B2: Symmetry and Relativity
Problem Set 1: Vectors, tensors, and groups
MT 2023 Weeks 1-2

Einstein summation convention is assumed, with indices taking values from 1 to N.

1. If we have two successive transformations from $u^i = u^i(x^1, x^2, \cdots x^N)$ to $v^i = v^i(y^1, y^2, \cdots y^N)$, and from v^i to $w^i = w^i(z^1, z^2, \cdots z^N)$, with $i = 1, 2, \cdots N$,
$$v^i = \frac{\partial y^i}{\partial x^j} u^j,$$
and
$$w^i = \frac{\partial z^i}{\partial y^j} v^j,$$
show that we can perform the transformation in one step via
$$w^i = \frac{\partial z^i}{\partial x^j} u^j.$$

2. Let B^i be the contravariant components of one vector, and C_i the covariant components of another. Show that the contraction of B^i with C_i is a scalar invariant.

   Furthermore, if $A^i{}_j$ are the components of a mixed tensor, show that $A^i{}_i$ transforms as a scalar invariant.

3. If $A^{ij}{}_k$ is a mixed tensor, $B^{ij}{}_k$ is another tensor of the same kind, and α and β are scalar invariants, show that $\alpha A^{ij}{}_k + \beta B^{ij}{}_k$ is yet another tensor of the same kind.

4. Consider rotations of x and y as components of a Euclidean vector. Show that the matrix
$$A^{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$$
transforms as a rank-2 tensor under such rotations. Show, on the other hand, that
$$B^{ij} = \begin{pmatrix} xy & y^2 \\ x^2 & -xy \end{pmatrix}$$
does not. [based on E Butkov, *Mathematical Physics*]

5. Show that if the components of a contravariant vector vanish in one coordinate system, they vanish in all coordinate systems. What can be said of two contravariant vectors whose components are equal in one coordinate system?

6. Let A_{ij} be a skew-symmetric tensor with $A_{ij} = -A_{ji}$, and S_{ij} a symmetric tensor with $S_{ij} = S_{ji}$. Show that the symmetry properties are preserved in coordinate transformations. Also show that the quantities with raised indices, A^{ij} and S^{ij}, possess the same properties. From this, show that $A^{ij} S_{ij} = 0$ and $A_{ij} S^{ij} = 0$.

7. Let $C^{k\ell} = A^{ijk}B^{\ell}{}_{ij}$ be a rank-2 contravariant tensor given by contracting the N^3 functions A^{ijk} with the tensor $B^{\ell}{}_{mn}$, which is symmetric in the mn indices but otherwise arbitrary, *i.e.*, $B^{\ell}{}_{mn} = B^{\ell}{}_{nm}$. Show that $A^{ijk} + A^{jik}$ is a rank-3 contravariant tensor. Give reasons why the same is not true for A^{ijk} or A^{jik} separately.

8. Show that $e^{in\theta}$, with θ a constant, is a representation of the group of integers n under the addition operator. If $\theta = \pi/N$, how many elements does the representation have, and in what sense is it still a representation of the infinite group of integers?

9. Working in a representation in which the basis vectors are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, show that rotations around the $\hat{\mathbf{z}}$ direction form a group.

   Show that the matrix

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   generates a finite rotation through angle θ using the exponential map $R(\theta) = e^{-i\theta J_3}$.

   Write down generators for rotations around the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions.

   Evaluate the product $(1 - i\theta_1 J_1)(1 - i\theta_2 J_2)$ or, alternatively, a finite rotation around $\hat{\mathbf{y}}$ followed by another rotation around $\hat{\mathbf{z}}$. What does the result indicate about the possibility of the rotations around two axes forming a group?

10. We can form another representation of the generators of 3D rotations using the rules

$$\begin{aligned} J_3|m\rangle &= m|m\rangle \\ J_\pm|m\rangle &= [j(j+1) - m(m \pm 1)]^{1/2}|m \pm 1\rangle \\ J_\pm &= J_1 \pm iJ_2 \end{aligned}$$

   with $j = 1$. Write down matrices representing the J_i generators using the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$.

   Verify that the generators satisfy the same Lie algebra as that of the $SO(3)$ group, *i.e.*,

$$[J_j, J_k] = i\sum_m \epsilon_{jkm} J_m.$$

   Now consider angular distributions which are linear combinations of the $j = 1$ spherical harmonics. Show that J_3 generates rotations of these distributions around $\hat{\mathbf{z}}$.

   Similarly, show that distribution evaluated in the yz plane are rotated around $\hat{\mathbf{x}}$ by J_1.

Additional questions

11. In this problem, we will consider a transformation from Cartesian to polar coordinate systems in two Euclidean dimensions. Let $x^1 = x$ and $x^2 = y$ for the Cartesian system, and $\hat{x}^1 = r$ and $\hat{x}^2 = \theta$ for the polar, with the transformation

$$\begin{aligned} x^1 &= r\cos\theta = \hat{x}^1 \cos\hat{x}^2 \\ x^2 &= r\sin\theta = \hat{x}^1 \sin\hat{x}^2 \end{aligned}$$

The metric for the Cartesian system is $g_{ij} = \delta_{ij}$. Derive the metric tensor $\hat{g}_{ij}$ for the polar coordinate system, its reciprocal $\hat{g}^{ij}$, and the covariant polar components $\hat{x}_1$ and $\hat{x}_2$ in terms of r and θ. Why might it *not* be appropriate to calculate a length from the origin to a point specified by finite values of r and θ using these covariant components?

Show that the components of the metrics g_{ij} and $\hat{g}_{ij}$ do not change under rotations of the coordinate system through a fixed angle α around the origin.

12. Write down a set of 3×3 matrices to represent the permutation group on three amplitudes, such that the action of swapping the second and third amplitudes is the matrix

$$(D_{132})^i{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Show that this matrix representation is reducible by the following steps:

(i) Find a common eigenvector for all the $D^i{}_j$ matrices.

(ii) Use the similarity transformation matrix $S^i{}_j$

$$S^i{}_j = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

to transform to a new basis. What is the common eigenvector in this basis?

(iii) Show that the transformation matrices in the new basis take on block-diagonal form.