B2: Symmetry and Relativity
Problem Set 4: Tensors and fields
MT 2022 Weeks 6-7

1. The 4-angular momentum of a single particle about the origin is defined as

$$L^{\mu\nu} \equiv X^\mu P^\nu - X^\nu P^\mu.$$

   (i) Prove that, in the absence of forces, $dL^{\mu\nu}/d\tau = 0$.

   (ii) Exhibit the relationship between the space-space part L^{ij} and the 3-angular momentum vector $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$.

   (iii) The total angular momentum of a collection of particles about the pivot R^λ is defined as

$$L^{\mu\nu}_{\rm tot}(R^\lambda) = \sum_i (X^\mu_i - R^\mu)P^\nu_i - (X^\nu_i - R^\nu)P^\mu_i$$

   where the sum runs over the particles (that is, X^μ and P^μ are 4-vectors, not 2nd-rank tensors; i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

2. Prove that the time rate of change of the angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ of a particle about an origin O is equal to the couple $\mathbf{r} \wedge \mathbf{f}$ of the applied force about O.

   If $L^{\mu\nu}$ is a particle's 4-angular momentum, and we define the 4-couple $G^{\mu\nu} \equiv X^\mu F^\nu - X^\nu F^\mu$, prove that $(d/d\tau)L^{\mu\nu} = G^{\mu\nu}$, and that the space-space part of this equation corresponds to the previous 3-vector result.

3. Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$ and ϕ such that

$$\begin{aligned} \mathbf{B} &= \nabla \wedge \mathbf{A} \\ \mathbf{E} &= -\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \nabla\phi. \end{aligned}$$

   (i) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ and ϕ.

   (ii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators, and Lorentz scalars (a *manifestly covariant* form).

4. How does a 2nd rank tensor change under a Lorentz transformation? By transforming the field tensor, and interpreting the result, prove that the electromagnetic field transforms as

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}) \\ \mathbf{B}'_{\perp} &= \gamma(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2) \end{aligned}$$

[Hint: you may find the algebra easier if you treat **E** and **B** separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]

Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.

5. The electromagnetic field tensor $F^{\mu\nu}$ (sometimes called the Faraday tensor) is defined such that the 4-force on a charged particle is given by

$$f^{\mu} = qF^{\mu\nu}U_{\nu}.$$

By comparing this to the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

which defines the electromagnetic and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $dP^{\mu}/d\tau$), show that the components of the field tensor are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

6. Show that the field equation

$$\partial_{\lambda} F^{\lambda\nu} = -\mu_0 \rho_0 U^{\nu}$$

is equivalent to

$$\partial^{\lambda}\partial_{\lambda} A^{\nu} - \partial^{\nu}(\partial_{\lambda} A^{\lambda}) = -\mu_0 J^{\nu}$$

where $J^{\nu} \equiv \rho_0 U^{\nu}$ (here ρ_0 is the proper charge density, and J^{ν} is the 4-current density). Comment.

7. Show that the following two scalar quantities are Lorentz invariant:

$$\begin{aligned} D &= B^2 - E^2/c^2 \\ \alpha &= \mathbf{B} \cdot \mathbf{E}/c. \end{aligned}$$

[Hint: for the second, introduce the dual field tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$.]

Show that if $\alpha = 0$ but $D \neq 0$ then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example such frame (by specifying its velocity relative to one in which the fields are **E**, **B**). Suggest a type of field for which both $\alpha = 0$ and $D = 0$.

8. Show that the space-space part of the energy-momentum tensor

$$T^{\mu\nu} = \varepsilon_0 c^2 \left(-F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right)$$

is

$$\sigma^{ij} = \frac{1}{2}\varepsilon_0 (E^k E_k + c^2 B^k B_k)\delta^{ij} - \varepsilon_0 (E^i E^j + c^2 B^i B^j)$$

(Greek indices run over space and time, and Latin indices over space only.)

Use the stress-energy tensor $T^{\mu\nu}$ to find the forces exerted by the magnetic field inside a long cylindrical solenoid of radius 3 cm and field 1 tesla. *Mu-metal* is an alloy of high magnetic permeability that can be used to provide shielding against magnetic fields. If a piece of mu-metal is placed against the end of a solenoid, it "confines" the magnetic field to the interior of the solenoid. By interpreting the stress-energy tensor for the field on each side of the mu-metal sheet, discover whether the latter is attracted or repelled by the solenoid, and find the net force.

Additional questions

9. Assuming the relation of fields $\mathbf{E}$ and $\mathbf{B}$ to potentials ϕ and $\mathbf{A}$, show that the field tensor can be written

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

(Note, the right hand side here is the 4-vector equivalent of a curl operation.) [Hint: use cyclic permutation to avoid unnecessary repetition.] Now write down $\partial^\lambda F^{\mu\nu}$ in terms of ∂ operators and A^μ. By keeping track of the sequence of indices, show that

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0.$$

(In an axiomatic approach, one could argue in the opposite direction, asserting the above as an axiom and then deriving the relation of fields to potentials.)

10. If $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ are two different electromagnetic fields, show that $\mathbf{E} \cdot \mathbf{E}' - c^2 \mathbf{B} \cdot \mathbf{B}'$ and $\mathbf{E} \cdot \mathbf{B}' + \mathbf{B} \cdot \mathbf{E}'$ are invariants.

Energy-momentum tensor

11. In a certain frame S_0 having 4-velocity U^μ, a 2nd-rank tensor $T^{\mu\nu}$ has but one non-zero component, $T^{00} = c^2$. Find the components of T in the general frame S, in which $U^\mu = \gamma_u (c, \mathbf{u})$.

12. Prove that for the electromagnetic stress-energy tensor, $T^\lambda{}_\lambda = 0$.

13. Prove that for the electromagnetic stress-energy tensor, $T^\mu{}_\lambda T^\lambda{}_\nu = (I\varepsilon_0/2)^2 \delta^\mu_\nu$ where $I^2 = (|\mathbf{E}|^2 - c^2 |\mathbf{B}|^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2 c^2$. [Hint: start by establishing the identity in a particular frame, such as one in which $\mathbf{E}$ is parallel to $\mathbf{B}$.]