Problem Sheet 4

Grav. radiation

1a) $$\phi = \frac{G}{5c^9} J^{(5)}_{ij} x^i x^j$$

$$J_{ij} = c^2 \int \rho (x_i x_j - \delta_{ij} r^2/3) dV = I_{ij} - \frac{1}{3}\delta_{ij} I_{kk}$$

$$\frac{dE}{dt} = \left\langle -c^2 \int \rho v_i \partial_i \phi \, dV \right\rangle = -\frac{G}{5c^9} \langle \dddot{J}_{ij} \dddot{J}_{ij} \rangle$$

- As suggested, we integrate by parts

$$-c^2 \int \rho v_i \partial_i \phi \, dV = \int c^2 \partial_i (\rho v_i) \phi \, dV$$

(with $\rho v_i \phi$ vanishing at ∞)

$$= -c^2 \int (\partial_t \rho) \phi \, dV = -c^2 \int \partial_t \rho \frac{G}{5c^9} J^{(5)}_{ij} x^i x^j dV$$

$$= -\frac{G}{5c^9} J^{(5)}_{ij} \partial_t \int c^2 \rho x^i x^j dV =$$

$$= -\frac{G}{5c^9} J^{(5)}_{ij} \frac{d}{dt} I_{ij}$$

$$\left\langle \dddot{J}^{(5)}_{ij} \frac{d}{dt} I_{ij} \right\rangle = \frac{1}{T}\int_{-T/2}^{T/2} dt\,(\ldots)$$

$\Rightarrow$ can integrate by parts in time with some assumpt. about the time deriv. vanish. at $T \to \pm\infty$

$$\Rightarrow \left\langle J^{(5)}_{ij} \frac{d}{dt} I_{ij} \right\rangle = \left\langle \dddot{J}_{ij}\, \dddot{I}_{ij} \right\rangle .$$

$$I_{ij} = J_{ij} + \frac{1}{3}\delta_{ij} I_{KK}$$

$$\dddot{J}_{ij}\,\dddot{I}_{ij} = \dddot{J}_{ij}\left(\dddot{J}_{ij} + \frac{1}{3}\delta_{ij}\dddot{I}_{KK}\right) =$$

$$= \dddot{J}_{ij}\,\dddot{J}_{ij}, \quad \text{since} \quad \dddot{J}_{ij}\,\delta_{ij} = \dddot{J}_{ii} = 0$$

(J_{ij} is traceless by construct.)

$$\Rightarrow \frac{dE}{dt} = -\frac{G}{5c^9}\left\langle J^{(5)}_{ij} \frac{d}{dt} I_{ij} \right\rangle =$$

$$= -\frac{G}{5c^9}\left\langle \dddot{J}_{ij}\,\dddot{J}_{ij} \right\rangle .$$

$$\frac{dL^k}{dt} = -\left\langle c^2 \int \varepsilon^{ijk} \rho\, x_i \partial_j \Phi\, dV \right\rangle =$$

$$= \left\langle -\int \varepsilon^{ijk} \rho c^2 x_i \partial_j \frac{G}{5c^9} J^{(5)}_{lm} x^l x^m\, dV \right\rangle =$$

$$= \left\langle -\frac{G J^{(5)}_{lm}}{5c^9} c^2 \int \varepsilon^{ijk} \rho\, x_i \partial_j (x^l x^m)\, dV \right\rangle =$$

$$= \left\langle -\frac{G J^{(5)}_{lm}}{5c^9} 2\, \varepsilon^{ilk} \underbrace{\int c^2 \rho\, x_i x_m\, dV}_{I_{im}} \right\rangle =$$

$$= \left\langle -\frac{2G}{5c^9} J^{(5)}_{lm} \varepsilon^{ilk} I_{im} \right\rangle =$$

$$= \left\langle -\frac{2G}{5c^9} \dddot{J}_{lm} \ddot{I}_{im} \varepsilon^{ilk} \right\rangle$$

Here we integrated by parts twice w.r.t. time — as before in part 1a

$$\ddot{I}_{im} = \ddot{J}_{im} + \delta_{im} \ddot{I}_{kk}/3$$

$$\dddot{J}_{lm} \ddot{I}_{im} \varepsilon^{ilk} = \varepsilon^{ilk} \dddot{J}_{lm} \ddot{J}_{im},$$ since $\varepsilon^{ilk} J_{lm} \delta_{im} \equiv 0$ $(J_{lm} = J_{ml})$.

$\Rightarrow$ result follows.

