^i}{\partial x^j} u^j,$$
and
$$w^i = \frac{\partial z^i}{\partial y^j} v^j,$$
show that we can perform the transformation in one step via
$$w^i = \frac{\partial z^i}{\partial x^j} u^j.$$

2. If $A^{ij}{}_k$ is a mixed tensor, $B^{ij}{}_k$ is another tensor of the same kind, and α and β are scalar invariants, show that $\alpha A^{ij}{}_k + \beta B^{ij}{}_k$ is yet another tensor of the same kind.

3. If $A^i{}_j$ are the components of a mixed tensor, show that $A^i{}_i$ transforms as a scalar invariant.

4. Assuming x and y transform as the components of a Euclidean vector, determine which of the following matrices are tensors:
$$A^{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}, \quad C^{ij} = \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix}$$
[based on E Butkov, *Mathematical Physics*]

5. Show that if the components of a contravariant vector vanish in one coordinate system, they vanish in all coordinate systems. What can be said of two contravariant vectors whose components are equal in one coordinate system?

6. Let A_{ij} be a skew-symmetric tensor with $A_{ij} = -A_{ji}$, and S_{ij} a symmetric tensor with $S_{ij} = S_{ji}$. Show that the symmetry properties are preserved in coordinate transformations. Also show that the quantities with raised indices, A^{ij} and S^{ij}, possess the same properties. From this, show that $A^{ij} S_{ij} = 0$ and $A_{ij} S^{ij} = 0$.

7. Let $C^{k\ell} = A^{ijk} B^{\ell}{}_{ij}$ be a rank-2 contravariant tensor given by contracting the N^3 functions A^{ijk} with the tensor $B^{\ell}{}_{mn}$, which is symmetric in the mn indices but otherwise arbitrary, *i.e.*, $B^{\ell}{}_{mn} = B^{\ell}{}_{nm}$. Show that $A^{ijk} + A^{jik}$ is a rank-3 contravariant tensor. Give reasons why the same is not true for A^{ijk} or A^{jik} separately.

8. In this problem, we will consider a transformation from Cartesian to polar coordinate systems in two Euclidean dimensions. Let $x^1 = x$ and $x^2 = y$ for the Cartesian system, and $\hat{x}^1 = r$ and $\hat{x}^2 = \theta$ for the polar, with the transformation

$$\begin{aligned} x^1 &= r\cos\theta = \hat{x}^1 \cos\hat{x}^2 \\ x^2 &= r\sin\theta = \hat{x}^1 \sin\hat{x}^2 \end{aligned}$$

The metric for the Cartesian system is $g_{ij} = \delta_{ij}$. Derive the metric tensor $\hat{g}_{ij}$ for the polar coordinate system, its reciprocal $\hat{g}^{ij}$, and the covariant polar components $\hat{x}_1$ and $\hat{x}_2$ in terms of r and θ. Why might it *not* be appropriate to calculate a length from the origin to a point specified by finite values of r and θ using these covariant components?

Show that the components of the metrics g_{ij} and $\hat{g}_{ij}$ do not change under rotations of the coordinate system through a fixed angle α around the origin.