2a) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$

- Second deriv. of $h_{\mu\nu} \sim T_{\mu\nu}$
- Eqs linear in $h_{\mu\nu}$
- $T_{\mu\nu} = T_{\nu\mu}$
- Ingredients: $h_{\mu\nu}$, $h = h_\rho{}^\rho$

$\partial_\mu \partial_\nu$, $\partial_\rho \partial^\rho \equiv \Box$, $\eta_{\mu\nu}$

One can build the following combinations:

$$\eta_{\mu\nu} \partial_\rho \partial^\rho h \qquad \partial_\mu \partial_\nu h \qquad \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma}$$

$$\partial_\rho \partial^\rho h_{\mu\nu} \qquad \partial_\mu \partial_\sigma h^\sigma{}_\nu + \partial_\nu \partial_\sigma h^\sigma{}_\mu$$

Nothing else satisfies the requir. above.

2b) E.o.m:

$$\Box h_{\mu\nu} + \alpha \left(\partial_\rho \partial_\mu h^\rho{}_\nu + \partial_\rho \partial_\nu h^\rho{}_\mu \right) + \beta \partial_\mu \partial_\nu h$$

$$+ \gamma \eta_{\mu\nu} \Box h + \delta \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} = C T_{\mu\nu}$$

Impose conserv. of energy-momentum,

$\partial^\mu T_{\mu\nu} = 0 \Rightarrow$

$$\underline{\Box \partial^\mu h_{\mu\nu}} + \underline{\alpha \Box \partial_\rho h^\rho_\nu} + \underline{\underline{\alpha \partial_\rho \partial_\nu \partial_\mu h^{\mu\rho}}} +$$

$$+ \underline{\underline{\underline{\beta \partial_\nu \Box h}}} + \underline{\underline{\underline{\gamma \partial_\nu \Box h}}} + \delta \underline{\underline{\partial_\nu \partial_\rho \partial_\sigma h^{\rho\sigma}}} = 0$$

Coefficients in front of indep. tensor structures should vanish $\Rightarrow$ $\alpha = -1$, $\delta + \alpha = 0$, $\beta + \gamma = 0$ $\Rightarrow$ $\alpha = -1$, $\delta = 1$, $\beta = -\gamma$.

$\Rightarrow$ the e.o.m. becomes

$$\Box h_{\mu\nu} - \partial_\rho \partial_\mu h^\rho_\nu - \partial_\rho \partial_\nu h^\rho_\mu + \beta \partial_\mu \partial_\nu h -$$

$$- \beta \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} = C T_{\mu\nu}$$

2c)

Taking the trace:

1) $$\Box h_\mu^\nu - \partial_\rho \partial_\mu h^{\rho\nu} - \partial_\rho \partial^\nu h^\rho_\mu + \beta \partial_\mu \partial^\nu h -$$

$$- \beta \delta_\mu^\nu \Box h + \delta_\mu^\nu \partial_\rho \partial_\sigma h^{\rho\sigma} = C T_\mu^\nu$$

2) $$\Box h - \partial_\rho \partial_\mu h^{\rho\mu} - \partial_\rho \partial^\mu h^\rho_\mu + \beta \Box h -$$

$$- 4\beta \Box h + 4 \partial_\rho \partial_\mu h^{\rho\mu} = C T^\mu_\mu$$

i.e. $\Box h\,(1-3\beta) + 2\,\partial_\mu \partial_\nu h^{\mu\nu} = C\,T^\mu{}_\mu$

$T^\mu{}_\mu = T^0{}_0 + T^i{}_i \approx T^0{}_0$ in the non-rel. limit since $T^0{}_0$ contains $\sim \rho c^2$ (i.e. mc^2) + kin. terms, $T^i{}_i \sim$ pressure (i.e. kin. terms only).

$T^0{}_0 = -T_{00}$.

$\Rightarrow$ the e.o.m. becomes

$$\partial_\mu \partial_\nu h^{\mu\nu} = \frac{1}{2}(3\beta-1)\,\Box h - \frac{1}{2} C\,T_{00}$$

2d) Static limit : $\Box h \to \nabla^2 h$ (no time dep.)

$$\Rightarrow \partial_\mu \partial_\nu h^{\mu\nu} = \frac{3\beta-1}{2}\nabla^2 h - \frac{1}{2} C\,T_{00}$$

Now consider the 00-comp. of the eom in 2b): (setting to zero all time derivatives)

$$\Rightarrow \nabla^2 h_{00} + \beta\,\nabla^2 h - \left(\frac{3\beta-1}{2}\nabla^2 h - \frac{1}{2} C\,T_{00}\right) = C\,T_{00}$$

$$\Rightarrow \nabla^2 h_{00} + \frac{1-\beta}{2} \nabla^2 h = \frac{C}{2} T_{00}$$

Compare this with Poisson's eq

$$\nabla^2 \Phi(r) = 4\pi G \rho$$

in Newtonian grav.

Since $\quad g_{00} = \eta_{00} + h_{00} = -1 - \frac{2\Phi}{c^2}$

(e.g. think of Schwarzschild sol.

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \ldots$$

$$\Rightarrow g_{00} = -1 - \frac{2\Phi}{c^2}, \quad \Phi = -\frac{GM}{r} \;).$$

we have

$$-\frac{2}{c^2} \nabla^2 \Phi + \frac{1-\beta}{2} \nabla^2 h = \frac{C}{2} \rho c^2$$

$\Rightarrow$ to recover Poisson, set $\beta = 1$,

$$C = -16\pi G / c^4.$$

$\Rightarrow$ we fixed all coefficients in e.o.m.

$$\Box h_{\mu\nu} - \partial_\rho \partial_\mu h^\rho{}_\nu - \partial_\rho \partial_\nu h^\rho{}_\mu +$$

$$+ \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Fierz, Pauli (1939)

The l.h.s. can be compared with the (linear in $h_{\mu\nu}$) terms of the expansion of $R_{\mu\nu}$ and $g_{\mu\nu} R$: with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \left(\Box h_{\mu\nu} + \partial_\mu \partial_\nu h + \dots \right)$$

exactly as above $+ O(h^2)$

$$\Rightarrow \quad 2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$ is

a non-linear completion.

<u>Note</u> : the sign in front of $\frac{8\pi G}{c^4} T_{\mu\nu}$ depends on the definition of Riemann

E.g. for $R^\mu{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\mu_{\lambda\nu} - \partial_\nu \Gamma^\mu_{\lambda\mu} + \Gamma\Gamma - \Gamma\Gamma$

the sign would be $+$. In your lecture notes, it is $R^\mu{}_{\lambda\mu\nu} = \partial_\nu \Gamma^\mu_{\lambda\mu} - \partial_\mu \Gamma + \Gamma\Gamma - \Gamma\Gamma$

3a. $h_{\mu\nu} = A_{\mu\nu} e^{ikx}$

e.o.m. $R_{\mu\nu} = 0$ (in vac)

Expand with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\underset{0}{\underset{\shortparallel}{R^{(0)}_{\mu\nu}}} + R^{(1)}_{\mu\nu} + \dots = 0$$

$$R^{(1)}_{\lambda\mu\nu\kappa} = \frac{1}{2}\Big(\partial_\kappa \partial_\mu h_{\lambda\nu} - \partial_\kappa \partial_\lambda h_{\mu\nu} - \partial_\mu \partial_\nu h_{\lambda\kappa} + \partial_\nu \partial_\lambda h_{\mu\kappa} \Big)$$

$$R^{(1)\sigma}{}_{\mu\sigma\kappa} = \frac{1}{2}\Big(\partial_\kappa \partial_\mu h - \partial_\kappa \partial^\sigma h_{\mu\sigma} - \partial_\mu \partial^\sigma h_{\sigma\kappa} + \Box h_{\mu\kappa} \Big) = R^{(1)}_{\mu\kappa} = 0$$

$$\Rightarrow \Big(k_\kappa k_\mu A_\sigma{}^\sigma - k_\kappa k^\sigma A_{\mu\sigma} - k_\mu k^\sigma A_{\sigma\kappa} + k^2 A_{\mu\kappa} \Big) e^{ikx} = 0$$

i.e. $$k_\kappa k_\mu h - k_\kappa k^\sigma h_{\mu\sigma} - k_\mu k^\sigma h_{\sigma\kappa} + k^2 h_{\mu\kappa} = 0$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} h/2$$

$$k_\varkappa k_\mu h - k_\varkappa k^\sigma \left(\bar{h}_{\mu\sigma} + \eta_{\mu\sigma} \frac{h}{2} \right) -$$

$$- k_\mu k^\sigma \left(\bar{h}_{\sigma\varkappa} + \eta_{\sigma\varkappa} \frac{h}{2} \right) + k^2 h_{\mu\varkappa} =$$

$$= \underline{k_\varkappa k_\mu h} - k_\varkappa k^\sigma \bar{h}_{\mu\sigma} - \underline{k_\varkappa k_\mu \frac{h}{2}} -$$

$$- k_\mu k^\sigma \bar{h}_{\sigma\varkappa} - \underline{k_\mu k_\varkappa} \frac{h}{2} + k^2 h_{\mu\varkappa}$$

$$\Rightarrow \boxed{k_\varkappa k^\sigma \bar{h}_{\mu\sigma} + k_\mu k^\sigma \bar{h}_{\sigma\varkappa} - k^2 h_{\mu\varkappa} = 0}$$

3b). if $k^2 \neq 0$: $h_{\mu\varkappa} = \dfrac{k_\varkappa k^\sigma}{k^2} \bar{h}_{\mu\sigma} + \dfrac{k_\mu k^\sigma}{k^2} \bar{h}_{\sigma\varkappa}$

Substitute in $R^{(1)}_{\lambda\mu\nu\varkappa}$:

$$R^{(1)}_{\lambda\mu\nu\varkappa} = \frac{1}{2} \Big[\underline{k_\varkappa k_\mu \frac{k_\nu k^\sigma}{k^2} \bar{h}_{\lambda\sigma}} + \underline{k_\varkappa k_\mu \frac{k_\lambda k^\sigma}{k^2}} \bar{h}_{\sigma\nu}$$

$$- \underline{k_\varkappa k_\lambda} \frac{k_\nu k^\sigma}{k^2} \bar{h}_{\mu\sigma} - \underline{k_\varkappa k_\lambda \frac{k_\mu k^\sigma}{k^2}} \bar{h}_{\sigma\nu} -$$

$$- \underline{k_\mu k_\nu \frac{k_\varkappa k^\sigma}{k^2} \bar{h}_{\lambda\sigma}} - \cancel{k_\mu \frac{k_\lambda k^\sigma k^\varkappa}{k^2} \bar{h}_{\sigma\varkappa}} +$$

$$+ \underline{\underline{k_\nu k_\lambda \frac{k_\varkappa k^\sigma}{k^2} \bar{h}_{\mu\sigma}}} + \cancel{k_\nu k_\lambda \frac{k_\mu k^\sigma}{k^2} \bar{h}_{\sigma\varkappa}} \Bigg] = 0$$

3c) $k^2 = 0 \Rightarrow k_\varkappa k^\sigma \bar{h}_{\mu\sigma} + k_\mu k^\sigma \bar{h}_{\sigma\varkappa} = 0$

$k_\varkappa$, k_μ are different components of k

$\Rightarrow k^\sigma \bar{h}_{\sigma\nu} = 0$ (consider e.g. eq. $a_i b_j + a_j b_i = 0$ for $a_i \neq 0$)

This just means that the solution $\sim e^{ikx}$ with $k^2 = 0$ is the solution compatible with the harmonic gauge - not surprising, since in that gauge the e.o.m. is $\Box h_{\mu\nu} = 0$ whose solution is $\sim e^{ikx}$. Of course, the same solution with some other gauge choice will look different.

4. Total energy emitted = luminosity × time

For the bound state with parameters given in the problem,

$$\langle L_{GW} \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5} \left[\frac{1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4}{(1-\epsilon^2)^{7/2}} \right]$$

We take the limit corresp. to a parabolic motion. See Section 6.8.1 of the lecture notes (or any other source) for details.

$$r(1+\cos\varphi) = L \qquad L = a(1-\epsilon^2)$$

$\epsilon \to 1$: parabola

$$r = \frac{L}{1+\cos\varphi} \qquad L = J^2/GM = \text{const}$$

Closest approach: $r_{min} = L/2$

Period: $T = 2\pi\sqrt{a^3/GM} \sim (1-\epsilon^2)^{-3/2}$

$$E = T\langle L_{GW}\rangle \sim a^{3/2} \frac{1}{a^5} \frac{1}{(1-\epsilon^2)^{7/2}} \sim$$

$\sim$ finite at $\epsilon \to 1$.

$$E = \frac{64\pi}{5} \frac{G^{7/2}}{c^5 L^{7/2}} m_1^2 m_2^2 M^{1/2} \left[1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4 \right]$$

$$r_{min} \equiv b = L/2$$

$$\Rightarrow E = \frac{85\pi\sqrt{2}}{24} \frac{G^{7/2} M^{1/2} m_1^2 m_2^2}{c^5 b^{7/2}}$$

in the limit $\epsilon \to 1$.